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Prabhakar Fractional Integral with Multivariate Mittag-Leffler Function and Hardy-Type Inequalities

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Abstract. In this research, we introduce a multi-parametric Mittag-Leffler function and provide a new definition of the extended Prabhakar fractional integral operator. This operator generalizes several previously introduce fractional integral operators. We establish some basic properties of the introduced operator and the multi-parametric Mittag-Leffler function. To mention applications of the newly-introduced generalized operator, we study Prabhakar-based Hardy-type integral inequalities.

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1. Introduction and Preliminaries

Fractional calculus is a necessity in this era due to its natural applications in all sciences. Scientists are interested in it because of its demonstrated applications in various fields. For example, Wu *et al.* presented a novel weighted fractional operator and used it to develop several fractional models in [1]. Meanwhile, in [2], Samraiz *et al.* established extended versions of some fractional operators and evaluated the Laplace transform. The reader can also find the new fractional operator and some applications in [3]. Integer-order integrals and derivatives are insufficient to explain all physical phenomena. Many common fractional operators have been briefly discussed, including the Riemann-Liouville fractional

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derivative operator, k-fractional Hilfer derivative, k-Riemann-Liouville fractional derivative, and applications of numerous fractional operators, such as [4–12]. The Rieman-Liouville fractional integrals of order $\ell \in \mathbb{C}$ are defined by the following equations with $\mathcal{R}(\ell) > 0$, i.e.,

$$(\mathcal{I}^{\ell}_{a+} f)(\xi) = \frac{1}{\Gamma(\ell)} \int_{a}^{\xi} \frac{f(s)}{(\xi - s)^{1-\ell}} ds \tag{1}$$

and

$$(\mathcal{I}^{\ell}_{b-} f)(\xi) = \frac{1}{\Gamma(\ell)} \int_{\xi}^{b} \frac{f(s)}{(s - \xi)^{1-\ell}} ds. \tag{2}$$

The operators in (1) and (2) are introduced as the left and right Riemann-Liouville fractional integrals.

Mathematical inequalities play a fundamental role in the theory of fractional-order integrals and derivatives. Some interesting inequalities for fractional integrals under different convexities have been studied in works such as [13–15]. Hardy's integral inequality is one of the most important inequalities in mathematics and has been investigated by many researchers. Wu *et al.* investigated a variety of fractional Hardy-type integral inequalities in [16]. In the available literature, G.H. Hardy [17] established the following inequality in 1918, which is of particular significance to us:

$$\|\mathcal{I}^{\ell}_{a_+} \mathcal{G}\|_{\wp} \leq \mathcal{K} \|\mathcal{G}\|_{\wp}, \quad \|\mathcal{I}^{\ell}_{b_-} \mathcal{G}\|_{\wp} \leq \mathcal{K} \|\mathcal{G}\|_{\wp}, \quad 1 \leq \wp \leq \infty,$$

and it involves bounded fractional integral operators.

In 1920, Hardy [18] defined the following fundamental integral inequality

$$\int_{0}^{\infty} \Big(\frac{1}{\xi} \int_{0}^{\xi} \mathcal{G}(s) ds \Big)^{\wp} d\xi \leq \Big(\frac{\wp}{\wp - 1} \Big)^{\wp} \int_{0}^{\infty} \mathcal{G}^{\wp}(\xi) d\xi,$$

where $\wp > 1$ and $\mathcal{G} \in L_{\wp}(0, \infty)$.

Several mathematicians have recently studied the Hardy inequality and its variants. Their work involves a variety of fractional integral and derivative operators. (see [19–22]). It has a variety of uses, the most important of which is to stabilize degenerate stationary waves. These arguments have motivated many academics working on integral inequalities. They are particularly interested in specific fractional calculus operators used to develop extensions and generalizations. Such inequalities may be applicable to more general fractional integral operators.

Now, we will go through several definitions and theorems used to complete this research path during recent years. This will provide us with a comprehensive perspective and make our main goals in this article clear to the reader.

Diaz *et al.* defined the k-gamma and k-beta functions in [23] by the following definitions.

Definition 1. *[23] Let $k > 0$ and $\mathcal{R}(\ell) > 0$. The integral form of k-gamma function is defined by*

$$\Gamma_k(\ell) = \int_0^\infty \xi^{\ell-1} e^{\frac{-\xi^k}{k}} d\xi.$$

Definition 2. *[23] Let $\mathcal{R}(\ell) > 0, k > 0$ and $\mathcal{R}(\zeta) > 0$. The integral form of the k-beta function is defined by*

$$\beta_k(\ell, \zeta) = \frac{1}{k} \int_0^1 \xi^{\frac{\ell}{k}-1}(1-\xi)^{\frac{\zeta}{k}-1} d\xi.$$

Both Γ_k and β_k have the following relation

$$\beta_k(\ell, \zeta) = \frac{\Gamma_k(\ell)\Gamma_k(\zeta)}{\Gamma_k(\ell+\zeta)}.$$

The following definition mentions to the (k, r)-Riemann-Liouville fractional integral introduced by Sarikaya *et al.* in [24].

Definition 3. *Let f be a continuous function on the interval $[a, b]$ with $r \in \mathbb{R}\backslash\{-1\}$. Then (k, r)-Riemann-Liouville integral of f of order $\ell > 0$ is defined by*

$$ {}_r^k I_a^\ell f(\xi) = \frac{(r+1)^{1-\frac{\ell}{k}}}{k\Gamma_k(\ell)} \int_a^\xi (\xi^{r+1} - s^{r+1})^{\frac{\ell}{k}-1} s^r f(s) ds. \tag{3}$$

The following theorems will be used in our study. Fubini's theorem is proved in [25, pp. 664] and is stated as follows:

Theorem 1. *Let $(\Lambda_1, \sum_1, \Theta_1)$, $(\Lambda_2, \sum_2, \Theta_2)$ be two-dimensional spaces. Let Θ_1 and Θ_2 be complete and $\mathcal{G} : \Lambda_1 \times \Lambda_2 \to (-\infty, \infty)$ be $(\Theta_1 \times \Theta_2)$-integrable. Then for Θ_1-a.e., where $s \in \Lambda_1$ the function $\mathcal{G}(s, \cdot)$ is Θ_2- integrable and the function $\int_{\Lambda_2} \mathcal{G}(\cdot, \xi) d\Theta_2(\xi)$ is Θ_1- integrable. Similarly, for Θ_2-a.e. where $\xi \in \Lambda_2$ the function $\mathcal{G}(\cdot, \xi)$ is Θ_1- integrable and the function $\int_{\Lambda_1} \mathcal{G}(s, \cdot) d\Theta_1(s)$ is Θ_2- integrable. Moreover*

$$\int_{\Lambda_1} \left(\int_{\Lambda_2} \mathcal{G}(s, \xi) d\Theta_2(\xi) \right) d\Theta_1(s) = \int_{\Lambda_2} \left(\int_{\Lambda_1} \mathcal{G}(s, \xi) d\Theta_1(s) \right) d\Theta_2(\xi). \tag{4}$$

The integral version of the Jensen's inequality for convex functions, introduced by Jensen in 1906 [26], is presented in the following theorem.

Theorem 2. *Let $(\Lambda_1, \sum_1, \Theta)$ be a probability space and if $\Phi \circ \mathcal{G} \in L^1(\Theta)$ on $[a, b]$ with $a \leq \mathcal{G}(s) \leq b$ for all $s \in \Lambda_1$, then the inequality holds true*

$$\Phi \left(\int_{\Lambda_1} \mathcal{G}(s) d\Theta(s) \right) \leq \int_{\Lambda_1} \Phi(\mathcal{G}(s)) d\Theta(s), \tag{5}$$

for every convex mapping $\Phi : [a,b] \to \mathbb{R}$.

Definition 4. *Let f be a function defined on $[a,b]$. The space $X^p_\lambda(a,b)$, $1 \leq p \leq \infty$ denotes all Lebesgue measurable functions for which $\parallel \varphi \parallel_{X^p_\lambda} < \infty$, i.e.,*

$$\parallel \varphi \parallel_{X^p_\lambda} = \Big[(s+1)\int_a^b \mid \lambda(t)\varphi(t) \mid^p \Omega^s(t)\Omega'(t)dt\Big]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

for $\lambda(t) \neq 0$, $s \in \mathbb{R}$, *and*

$$\parallel \varphi \parallel_{X^\infty_\lambda} = ess\, sup_{a \leq \xi \leq b} \mid \lambda(t)\varphi(t) \mid < \infty.$$

In [27] the well-known Minkowski's integral inequality is stated as follows.

