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A Model of Intuitionistic Fuzzy Soft Super-Hypergraphs with Illustrative Applications

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Abstract. Graph theory provides a fundamental framework for modeling relationships using vertices and edges. Hypergraphs extend this framework by permitting *hyperedges* that connect any number of vertices simultaneously, and Super-hypergraphs further generalize hypergraphs through iterated powerset constructions to represent hierarchical connections. Concurrently, a variety of uncertainty-modeling paradigms—such as fuzzy sets, soft sets, intuitionistic fuzzy sets, intuitionistic fuzzy offsets, hyper-intuitionistic fuzzy sets, and intuitionistic fuzzy multidirected sets—have enriched classical graph concepts.
In this paper, we introduce the *Intuitionistic Fuzzy Soft n-Super-hypergraph*, a unified n-th order structure that seamlessly integrates intuitionistic fuzzy Super-hypergraphs with soft Super-hypergraphs. We present rigorous definitions, explore key structural properties, and highlight potential applications in decision-making under uncertainty, demonstrating how this framework effectively captures both complex hierarchical organization and nuanced uncertainty in real-world networks.

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Key Words and Phrases: Super-hypergraph, hypergraph, fuzzy Graph, soft Graph, nth-PowerSet, intuitionistic Fuzzy HyperGraph, intuitionistic Fuzzy Graph.

1. Introduction

1.1. From Graphs to Super-hypergraphs

Classical graphs model pairwise relationships by representing elements as vertices and their connections as edges [1]. To capture interactions among larger groups, *hypergraphs* allow each hyperedge to join any nonempty subset of vertices, enabling representation of higher-order relations [2]. Extending this idea further, a *Super-hypergraph* builds successive layers of connectivity via iterated powerset operations, thereby encoding nested, hierarchical structures within complex networks [3]. Graphs, hypergraphs, and Super-hypergraphs

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serve as versatile tools across artificial intelligence, network science, data mining, chemistry, physics, and beyond, offering intuitive visualizations and rigorous foundations for analyzing real-world systems [4]. By incorporating multiple abstraction levels, Super-hypergraphs in particular provide a powerful framework for dissecting and understanding multi-tiered interactions. Table 1 provides a brief overview of graphs, hypergraphs, and Super-hypergraphs.

Table 1: Compact overview of graphs, hypergraphs, and Super-hypergraphs

Model	Basic structure	Short description
Graph	$G = (V, E),\ \ E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$	Vertices V connected by pairwise edges E; models binary relationships such as communication links, friendship ties, or simple transport networks.
Hypergraph	$H = (V, E),\ \ E \subseteq \{e \subseteq V \mid e \neq \varnothing\}$	Generalizes graphs by allowing each hyperedge $e \in E$ to join any nonempty subset of vertices; captures higher-order interactions such as teams, transactions, or co-authorship groups.
Super-hypergraph	$\mathrm{SuHG}^{(n)} = (V, E),\ \ \mathrm{V},\ \mathrm{E} \subseteq \mathrm{PSET}^{n}(V_0)$	Built from iterated powersets over a base set V_0; supervertices and superedges represent hierarchical or multi-level groupings (for example employees–teams–departments–divisions) in complex networks.

1.2. Modeling Uncertainty in Networks

Real-world networks often involve ambiguity or incomplete information. To address this challenge, classical set-based notions such as fuzzy sets [5], intuitionistic fuzzy sets [6], hyperfuzzy sets [7], neutrosophic sets [8], quadripartitioned neutrosophic sets [9], and plithogenic sets [10]—have been extended to the graph domain. These extensions yield models such as fuzzy graphs [11], intuitionistic fuzzy graphs [12], neutrosophic graphs [13], hesitant fuzzy graphs [14], plithogenic graphs [15], and soft graphs [16], all of which incorporate degrees of membership, indeterminacy, or conflict to more faithfully represent uncertain relationships.

In order to model even richer and more context-dependent forms of uncertainty, fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and plithogenic sets have further been extended to their corresponding soft counterparts, namely fuzzy soft sets[17, 18], intuitionistic fuzzy soft sets[19], neutrosophic soft sets[20, 21], and plithogenic soft sets [22]. At the graph level, this has led to fuzzy soft graphs[23], intuitionistic fuzzy soft graphs [24], neutrosophic soft graphs[25–27], and plithogenic soft graphs[28], which have been very actively studied in decision-making and related application areas (e.g.[29–31]). Consequently, research on fuzzy soft graphs, intuitionistic fuzzy soft graphs, and their variants plays a crucial role in representing uncertainty in real-world networked systems.