Let (Λ_1, Θ_1) and (Λ_2, Θ_2) be two σ-finite measure spaces and $F : \Lambda_1 \times \Lambda_2 \to \mathbb{R}$ be measurable. Then for $1 \leq \jmath < \infty$,

$$\left[\int_{\Lambda_2} \left|\int_{\Lambda_1} \mathcal{G}(\xi,s)d\Theta_1(\xi)\right|^\jmath d\Theta_2(s)\right]^{\frac{1}{\jmath}} \leq \int_{\Lambda_1} \left(\int_{\Lambda_2} |\mathcal{G}(\xi,s)|^\jmath d\Theta_2(s)\right)^{\frac{1}{\jmath}} d\Theta_1(\xi).$$

Let $(\Lambda_1, \sum_1, \Theta_1)$ and $(\Lambda_2, \sum_2, \Theta_2)$ be two finite σ-measures. Let $E(f,k)$ specifies the function's class $\mathcal{Z} : \Lambda_1 \to \mathbb{R}$ in regard to the representation

$$\mathcal{Z}(\xi) := \int\limits_{\Lambda_2} \mathcal{H}(\xi,s)\mathcal{F}(s)d\Theta_2(s)$$

and the operator $\mathcal{A}_\mathcal{H}$ be given by

$$\mathcal{A}_\mathcal{H}\mathcal{F}(\xi) := \frac{1}{\mathcal{K}(\xi)} \int\limits_{\Lambda_2} \mathcal{H}(\xi,s)\mathcal{F}(s)d\Theta_2(s), \tag{6}$$

where $\mathcal{H} : \Lambda_1 \times \Lambda_2 \to \mathcal{R}$ is measurable kernel with $\mathcal{H} \geq 0$ and $\mathcal{F} : \Lambda_2 \to \mathbb{R}$ is measurable, and

$$0 < \mathcal{K}(\xi) := \int\limits_{\Lambda_2} \mathcal{H}(\xi,s)d\Theta_2(s). \tag{7}$$

The following results are given in [28].

Theorem 3. *Let $(\Lambda_1, \sum_1, \Theta_1)$ and $(\Lambda_2, \sum_2, \Theta_2)$ be σ-finite measure spaces and $0 < \jmath \leq \hbar < \infty$. Let a weight function $\mathcal{U}$ be defined on Λ_1 and a measurable positive function $\mathcal{W}$ be defined on Λ_2. Also $\mathcal{H}$ be a non-negative measurable function on $\Lambda_1 \times \Lambda_2$ and $\mathcal{K}$ is given by (7). Moreover $\mathcal{K}(\xi) > 0$ for all $\xi \in \Lambda_1$ and the function $\xi \mapsto \mathcal{U}(\xi)(\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)})^{\frac{\hbar}{\jmath}}$ is integrable on Λ_1 for each fixed element $s \in \Lambda_2$, and a convex map Φ is given on a non-negative interval $I \subseteq \mathbb{R}$. If*

$$\mathcal{A}_1 := \sup_{s \in \Lambda_2} \mathcal{W}^{\frac{-1}{\jmath}}(s) \left(\int\limits_{\Lambda_1} \mathcal{U}(\xi) \left(\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}\right)^{\frac{\hbar}{\jmath}} d\Theta_1(\xi)\right)^{\frac{1}{\hbar}} < \infty,$$

then for some $\mathrm{E}_1 > 0$*, the inequality*

$$\left(\int\limits_{\Lambda_1} \mathcal{U}(\xi)\left[\Phi(\mathcal{A}_{\mathcal{H}}f(\xi))\right]^{\frac{\hbar}{\jmath}} d\Theta_1(\xi)\right)^{\frac{1}{\hbar}} \leq \mathrm{E}_1 \left(\int\limits_{\Lambda_2} \mathcal{W}(s)\Phi(f(s))d\Theta_2(s)\right)^{\frac{1}{\jmath}} \tag{8}$$

is valid for measurable mapping $f : \Lambda_2 \to \mathbb{R}$ *with* $Imf \subseteq I$ *in which* $\mathcal{A}_{\mathcal{H}}f$ *is defined by* (6)*. Moreover, if* E_1 *is the least positive number for* (8) *to be held, then* $\mathrm{E}_1 \leq \mathcal{A}_1$*.*

Corollary 1. *[28] Let* $-\infty < \hbar \leq \jmath < 0$ *and consider the conditions of Theorem 3. For a convex function* $\Phi > 0$*, if*

$$\mathcal{A}_2 = \inf_{s\in\Lambda_2} \mathcal{W}^{\frac{-1}{\jmath}}(s)\left(\int\limits_{\Lambda_1} \mathcal{U}(\xi)\left(\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}\right)^{\frac{\hbar}{\jmath}} d\Theta_1(\xi)\right)^{\frac{1}{\hbar}} < \infty,$$

then for some positive real numbers E_2 *the inequality*

$$\left(\int\limits_{\Lambda_1} \mathcal{U}(\xi)\left[\Phi(\mathcal{A}_{\mathcal{H}}f(\xi))\right]^{\frac{\hbar}{\jmath}} d\Theta_1(\xi)\right)^{\frac{1}{\hbar}} \geq \mathrm{E}_2 \left(\int\limits_{\Lambda_2} \mathcal{W}(s)(f(s))d\Theta_2(s)\right)^{\frac{1}{\jmath}}, \tag{9}$$

is valid for measurable function $f : \Lambda_2 \to \mathbb{R}$ *with* $Imf \subseteq I$*. In addition, if* E_2 *is the largest positive number for* (9) *to be held, then* $\mathrm{E}_2 \geq \mathcal{A}_2$*.*

The Cizmesija *et al.* proved a theorem in [29] which is stated here.

Theorem 4. *Let* $(\Lambda_1, \sum_1, \Theta_1)$ *and* $(\Lambda_2, \sum_2, \Theta_2)$ *be* σ*-finite measure spaces. Let a weight function* $\mathcal{U}$ *be defined on* Λ_1 *and* $\mathcal{H} \geq 0$ *be a measurable function on* $\Lambda_1 \times \Lambda_2$*. Let* $\mathcal{K}$ *be defined on* Λ_1 *by* (7)*. Suppose that* $\mathcal{K}(\xi)$ *for all* $\xi \in \Lambda_1$*, and the function* $\xi \mapsto \mathcal{U}(\xi)\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}$ *is integrable on* Λ_1 *for each fixed* $s \in \Lambda_2$*, and that* $\mathcal{V}$ *is given on* Λ_2 *as*

$$\mathcal{V}(s) := \int\limits_{\Lambda_1} \mathcal{U}(\xi)\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)} d\Theta_1(\xi) < \infty. \tag{10}$$

If Φ*, on the interval* $I \subseteq \mathbb{R}$*, is convex, then*

$$\int_{\Lambda_1} \mathcal{U}(\xi)\Phi\left(\mathcal{A}_{\mathcal{H}}f(\xi)\right) d\Theta_1(\xi) \leq \int_{\Lambda_2} \mathcal{V}(s)\Phi\left(f(s)\right) d\Theta_2(s), \tag{11}$$

holds for measurable function $f_i : \Lambda_2 \to \mathbb{R}$ *s.t.* $Imf \subseteq I$*, and* $\mathcal{A}_{\mathcal{H}}$ *is formulated as* (6)*.*

Theorem 5. *[29] Let* $(\Lambda_1, \sum_1, \Theta_1)$ *and* $(\Lambda_2, \sum_2, \Theta_2)$ *be* σ*-finite measure spaces, a weight function* $\mathcal{U}$ *be defined on* Λ_1*. Consider a measurable function* $\mathcal{H}$ *which is non-negative on*

$\Lambda_1 \times \Lambda_2$. *Assume that the function* $\xi \mapsto \mathcal{U}(\xi)\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}$ *is integrable on* Λ_1 *for each fix* $s \in \Lambda_2$ *and* $\mathcal{P}$ *defined on* Λ_2 *has the form*

$$\mathcal{P}(s) := f_2(s) \int\limits_{\Lambda_1} \mathcal{U}(\xi)\frac{\mathcal{H}(\xi,s)}{\mathcal{Z}_2(\xi)} d\Theta_1(\xi) < \infty.$$

For a convex function $\Phi : I \to \mathbb{R}$, *if* $\frac{\mathcal{Z}_1(\xi)}{\mathcal{Z}_2(\xi)}, \frac{f_1(s)}{f_2(s)} \in I$, *then the inequality*

$$\int_{\Lambda_1} \mathcal{U}(\xi)\Phi\left(\frac{\mathcal{Z}_1(\xi)}{\mathcal{Z}_2(\xi)}\right) d\Theta_1(\xi) \leq \int_{\Lambda_2} \mathcal{P}(s)\Phi\left(\frac{f_1(s)}{f_2(s)}\right) d\Theta_2(s), \tag{12}$$

is valid $\forall\, \mathcal{G}_i \in D(f_i, k)$ $(i = 1, 2)$ *and each measurable mappings* $f_i : \Lambda_2 \to \mathbb{R}$.

Remark 1. *If* Φ *on the other hand, is exactly convex on* I *and if* $\frac{f_1(\xi)}{f_2(\xi)}$ *is variable, the inequality presented in* (12) *is therefore strict.*

The following theorem is connected to any arbitrary convex function (see [30]).

Theorem 6. *Consider the hypotheses of Theorem 4 and a convex mapping* Φ *on* $I \subseteq \mathbb{R}$. *If* $\varphi : I \to \mathbb{R}$ *be so that* $\varphi \in \partial\Phi(\xi)$ *for each* $\xi \in IntI$ *, then*

$$\begin{aligned}\int_{\Lambda_2} &\mathcal{V}(s)\Phi\left(f(s)\right) d\Theta_2(s) - \int_{\Lambda_1} \mathcal{U}(\xi)\Phi\left(\mathcal{A}_{\mathcal{H}}f(\xi)\right)\right) d\Theta_1(\xi) \\ &\geq \int_{\Lambda_1} \frac{\mathcal{U}(\xi)}{\mathcal{K}(\xi)} \int_{\Lambda_2} \mathcal{H}(\xi,s) \left\| \Phi\left(f(s)\right) - \Phi\left(\mathcal{A}_{\mathcal{H}}f(\xi)\right)\right| \\ &\quad - \left|\varphi\left(\mathcal{A}_{\mathcal{H}}f(\xi)\right)\right| \cdot \left|f(s) - \mathcal{A}_{\mathcal{H}}f(\xi)\right\| d\Theta_2(s) d\Theta_1(\xi),\end{aligned}$$

is fulfilled for any measurable mapping $f : \Lambda_2 \to \mathbb{R}$ *s.t.* $f(s) \in I$ *for each fixed* $s \in \Lambda_2$. *If on an interval* $I \subseteq \mathbb{R}, \Phi$ *is monotone and convex, then*

$$\begin{aligned}\int_{\Lambda_2} &\mathcal{V}(s)\Phi\Big(f(s)\Big) d\Theta_2(s) - \int_{\Lambda_1} \mathcal{U}(\xi)\Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big) d\Theta_1(\xi) \\ &\geq \Big| \int_{\Lambda_1} \frac{\mathcal{U}(\xi)}{\mathcal{K}(\xi)} \int_{\Lambda_2} sgn\Big(f(s) - (\mathcal{A}_{\mathcal{H}}f(\xi)\Big) \mathcal{H}(\xi,s) \Big[\Phi\Big(f(s)\Big) - \Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big) \\ &\quad - \Big|\varphi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big| . \Big(f(s) - \mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big] d\Theta_2(s) d\Theta_1(\xi) \Big|,\end{aligned}$$

is valid for any measurable function $f : \Lambda_2 \to \mathbb{R}$ *s.t.* $f(s) \in I$ *for each fixed* $s \in \Lambda_2$, *in which* $\mathcal{A}_{\mathcal{H}}f$ *is given by* (6).

The following theorem, established by Kaijser *et al.* in [31], proves a refinement of a Hardy-type inequality.

Theorem 7. *Let a weight mapping $\mathcal{U}$ be given on $(0,b)$ and $\xi \mapsto \frac{\mathcal{U}(\xi)}{\xi}\frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}$ be integrable on (s,b) $\forall$ fixed $s \in (0,b)$, and define $\check{\mathcal{W}} : (0,b) \to \mathbb{R}$ as*

$$\check{\mathcal{W}}(s) := s\int_s^b \frac{\mathcal{H}(\xi,s)}{\mathcal{K}(\xi)}\mathcal{U}(\xi)\frac{d(\xi)}{\xi},$$

with $0 < b < \infty$, $\mathcal{H} : (0,b)^2 \to \mathbb{R}$ be a positive function and

$$\mathcal{K}(\xi) := \int_0^\xi \mathcal{H}(\xi,s)ds > 0.$$

If a convex function Φ, on an interval $I \subseteq \mathbb{R}$, is so that $\varphi(\xi) \in \partial\Phi(\xi)$ for every $\xi \in IntI$, then

$$\int_0^b \check{\mathcal{W}}(s)\Phi\Big(f(s)\Big)\frac{ds}{s} - \int_0^b \mathcal{U}(\xi)\Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\frac{d\xi}{\xi} \geq \int_0^b \frac{\mathcal{U}(\xi)}{\mathcal{K}(\xi)}\int_0^\xi \mathcal{H}(\xi,s) \tag{13}$$

$$\times \bigg|\Big|\Phi\Big(f(s)\Big) - \Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big| - \Big|\varphi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big|.\Big|f(s) - \mathcal{A}_{\mathcal{H}}f(\xi)\Big|\bigg|ds\frac{d\xi}{\xi},$$

holds for any measurable function $f : (0,b) \to \mathbb{R}$ having values in I. If Φ is a concave function, in that case, the sequence of integrals on the L.H.S. of (13) *is inverted. If Φ is monotone convex, then*

$$\int_0^b \check{\mathcal{W}}(s)\Phi\Big(f(s)\Big)\frac{ds}{s} - \int_0^b \mathcal{U}(\xi)\Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\frac{d\xi}{\xi}$$

$$\geq \bigg|\int_0^b \frac{\mathcal{U}(\xi)}{\mathcal{K}(\xi)}\int_0^\xi sgn\Big(f(s) - \mathcal{A}_{\mathcal{H}}f(\xi)\Big)\mathcal{H}(\xi,s)\Big[\Phi\Big(f(s)\Big) - \Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)$$

$$- \Big|\varphi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big|.\Big(f(s) - \mathcal{A}_{\mathcal{H}}f(\xi)\Big)\Big]ds\frac{d\xi}{\xi}\bigg|,$$

holds on the interval $I \subseteq \mathbb{R}$ for any measurable function $f : (0,b) \to \mathbb{R}$.

The mean value theorem presented in the sequel is proved in [32].