Although some portions of this discussion are inevitably repetitive, Fuzzy Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs play highly significant roles in decision support systems and, more broadly, in the field of decision-making [32, 33]. For example, Neutrosophic TOPSIS has been investigated in [34], Intuitionistic fuzzy MACONT in [35], Neutrosophic ANP in [36, 37], Intuitionistic Fuzzy ANP in [38], and Neutrosophic DEMATEL in [39], among many other methods developed for a wide range of real-world applications. From this perspective, when practical applications are taken into account, the study of Intuitionistic Fuzzy HyperGraphs and Intuitionistic Fuzzy SuperHyperGraphs is also of clear importance.

1.3. Our Contribution

As discussed above, fuzzy soft graphs, intuitionistic fuzzy soft graphs, and related models are important tools for capturing uncertainty in complex networks. However, their counterparts in the setting of Super-hypergraphs have not yet been systematically explored. To bridge this gap, in this paper we propose the notion of an *Intuitionistic Fuzzy Soft n-Super-hypergraph*, a unified framework that systematically combines the properties of intuitionistic fuzzy Super-hypergraphs and soft Super-hypergraphs within an n-th order hierarchical structure. This new formulation is designed to support more sophisticated modeling of uncertainty and multi-level relations, thereby enabling enhanced decision-making in complex networked environments. For reference, Table 2 presents a comparison of Soft, Fuzzy Soft, and Intuitionistic Fuzzy Soft n-Super-hypergraphs.

Table 2: Compact comparison of Soft, Fuzzy Soft, and Intuitionistic Fuzzy Soft n-Super-hypergraphs

Model	Uncertainty type	Brief description
Soft n-Super-hypergraph	Qualitative, set-valued	Parameterized family of sub-n-Super-hypergraphs: each parameter selects a set of n-supervertices and the induced n-superedges, without numeric degrees.
Fuzzy Soft n-Super-hypergraph	Single membership degree in $[0, 1]$	Soft n-Super-hypergraph equipped, for each parameter, with fuzzy membership grades on n-supervertices and n-superedges, encoding the strength or reliability of hierarchical relations.
Intuitionistic Fuzzy Soft n-Super-hypergraph	Membership and non-membership in $[0, 1]$	Fuzzy Soft n-Super-hypergraph further endowed with non-membership degrees and an explicit hesitation margin, allowing finer modeling of agreement, disagreement, and uncertainty in multi-level networks.

2. Preliminaries

This section gathers the core definitions and notation used throughout this work.

2.1. Super-hypergraph

In graph theory, a *hypergraph* generalizes an ordinary graph by permitting *hyperedges* that join any number of vertices, which is useful for capturing complex, multi-way relationships across diverse fields [40]. A *Super-hypergraph* enriches this idea further by applying iterated powerset operations to the vertex set, a concept that has attracted recent interest [41, 42]. Throughout, the parameter n denotes a nonnegative integer used both in the n-th powerset construction and in defining an n-Super-hypergraph.

Definition 1 (Powerset)**.** *[43] The* powerset *of a set S, denoted* $\mathrm{PSET}(S)$*, is the collection of every subset of S, including $\varnothing$ and S itself:*

$$\mathrm{PSET}(S) = \{ A : A \subseteq S \}.$$

Definition 2 (Hypergraph)**.** *[40, 44] A* hypergraph $H = (V, E)$ *consists of*

- *a nonempty vertex set V,*
- *a collection E of nonempty subsets of V, called* hyperedges*.*

This structure allows each hyperedge to link any number of vertices. In this paper, both V and E are finite sets.

Example 1 (Hypergraph Model of Retail Market Basket Transactions)**.** *Let*

$$V = \{\text{Milk, Bread, Eggs, Cheese, Butter}\}$$

be a set of grocery items. Each customer's shopping basket is naturally represented as a hyperedge connecting all items purchased in that transaction. For example:

$$e_1 = \{\text{Milk}, \text{Bread}, \text{Eggs}\}, \quad e_2 = \{\text{Bread}, \text{Butter}\},$$

$$e_3 = \{\text{Milk}, \text{Cheese}\}, \quad e_4 = \{\text{Eggs}, \text{Butter}, \text{Cheese}\}.$$

Define

$$E = \{e_1,\, e_2,\, e_3,\, e_4\}.$$

Then

$$H = (V,\, E)$$

is a finite hypergraph that encodes co-purchase relationships. Mining this hypergraph for frequent hyperedges (itemsets) helps retailers design product bundles, optimize shelf layout, and target promotions based on actual consumer behavior.

Figure 1 illustrates the hypergraph model of retail market basket transactions, where each vertex represents a grocery item and each hyperedge corresponds to one customer basket.