Theorem 8. *Let $(\Lambda_1, \sum_1, \Theta_1)$ and $(\Lambda_2, \sum_2, \Theta_2)$ be measure spaces with positive σ-finite measures, $\mathcal{U}$ be a weight mapping on Λ_1, the interval I be compact in $\mathcal{R}$, $h \in \mathcal{C}^2(I)$, and $f : \Lambda_2 \to \mathbb{R}$ a measurable function such that $Imf \subseteq I$. Then $\exists\, \varrho \in I$ s.t.*

$$\int_{\Lambda_2} \mathcal{V}(s)h\Big(f(s)\Big)d\Theta_2(s) - \int_{\Lambda_1} \mathcal{U}(\xi)h\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)d\Theta_1(\xi)$$

$$= \frac{h''(\varrho)}{2}\Big[\int_{\Lambda_2}\mathcal{V}(s)f^2(s)d\Theta_2(s) - \int_{\Lambda_1}\mathcal{U}(\xi)\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)^2 d\Theta_1(\xi)\Big].$$

Theorem 9. *[32] Consider the conditions of Theorem 8. Furthermore, let $j, h \in \mathcal{C}^2(I)$ be so that $h''(\xi) \neq 0$ for all $\xi \in I$ and*

$$\int_{\Lambda_2}\mathcal{V}(s)h\Big(f(s)\Big)d\Theta_2(s) - \int_{\Lambda_1}\mathcal{U}(\xi)h\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)d\Theta_1(\xi) \neq 0.$$

Then $\exists\, \varrho \in I$ such that

$$\frac{j''(\varrho)}{h''(\varrho)} = \frac{\int_{\Lambda_2}\mathcal{V}(s)j\Big(f(s)\Big)d\Theta_2(s) - \int_{\Lambda_1}\mathcal{U}(\xi)j\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)d\Theta_1(\xi)}{\int_{\Lambda_2}\mathcal{V}(s)h\Big(f(s)\Big)d\Theta_2(s) - \int_{\Lambda_1}\mathcal{U}(\xi)h\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)d\Theta_1(\xi)}.$$

The main goal of this research study is to fill the existing gaps related to a new kind of multi-parametric integral operators in the sense of Prabhakar. Moreover, another novelty of this study is to investigate a generalized version of some well-known integral inequalities through the multi-parametric generalized Prabhakar fractional integral. After recalling some notions in the direction of the purposes of the manuscript (in the present section), we introduce the multi-parametric generalized Prabhakar fractional integral and its related properties in Section 2. To demonstrate the useful applications of these new multi-parametric operators, we derive the Prabhakar-based Hardy-type integral inequalities and other variants in Section 3. Finally, in Section 4, we present the conclusions.

2. Generalized Multi-Parametric Prabhakar Operators

In this section, we first define the multi-parametric Mittag-Leffler function with seven parameters. Second, we define the extended Prabhakar fractional integral operator using this generalized form and establish some properties.

Definition 5. *Consider $\ell, \beta, \gamma, \Lambda \in \mathbb{C}$, $k \in \mathbb{R}^+$, $n \in \mathbb{N}$, $q, \rho > 0$ with $q \leq \mathcal{R}(\ell) + \rho$, and*

$$\min\{\mathcal{R}(\ell), \mathcal{R}(\beta), \mathcal{R}(\gamma), \mathcal{R}(\Lambda)\} > 0,$$

then the k-generalized Mittag-Leffler function is defined as

$$E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\theta) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\ell n+\beta)}\frac{\theta^n}{(\Lambda)_{\rho n,k}}, \tag{14}$$

where $(\gamma)_{qn,k}$ denotes the Pochhammer symbol and can be represented as $\frac{\Gamma_k(qn+\gamma)}{\Gamma_k(\gamma)}$.

By specifying the following settings, the generalized multi-parametric function (14) covers the following prior generalizations of the standard Mittag-Leffler function.

Remark 2. *(i) If $k = 1$, the function (14) becomes*

$$E^{\gamma,\Lambda,q}_{\ell,\beta,\rho}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\ell n + \beta)} \frac{\theta^n}{(\Lambda)_{\rho n}},$$

which is defined in [33].

(ii) If $\rho = q = k = 1$, the equation (14) *reduces to*

$$E^{\gamma,\Lambda}_{\ell,\beta}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(\ell n + \beta)} \frac{\theta^n}{(\Lambda)_{n}},$$

which is stated by Salim in [34].

(iii) If we choose $k = \Lambda = \rho = 1$, (14) *represents*

$$E^{\gamma,q}_{\ell,\beta}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\ell n + \beta)} \frac{\theta^n}{n!},$$

that was explored by Shukla et al. in [35]. Srivastava et al. depict the features of such an extended function as well as its presence for a larger set of parameters in [36].

(iv) For $\rho = \Lambda = q = 1$, we have

$$E^{\gamma}_{k,\ell,\beta}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\ell n + \beta)} \frac{\theta^n}{n!},$$

which is given in [37].

(v) If $\rho = \Lambda = q = k = 1$, it represents

$$E^{\gamma}_{\ell,\beta,}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(\ell n + \beta)} \frac{\theta^n}{n!},$$

which is defined by Prabhakar in [38].

(vi) It turns into the Wiman's function in [39] for $\rho = \gamma = \Lambda = q = k = 1$. In addition, by taking $\beta = 1$, the standard Mittag-Leffler function E_ℓ will be derived.

The modified fractional integral operator based on the Mittag-Leffler function with seven parameters (14) is defined as follows:

Definition 6. *Let $r \in \mathbb{R}\backslash\{-1\}$, $\ell, \beta, \gamma, \Lambda, \omega \in \mathbb{C}$, $k \in \mathbb{R}^+$, $n \in \mathbb{N}$, $q, \rho > 0$ with $q \leq \mathcal{R}(\ell) + \rho$, and*

$$\min\{\mathcal{R}(\ell), \mathcal{R}(\beta), \mathcal{R}(\gamma), \mathcal{R}(\Lambda)\} > 0,$$

then the modified (k,r)-fractional integral operator of order β is given by

$$({}_r^k\mathcal{I}_{a^+,\ell,\beta,\rho}^{\omega,\gamma,\Lambda,q}f)(\xi) = \frac{(r+1)^{1-\frac{\beta}{k}}}{k}\int\limits_a^\xi (\xi^{r+1}-s^{r+1})^{\frac{\beta}{k}-1}s^r E_{k,\ell,\beta,\rho}^{\gamma,\Lambda,q}(\omega(\xi^{r+1}-s^{r+1})^{\frac{\ell}{k}})f(s)ds. \quad (15)$$

We obtain the modified integral operator incorporating the k-Mittag-Leffler function defined in [2] if $\rho = \Lambda = q = 1$.

Now, in the following definition, we define the more general form of operator described in (15). We call it the generalized multi-parametric Prabhakar fractional integral.

Definition 7. *Let $r \in \mathbb{R}\backslash\{-1\}, \ell, \beta, \gamma, \Lambda, \omega \in \mathbb{C}$, $k \in \mathbb{R}^+$, $n \in \mathbb{N}$, $q, \rho > 0$ with $q \leq \mathcal{R}(\ell)+\rho$, and*

$$\min\{\mathcal{R}(\ell), \mathcal{R}(\beta), \mathcal{R}(\gamma), \mathcal{R}(\Lambda)\} > 0.$$

The generalized multi-parametric Prabhakar fractional integral operator of f w.r.t. another strictly increasing function g is defined as

$$\begin{aligned}({}_r^k\mathcal{I}_{a^+,\ell,\beta,\rho;g}^{\omega,\gamma,\Lambda,q}f)(\xi) = \frac{(r+1)^{1-\frac{\beta}{k}}}{k}\int\limits_a^\xi &(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\beta}{k}-1}g^r(s)\\ &\times E_{k,\ell,\beta,\rho}^{\gamma,\Lambda,q}(\omega(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\ell}{k}})g'(s)f(s)ds. \qquad (16)\end{aligned}$$

Note that the new generalized integral (16) covers several cases that exist in the literature. The fractional integral operator (15) is achieved if $g(\xi) = \xi$. The (k,r)-Prabhakar fractional operator described by Samraiz *et al.* in [2] corresponds to the case $g(\xi) = \xi, \rho = \Lambda = q = 1$. If we choose $g(\xi) = \xi, \rho = \Lambda = q = 1$ and $r = 0$, we get the k-Prabhakar integral operator described by Dorrego in [40]. By taking $g(\xi) = \xi, k = 1, \rho = \Lambda = q = 1$, and $r = 0$, we get the Prabhakar integral operator introduced by Prabhakar [38]. Note that the aforementioned operator simplifies to the (k,r)-Riemann-Liouville fractional integral described by Sarikaya *et al.* in [24] for $\gamma = 0$ and $g(\xi) = \xi$.

In the next theorem, we prove that our introduced operator is bounded in the space X_w^p given in [41].

Theorem 10. *Let $r \in \mathcal{R}\backslash\{-1\}$, $\ell, \beta, \gamma, \Lambda, \omega \in \mathcal{C}$, $k \in \mathcal{R}^+$, $n \in \mathbb{N}$,*

$$\min\{\mathcal{R}(\ell), \mathcal{R}(\beta), \mathcal{R}(\gamma), \mathcal{R}(\Lambda)\} > 0,$$

and $\rho, q > 0$ and $q \leq \mathcal{R}(\ell)+\rho$. Then the generalized Prabhakar fractional integral operator of a function f with respect to another strictly increasing function g is bounded; i.e.,

$$\left\|({}_r^k\mathcal{I}_{a,\ell,\beta,\rho;g}^{\omega,\gamma,\Lambda,q}f)\right\|_{X^p} \leq (r+1)^{-\frac{\beta}{kp}}(g^{r+1}(t)-g^{r+1}(a))^{\frac{\beta}{k}-1}E_{k,\ell,\beta,\rho}^{\gamma,\Lambda,q}(\omega(g^{r+1}(t)-g^{r+1}(a))^{\frac{\ell}{k}})||f||_{X^p}.$$

Proof. By using Definitions 4 and 16, we have

$$\left\|\left({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a,\ell,\beta,\rho;g}f\right)(\xi)\right\|_{X^p}=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}$$

$$\times\left\|\int\limits_a^{\xi}(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\beta}{k}-1}g^r(s)E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\ell}{k}})g'(s)f(s)ds\right\|_{X^p}$$

$$=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\left\|\int_a^{\xi}(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\beta+\ell n}{k}-1}g^r(s)g'(s)f(s)ds\right\|_{X^p}$$

$$=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\left((r+1)\int_a^t\left|\int_a^{\xi}(g^{r+1}(\xi)-g^{r+1}(s))^{\frac{\beta+\ell n}{k}-1}g^r(s)g'(s)f(s)ds\right|^p g^r(\xi)g'(\xi)d\xi\right)^{\frac{1}{p}}.$$

By substituting $g^{r+1}(\xi)=\alpha$ and $g^{r+1}(s)=\delta$, we have

$$=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\left(\int_{g^{r+1}(a)}^{g^{r+1}(t)}\left|\int_{g^{r+1}(a)}^{g^{r+1}(\xi)}\left(\alpha-\delta\right)^{\frac{\beta+\ell n}{k}-1}f(g^{-1}(\delta^{\frac{1}{r+1}}))d\delta\right|^p d\alpha\right)^{\frac{1}{p}}.$$

Using the generalized Minkowski's inequality, we have

$$\leq\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\int_{g^{r+1}(a)}^{g^{r+1}(t)}\left(|f(g^{-1}(\delta^{\frac{1}{r+1}}))|^p\int_{\delta}^{g^{r+1}(t)}\left(\alpha-\delta\right)^{(\frac{\beta+\ell n}{k}-1)p}d\alpha\right)^{\frac{1}{p}}d\delta$$

$$=\frac{(r+1)^{-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\int_{g^{r+1}(a)}^{g^{r+1}(t)}\left(|f(g^{-1}(\delta^{\frac{1}{r+1}}))|^p\frac{\left(g^{r+1}(t)-\delta\right)^{(\frac{\beta+\ell n}{k}-1)p+1}}{(\frac{\beta+\ell n}{k}-1)p+1}\right)^{\frac{1}{p}}d\delta.$$

If we use the Hölder's inequality, then we have

$$\leq \frac{(r+1)^{-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\left(\int_{g^{r+1}(a)}^{g^{r+1}(t)}|f(g^{-1}(\delta^{\frac{1}{r+1}}))|^p d\delta\right)^{\frac{1}{p}}\int_{g^{r+1}(a)}^{g^{r+1}(t)}\left(\left(\frac{\left(g^{r+1}(t)-\delta\right)^{(\frac{\beta+\ell n}{k}-1)p+1}}{((\frac{\beta+\ell n}{k}-1)p+1}\right)^{\frac{q}{p}}d\delta\right)^{\frac{1}{q}}$$

$$\leq \frac{(r+1)^{-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\left(\int_{g^{r+1}(a)}^{g^{r+1}(t)}|f(g^{-1}(\delta^{\frac{1}{r+1}}))|^p d\delta\right)^{\frac{1}{p}}\frac{\left(g^{r+1}(t)-g^{r+1}(a)\right)^{\frac{\beta+\ell n}{k}}}{\frac{\beta+\ell n}{k}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now substituting $(g^{-1}(\delta^{\frac{1}{r+1}}))=s$, we have

$$=\frac{(r+1)^{-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\frac{\left(g^{r+1}(t)-g^{r+1}(a)\right)^{\frac{\beta+\ell n}{k}}}{\frac{\beta+\ell n}{k}}\left((r+1)\int_a^t|f(s)|^p g^r(s)g'(s)ds\right)^{\frac{1}{p}}$$

$$=(r+1)^{-\frac{\beta}{k}}(g^{r+1}(t)-g^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+1,\rho}(\omega(g^{r+1}(t)-g^{r+1}(a))^{\frac{\ell}{k}})||f||_{X^p}.$$

The proof is completed.

We investigate some other properties. First, we give an identity.

Lemma 1. *Let* $r\in\mathbb{R}\backslash\{-1\},\ell,\beta,\upsilon,\sigma,\gamma,\Lambda,\lambda,\omega\in\mathbb{C},k\in\mathbb{R}^+,n\in\mathbb{N};\mathcal{R}(\ell)>0,\mathcal{R}(\beta)>0,\mathcal{R}(\upsilon)>0,\mathcal{R}(\sigma)>0,\mathcal{R}(\lambda)>0$. *Then we have the identity*

$$\frac{1}{k}\int_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s)$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})E^{\sigma,\lambda,q}_{k,\ell,\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})ds$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{r+1}E^{\gamma+\sigma,\Lambda+\lambda,q}_{k,\ell,\beta+\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}}).$$

Proof. Consider

$$\frac{1}{k}\int_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s)$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})E^{\sigma,\lambda,q}_{k,\ell,\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})ds$$

$$=\frac{1}{k}\int_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)$$

$$\times\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\ell n+\beta)}\frac{\omega^n}{(\Lambda)_{\rho n,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell n}{k}}$$

$$\times\sum_{m=0}^{\infty}\frac{(\sigma)_{qm,k}}{\Gamma_k(\ell m+\upsilon)}\frac{\omega^m}{(\lambda)_{\rho m,k}}(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\ell m}{k}}ds:=I.$$

Substituting $u=\frac{\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s)}{\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a)}$ and using Definition 2 we have

$$I=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{k(r+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^{n+m}}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\frac{(\sigma)_{qm,k}}{\Gamma_k(\ell m+\upsilon)(\lambda)_{\rho m,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell(m+n)}{k}}\int_0^1 u^{\frac{\beta+\ell n}{k}-1}(1-u)^{\frac{\upsilon+\ell m}{k}-1}du$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^{n+m}}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}$$

$$\times\frac{(\sigma)_{qm,k}}{\Gamma_k(\ell m+\upsilon)(\lambda)_{\rho m,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell(m+n)}{k}}B_k(\ell n+\beta,\ell m+\upsilon)$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)}\sum_{m=0}^{\infty}\frac{\omega^m(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell m}{k}}}{\Gamma_k(\ell m+\beta+\upsilon)}\sum_{n=0}^{m}\frac{(\sigma)_{q(m-n),k}(\gamma)_{qn,k}}{(\Lambda)_{\rho n,k}(\lambda)_{\rho(m-n),k}}$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)}E^{\gamma+\sigma,\Lambda+\lambda,q}_{k,\ell,\beta+\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}}).$$

This ends the argument.

Proposition 1. *(Semi-group property) Let* $r\in\mathbb{R}\backslash\{-1\},\ell,\beta,\upsilon,\sigma,\gamma,\Lambda,\lambda,\omega\in\mathbb{C},k\in\mathbb{R}^+,n\in\mathbb{N};\mathcal{R}(\ell),\mathcal{R}(\beta),\mathcal{R}(\upsilon),\mathcal{R}(\sigma),\mathcal{R}(\lambda)>0$. *Then for any* $f\in L^1[0,B]$ *and* $\xi\in[0,B]$, *we have*

$$ {}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;\mathfrak{g}}\left[{}^{k}_{r}\mathcal{I}^{\omega,\sigma,\lambda,q}_{a^+,\ell,\upsilon,\rho;\mathfrak{g}}f\right](\xi)=({}^{k}_{r}\mathcal{I}^{\omega,\gamma+\sigma,\Lambda\lambda,q}_{a^+,\ell,\beta+\upsilon,\rho;\mathfrak{g}}f)(\xi). \tag{17}$$

Proof. By utilizing Definition 7, Lemma 1 and Dirichlet formula for interchanging the order of integration, we get

$$ {}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;\mathfrak{g}}[{}^{k}_{r}\mathcal{I}^{\omega,\sigma,\lambda,q}_{a^+,\ell,\upsilon,\rho;\mathfrak{g}}f](\xi)$$

$$= \frac{(r+1)^{1-\frac{\beta}{k}}}{k} \int_0^{\xi} (\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} \mathfrak{g}^r(s)\mathfrak{g}'(s) E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})$$

$$\times \frac{(r+1)^{1-\frac{\upsilon}{k}}}{k} \int_0^{s} (\mathfrak{g}^{r+1}(s) - \mathfrak{g}(\theta)^{r+1})^{\frac{\upsilon}{k}-1} \mathfrak{g}^r(\theta)\mathfrak{g}'(\theta)$$

$$\times E^{\sigma,\lambda,q}_{k,\ell,\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(s) - \mathfrak{g}(\theta)^{r+1})^{\frac{\ell}{k}}) f(\theta) d\theta ds$$

$$= \frac{(r+1)^{2-(\frac{\beta+\upsilon}{k})}}{k^2} \int_0^{\xi} \mathfrak{g}^r(\theta)\mathfrak{g}'(\theta) f(\theta) \int_{\theta}^{\xi} (\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}$$

$$\times (\mathfrak{g}^{r+1}(s) - \mathfrak{g}(\theta)^{r+1})^{\frac{\upsilon}{k}-1} \mathfrak{g}^r(s)\mathfrak{g}'(s) E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})$$

$$\times E^{\sigma,\lambda,q}_{k,\ell,\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(s) - \mathfrak{g}^{r+1}(\theta))^{\frac{\ell}{k}}) ds d\theta$$

$$= \frac{(r+1)^{1-(\frac{\beta+\upsilon}{k})}}{k} \int_0^{\xi} (\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(\theta))^{\frac{\beta+\upsilon}{k}-1}$$

$$\times E^{\gamma+\sigma,\Lambda+\lambda,q}_{k,\ell,\beta+\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(\theta))^{\frac{\ell}{k}}) \mathfrak{g}^r(\theta)\mathfrak{g}'(\theta) f(\theta) d\theta$$

$$= \left({}^{k}_{r}\mathcal{I}^{\omega,\gamma+\sigma,\Lambda\lambda,q}_{a^+,\ell,\beta+\upsilon,\rho;\mathfrak{g}} f \right)(\xi).$$

This completes the proof.

Lemma 2. *Let* $r \in \mathbb{R}\backslash\{-1\}, \ell, \beta, \upsilon, \gamma, \Lambda, \omega \in \mathbb{C}, k \in \mathbb{R}^+, n \in \mathbb{N}$ *and* $\mathcal{R}(\ell), \mathcal{R}(\beta), \mathcal{R}(\upsilon) > 0$. *Then*

$$\left({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;\mathfrak{g}} (\mathfrak{g}^{r+1}(s) - \mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1} \right)(\xi) = \frac{\Gamma_k(\upsilon)(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)^{\frac{\beta}{k}}}$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta+\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}}).$$

Proof. By using Definition 7, we get

$$\left({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;\mathfrak{g}} (\mathfrak{g}^{r+1}(s) - \mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1} \right)(\xi)$$

$$= \frac{(r+1)^{1-\frac{\beta}{k}}}{k} \int_a^{\xi} (\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} (\mathfrak{g}^{r+1}(s) - \mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1} \mathfrak{g}^r(s)\mathfrak{g}'(s)$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})ds$$

$$=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\ell n+\beta)}\frac{\omega^n}{(\Lambda)_{\rho n,k}}\int_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta+\ell n}{k}-1}$$

$$\times(\mathfrak{g}^{r+1}(s)-\mathfrak{g}^{r+1}(a))^{\frac{\upsilon}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s)ds:=J.$$

Substituting $u=\dfrac{\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s)}{\mathfrak{g}^{r+1}(\xi)-\mathrm{g}^{r+1}(a)}$ and using Definition 2, we have

$$J=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{k(r+1)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell n}{k}}\int_0^1 u^{\frac{\beta+\ell n}{k}-1}(1-u)^{\frac{\upsilon}{k}-1}du$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell n}{k}}B_k(\ell n+\beta,\upsilon)$$

$$=\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta)(\Lambda)_{\rho n,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell n}{k}}\frac{\Gamma_k(\ell n+\beta)\Gamma_k(\upsilon)}{\Gamma_k(\ell n+\beta+\upsilon)}$$

$$=\frac{\Gamma_k(\upsilon)(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}\omega^n}{\Gamma_k(\ell n+\beta+\upsilon)(\Lambda)_{\rho n,k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell n}{k}}$$

$$=\frac{\Gamma_k(\upsilon)(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta+\upsilon}{k}-1}}{(r+1)^{\frac{\beta}{k}}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+\upsilon,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}}).$$

Hence the proof is done.

3. Applications: Prabhakar-based Hardy-type Inequalities

In this section, we present the Prabhakar-based Hardy-type fractional integral inequalities involving the generalized multi-parametric Prabhakar operator defined by (16). We begin with the following lemma.

Lemma 3. *Let* $k>0, r\in\mathbb{R}\backslash\{-1\}, \ell,\beta,\upsilon,\gamma,\Lambda,\omega\in\mathbb{C}, n\in\mathbb{N}; \mathcal{R}(\ell)>0, \mathcal{R}(\beta)+k>0, \mathcal{R}(\gamma)>0, \mathcal{R}(\Lambda)>0$. *Then*

$$\mathcal{K}(\xi)=(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}},\tag{18}$$

and

$$\mathcal{A}_{\mathcal{H}}f(\xi)=\frac{(r+1)}{k}\int_a^{\xi}\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}g^r(s)\mathfrak{g}'(s)ds.\tag{19}$$

Proof. Since

$$\mathcal{H}(\xi,s)=\begin{cases}\frac{(r+1)^{1-\frac{\beta}{k}}}{k}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s) & \\ \times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}}), & \text{for } a\leq s\leq \xi; \\ 0, & \text{for } \xi<s\leq b,\end{cases} \tag{20}$$

by using (7) and (14), we get

$$\begin{aligned}\mathcal{K}(\xi)&=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\int\limits_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)\\ &\quad\times\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\ell n+\beta)}\frac{\omega^n(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell n}{k}}}{(\Lambda)_{\rho n,k}}ds\\ &=\frac{(r+1)^{1-\frac{\beta}{k}}}{k}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn,k}}{\Gamma_k(\ell n+\beta)}\frac{\omega^n}{(\Lambda)_{\rho n,k}}\int\limits_a^{\xi}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}+\frac{\ell n}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)ds.\end{aligned}$$

By simple integration, we obtain equation (18). Furthermore, using (18) and (20) into (6), we have relation (19).

Theorem 11. *Let $\mathcal{W}$ be a measurable positive function on $(a,b), 0<p\leq q<\infty, \ell>0, r\in\mathbb{R}\backslash\{-1\}$ and consider $\mathcal{U}$ as a weight mapping on (a,b). If*

$$\begin{aligned}\mathcal{A}_1:=&\sup_{s\in(a,b)}\mathcal{W}^{-\frac{1}{p}}(s)\Big(\frac{(r+1)\mathfrak{g}^{r}(s)\mathfrak{g}'(s)}{k}\Big)^{\frac{1}{p}}\\ &\times\Big[\int\limits_s^b\mathcal{U}(\xi)\Big(\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)^{\frac{q}{p}}d\xi\Big]^{\frac{1}{q}}<\infty,\end{aligned} \tag{21}$$

then for some positive number E_1 and for a non-negative convex mapping Φ on$I\subseteq\mathbb{R}$, we get

$$\begin{aligned}&\Big[\int\limits_a^b\mathcal{U}(\xi)\Big(\Phi\Big(\frac{(r+1)^{-\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big)^{\frac{q}{p}}d\xi\Big]^{\frac{1}{q}}\\ &\qquad\leq\mathrm{E}_1\Big(\int\limits_a^b\mathcal{W}(s)\Phi(f(s))ds\Big)^{\frac{1}{p}},\end{aligned} \tag{22}$$

for measurable mapping $f : (a, b) \to \mathbb{R}$. Furthermore, if $\mathrm{E}_1 > 0$ is the least constant to exist (22)*, then $\mathrm{E}_1 \leq \mathcal{A}_1$.*

Proof. By applying the Jensen's and then Minkowski's inequality, we obtain

$$\Big[\int_a^b \mathcal{U}(\xi)\Big(\Phi\Big(\frac{({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big)^{\frac{q}{p}}d\xi\Big]^{\frac{1}{q}}$$

$$\leq \Big[\int_a^b \mathcal{U}(\xi)\Big(\frac{(r+1)}{k(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times \int_a^\xi (\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s)E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})\Phi(f(s))ds\Big)^{\frac{q}{p}}d\xi\Big]^{\frac{1}{q}}$$

$$\leq \Big[\int_a^b \Big(\Big(\mathcal{W}^{-\frac{1}{p}}(s)\Big(\frac{(r+1)\mathfrak{g}^r(s)\mathfrak{g}'(s)}{k}\Big)^{\frac{1}{p}}\Big)^q \int_s^b$$

$$\times \mathcal{U}(\xi)\Big[\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}d(\xi)\Big]^{\frac{q}{p}}\Big)^{\frac{p}{q}}\mathcal{W}(s)\Phi(f(s))ds\Big]^{\frac{1}{p}}$$

$$\leq \mathcal{A}_1\Big(\int_a^b \mathcal{W}(s)\Phi(f(s))ds\Big)^{\frac{1}{p}}.$$

Thus, the proof is concluded.

Corollary 2. *Consider $-\infty < q \leq p < 0$ and the hypotheses of Theorem* 11*. If*

$$\mathcal{A}_2 := \inf_{s\in(a,b)} \mathcal{W}^{-\frac{1}{p}}(s)\Big(\frac{(r+1)\mathfrak{g}^r(s)\mathfrak{g}'(s)}{k}\Big)^{\frac{1}{p}}$$

$$\times \Big(\int_s^b \mathcal{U}(\xi)\Big(\frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)^{\frac{q}{p}}d\xi\Big)^{\frac{1}{q}} < \infty,$$

then for some positive real number E_2 and convex function $\Phi > 0$, the inequality

$$\Big(\int_a^b \mathcal{U}(\xi)\left[\Phi\left(\frac{({}^{k}_{r}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\right)\right]^{\frac{q}{p}}d\xi\Big)^{\frac{1}{q}}$$

$$\geq \mathrm{E}_2 \left(\int_a^b \mathcal{W}(s)\Phi(f(s))ds \right)^{\frac{1}{p}} \tag{23}$$

is valid for any measurable function $f : (a,b) \to \mathbb{R}$. In addition, if E_2 is the largest positive number to hold (23), then $\mathrm{E}_2 \geq A_2$.

Utilization of Theorem 4 for the newly introduced multi-parametric Prabhakar integral operator given in (7), we state this theorem.

Theorem 12. *Assume $\ell,\beta,\gamma,\Lambda,\rho,q$ are the same notions in Definition 7, and the weight mapping $\mathcal{U}$ is given on (a,b). For each fixed $s \in (a,b)$, define*

$$\mathcal{V}(s) := \frac{(r+1)}{k} \int_s^b \mathcal{U}(\xi) \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s)d(\xi) < \infty. \tag{24}$$

If the function Φ is convex on $I \subseteq \mathbb{R}$, then for all measurable mapping $f \in L[a,b]$ with $Imf \subseteq I$, we have

$$\int_a^b \mathcal{U}(\xi)\Phi\Big(\frac{(r+1)}{k} \int_a^\xi \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s)f(s)ds\Big)d\xi$$

$$\leq \int_a^b \mathcal{V}(s)\Phi\Big(f(s)\Big)ds. \tag{25}$$

Proof. By Theorem 4 with $\Lambda_1 = \Lambda_2 = (a,b), d\Theta_1(\xi) = d\xi, d\Theta_2(s) = ds$ and with the help of the Jensen's inequality along with Fubini's theorem, we have

$$\int_a^b \mathcal{U}(\xi)\Phi\Big(\mathcal{A}_{\mathcal{H}}f(\xi)\Big)d\xi$$

$$= \int_a^b \mathcal{U}(\xi)\Phi\Big(\frac{(r+1)}{k} \int_a^\xi \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s)f(s)ds\Big)d\xi$$

$$\leq \int_a^b \frac{(r+1)}{k} \int_s^b \mathcal{U}(\xi)\Big(\frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s)d\xi\Big)\Phi\Big(f(s)\Big)ds$$

$$= \int_a^b \mathcal{V}(s)\Phi\Big(f(s)\Big)ds,$$

and the desired inequality is derived.

Theorem 13. *Assume $k, \ell, \beta, \gamma, \Lambda, \rho, q$ are the same notions in Definition 7 and the weight function $\mathcal{U}$ is given on (a,b). For a fixed $s \in (a,b)$, define*

$$\mathcal{P}(s) := f_2(s) \int\limits_s^b \mathcal{U}(\xi) \frac{\frac{(r+1)^{1-\frac{\beta}{k}}}{k} (\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} \mathfrak{g}^r(s)\mathfrak{g}'(s)}{({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_2)(\xi)}$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}}) d\xi < \infty.$$

If the function $\Phi : I \to \mathbb{R}$ is convex and $\dfrac{({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_1)(\xi)}{({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_2)(\xi)}, \dfrac{f_1(s)}{f_2(s)} \in I$, then the inequality

$$\int\limits_a^b \mathcal{U}(\xi)\Phi\Big(\frac{({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_1)(\xi)}{({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_2)(\xi)}\Big) d\xi \leq \int\limits_a^b \mathcal{P}(s)\Phi\Big(\frac{f_1(s)}{f_2(s)}\Big) ds \tag{26}$$

holds.

Proof. By using Theorem 5 with $\Lambda_1 = \Lambda_2 = (a,b), d\Theta_1(\xi) = d\xi, d\Theta_2(s) = ds, \mathcal{Z}_1(\xi) = (({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_1)(\xi), \mathcal{Z}_2(\xi) = (({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f_2)(\xi)$, we immediately obtain inequality (26).

The next theorem proves the new refined weighted Hardy-type integral inequality for an arbitrary convex mapping based on the generalized multi-parametric Prabhakar integral provided by (7).

Theorem 14. *Consider the hypothses of Theorem 12. Furthermore, let a convex mapping Φ on $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be so that $\varphi(\xi) \in \partial\Phi(\xi)$ for all $\xi \in IntI$. Then*

$$\int_a^b \mathcal{V}(s)\Phi\Big(f(s)\Big)d(s) - \int_a^b \mathcal{U}(\xi)\Phi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}_r^k\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g} f)(\xi)}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)d(\xi)$$

$$\geq \int_a^b \frac{\mathcal{U}(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\int_a^\xi \frac{(r+1)^{1-\frac{\beta}{k}}}{k}(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s) E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})$$

$$\Big\|\Phi\Big(f(s)\Big) - \Phi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}_r^k\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho} f)(\xi)}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big|$$

$$- \Big|\varphi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}_r^k\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho} f)(\xi)}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big|$$

$$\times \Big|f(s) - \frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big\|d(s)d(\xi), \tag{27}$$

holds for all measurable mappings $f : \Lambda_2 \to \mathbb{R}$ *with* $f(s) \in I$, $\forall\, s \in (a,b)$. *If the function* Φ *is monotone and convex on*$I \subseteq \mathbb{R}$, *then*

$$\int_a^b \mathcal{V}(s)\Phi\Big(f(s)\Big)d(s) - \int_a^b \mathcal{U}(\xi)\Phi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)d(\xi)$$

$$\geq \int_a^b \frac{\mathcal{U}(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times \int_a^\xi sgn\Big(f(s) - \frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)$$

$$\frac{(r+1)^{1-\frac{\beta}{k}}}{k}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})$$

$$\times \Big[\Phi\Big(f(s)\Big) - \Phi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)$$

$$-\Big|\varphi\Big(\frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big| \tag{28}$$

$$\times \Big(f(s) - \frac{(r+1)^{\frac{\beta}{k}}({}^{k}_{r}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big]d(s)d(\xi)\Big|$$

for each measurable function $f : (a,b) \to \mathbb{R}$ *with* $f(s) \in I, \forall\, s \in (a,b)$.

Proof. By combining $\Lambda_1 = \Lambda_2 = (a,b)$, $d\Theta_1(\xi) = d\xi$, $d\Theta_2(s) = ds$ with Theorem 6, and $\mathcal{H}(\xi,s)$ given in (20), we derive (27) and (28).

The 1-dimensional structure yeilds refined Hardy-type and Polya-Knopp-type inequalities. Kaijser *et al.* in [31] presented a refinement of a Hardy-type inequality that can be generalized for fractional multi-parametric Prabhakar integral operator given in (16).

Theorem 15. *Consider a weight mapping* $\mathcal{U} > 0$ *defined on* (a,b) *and let* $k,\ell,\beta,\gamma,\Lambda,p,q$ *be the same notions in Definition 7. For each fixed* $s \in (a,b)$, *define*

$$\check{\mathcal{W}}(s) := s\int_s^b \frac{\frac{(r+1)}{k}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})\mathcal{U}(\xi)\frac{d(\xi)}{\xi}.$$

If on $I\subseteq\mathbb{R}$, Φ is convex and $\varphi: I\to\mathbb{R}$ is so that $\varphi(\xi)\in\partial\Phi(\xi), \forall\,\xi\in IntI$, then

$$\int_a^b \check{\mathcal{W}}(s)\Phi\Big(f(s)\Big)\frac{ds}{s}-\int_a^b\mathcal{U}(\xi)\Phi\Big(\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\frac{d\xi}{\xi}$$

$$\geq\int_a^b\frac{\mathcal{U}(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times\int_0^\xi\frac{(r+1)^{1-\frac{\beta}{k}}}{k}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^r(s)\mathfrak{g}'(s)E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}}) \quad (29)$$

$$\times\Bigg\|\Phi\Big(f(s)\Big)-\Phi\Big(\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Bigg|$$

$$-\Bigg|\varphi\Big(\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Bigg|$$

$$\times\Bigg|f(s)-\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Bigg\|ds\frac{d\xi}{\xi},$$

is valid for all measurable function $f:(a,b)\to\mathcal{R}$ having values in I. If Φ is concave, in that case, the order of integrals on the L.H.S. of (29) is inverted. If Φ is convex and monotone on $I\subseteq\mathbb{R}$, then

$$\int_a^b \mathcal{W}(s)\Phi\Big(f(s)\Big)\frac{ds}{s}-\int_a^b\mathcal{U}(\xi)\Phi\Big(\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\frac{d\xi}{\xi}$$

$$\geq\Big|\int_a^b\frac{\mathcal{U}(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times\int_a^\xi sgn\Big(f(s)-\frac{({}^k_r\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)$$

$$(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})$$

$$\times \Big[\Phi\Big(f(s)\Big) - \Phi\Big(\frac{({}_{r}^{k}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)$$

$$-\Big|\varphi\Big(\frac{({}_{r}^{k}\mathcal{I}^{\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho;g}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big|$$

$$\times \Big(f(s) - \frac{({}_{r}^{k}\mathcal{I}^{g,\omega,\gamma,\Lambda,q}_{a^+,\ell,\beta,\rho}f)(\xi)}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\Big)\Big]ds\frac{d\xi}{\xi}\Big|,$$

is fulfilled for all measurable function $f:(a,b)\to\mathcal{R}$ having values in I.

Proof. By using Theorem 7 with $(0,b)=(a,b)$, value of non-negative kernel $\mathcal{H}(\xi,s)$ given in (20) and

$$\mathcal{A}_{\mathcal{H}}f(\xi) = \frac{1}{(r+1)^{-\frac{\beta}{k}}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}$$

$$\times \int_a^{\xi} \frac{(r+1)^{1-\frac{\beta}{k}}}{k}(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})f(s)ds,$$

give the required result.

Next, the mean value theorem given in [32] is proved for the generalized multi-parametric Prabhakar fractional integral operator (7).

Theorem 16. *Consider the hypothses of Theorem 12. Let I be compact in $\mathbb{R}$, $h''\in\mathcal{C}^2(I)$, and $f:(a,b)\to\mathbb{R}$ be a measurable mapping with $Imf\subseteq I$. Then $\exists\,\varrho\in I$ s.t.*

$$\int_a^b \mathcal{V}(s)h\Big(f(s)\Big)d(s) - \int_a^b \mathcal{U}(\xi)h\Big(\frac{(r+1)}{k}$$

$$\times \int_a^{\xi} \frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)f(s)ds\Big)d\xi$$

$$= \frac{h''(\varrho)}{2}\Big[\int_a^b \mathcal{V}(s)f^2(s)d(s) - \int_a^b \mathcal{U}(\xi)\Big(\frac{(r+1)}{k}$$

$$\times \int_a^{\xi} \frac{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1}E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}}E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi)-\mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})}\mathfrak{g}^{r}(s)\mathfrak{g}'(s)f(s)ds\Big)^2 d\xi\Big]. \tag{30}$$

Proof. By using Theorem 8 with $\Lambda_1 = \Lambda_2 = (a, b), d\Theta_1(\xi) = d\xi, d\Theta_2(s) = ds$ and using value of $\mathcal{A}_{\mathcal{H}} f(\xi)$ from (19), we derive (30).

Theorem 17. *Let all conditions of Theorem 16 are fulfilled. Also,* $j, h \in \mathcal{C}^2(I)$ *and* $h(\xi) \neq 0, \forall \xi \in I$. *If*

$$\int_a^b \mathcal{V}(s) h\Big(f(s)\Big) d(s) - \int_a^b \mathcal{U}(\xi) h\Big(\frac{(r+1)}{k}$$

$$\times \int\limits_a^\xi \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s) f(s) ds\Big) d\xi \neq 0,$$

then $\exists\, \varrho \in I$ *s.t.*

$$\frac{j''(\varrho)}{h''(\varrho)} = \frac{\int_a^b \mathcal{V}(s) j\Big(f(s)\Big) ds - \int_a^b \mathcal{U}(\xi)}{\int_a^b \mathcal{V}(s) h\Big(f(s)\Big) ds - \int_a^b \mathcal{U}(\xi)}$$

$$\times \frac{j\Big(\frac{(r+1)}{k} \int\limits_a^\xi \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s) f(s) ds\Big) d\xi}{h\Big(\frac{(r+1)}{k} \int\limits_a^\xi \frac{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\beta}{k}-1} E^{\gamma,\Lambda,q}_{k,\ell,\beta,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(s))^{\frac{\ell}{k}})}{(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\beta}{k}} E^{\gamma,\Lambda,q}_{k,\ell,\beta+k,\rho}(\omega(\mathfrak{g}^{r+1}(\xi) - \mathfrak{g}^{r+1}(a))^{\frac{\ell}{k}})} \mathfrak{g}^r(s)\mathfrak{g}'(s) f(s) ds\Big) d\xi}. \tag{31}$$

Proof. Applying Theorem 9 with $\Lambda_1 = \Lambda_2 = (a, b), d\Theta_1(\xi) = d\xi, d\Theta_2(s) = ds$ and using value of $\mathcal{A}_{\mathcal{H}} f(\xi)$ from (19), we get (31), and the proof is completed.

4. Conclusion

Inequalities are prevalent in practically every aspect of life, and fractional calculus depicts nature better than integer-order calculus. This article examines the generalized multi-parametric Mittag-Leffler function and extended forms of fractional integral operators in the Prabhakar settings. Some important characteristics, such as the semi-group, are also explored. Jensen's and Minkowski's inequalities have numerous applications in optimization theory, and they are used to investigate Hardy-type integral inequalities involving the newly described multi-parametric Prabhakar integral operator. In conclusion, we may claim that this study will motivate scholars to explore new fractional operators and Hardy-type fractional integral inequalities.

Declarations:

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Availability of data and material

No data were generated or analyzed during the current study.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final version.

Competing interests

The authors declare that they have no conflicts of interest.

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