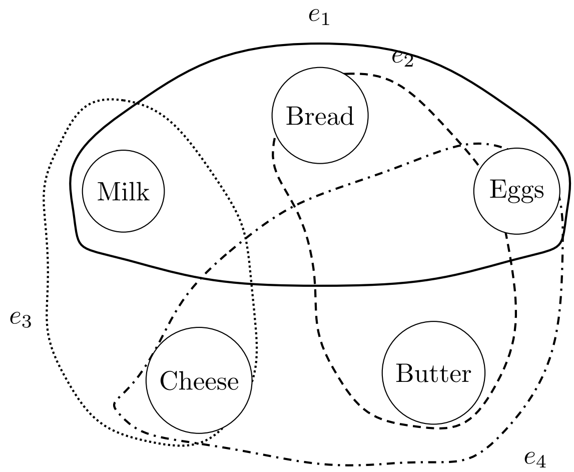

Figure 1: Hypergraph model of retail market basket transactions. Each vertex denotes an item, and each hyperedge represents a customer basket (transaction).

Definition 3 (n-th Powerset). *(cf.[45]) The* n-th powerset *of a set* X*, written* $\mathrm{PSET}_n(X)$*, is defined by iteration:*

$$\mathrm{PSET}_1(X) = \mathrm{PSET}(X), \quad \mathrm{PSET}_{k+1}(X) = \mathrm{PSET}\big(\mathrm{PSET}_k(X)\big), \quad k \geq 1.$$

Likewise, the n-th nonempty powerset $\mathrm{PSET}^*_n(X)$ *is obtained by removing the empty set at each stage:*

$$\mathrm{PSET}^*_1(X) = \mathrm{PSET}(X) \setminus \{\varnothing\}, \quad \mathrm{PSET}^*_{k+1}(X) = \mathrm{PSET}\big(\mathrm{PSET}^*_k(X)\big) \setminus \{\varnothing\}.$$

Definition 4 (n-Super-hypergraph). *[3] Let* V_0 *be a finite nonempty base set. For each integer* $k \geq 0$*, define*

$$\mathrm{PSET}^0(V_0) = V_0, \qquad \mathrm{PSET}^{k+1}(V_0) = \mathrm{PSET}\big(\mathrm{PSET}^k(V_0)\big).$$

An n-Super-hypergraph *is an ordered pair*

$$\mathrm{SuHG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathrm{PSET}^n(V_0), \qquad E \subseteq \mathrm{PSET}(V) \setminus \{\varnothing\}.$$

The elements of V *are called* n-supervertices*, and the elements of* E *are called* n-superedges*. Thus, each* n*-superedge is a nonempty collection of* n*-supervertices.*

Example 2 (A 2-Super-hypergraph for Internet autonomous systems). *Let the base set* V_0 *be the set of routers in a given region:*

$$V_0 = \{r_1, r_2, \ldots, r_{10}\}.$$

Then $\mathrm{PSET}^1(V_0)$ *consists of subsets of routers, which may be interpreted as local area networks (LANs). For example,*

$$L_1 = \{r_1, r_2, r_3\}, \qquad L_2 = \{r_4, r_5\}, \qquad L_3 = \{r_6, r_7, r_8, r_9\}.$$

Next, $\mathrm{PSET}^2(V_0)$ *consists of sets of LANs, which we interpret as autonomous systems (ASs). For instance,*

$$\mathrm{AS}_A = \{L_1, L_2\}, \qquad \mathrm{AS}_B = \{L_2, L_3\}, \qquad \mathrm{AS}_C = \{L_1, L_3\}.$$

Define

$$V = \{\mathrm{AS}_A, \mathrm{AS}_B, \mathrm{AS}_C\} \subseteq \mathrm{PSET}^2(V_0).$$

Now let the superedges represent multilateral peering agreements among autonomous systems:

$$e_1 = \{\mathrm{AS}_A, \mathrm{AS}_B\}, \qquad e_2 = \{\mathrm{AS}_A, \mathrm{AS}_B, \mathrm{AS}_C\}.$$

Set

$$E = \{e_1, e_2\} \subseteq \mathrm{PSET}(V) \setminus \{\varnothing\}.$$

Hence,

$$\mathrm{SuHG}^{(2)} = (V, E)$$

is a 2*-Super-hypergraph modeling the AS-level Internet topology, where each* 2*-supervertex is an autonomous system (that is, a set of LANs) and each* 2*-superedge is a nonempty group of ASs participating in a peering arrangement.*

Figure 2 illustrates this 2*-Super-hypergraph representation of the AS-level Internet topology.*

Example 3 (A 3-Super-hypergraph for corporate organizational structure)**.** *Consider a three-level hierarchy in a multinational corporation.*

- ***Level 0 (employees):***

  $$V_0 = \{E_1, E_2, E_3, E_4, E_5, E_6\},$$

  where each E_i *denotes an individual employee.*

- ***Level 1 (teams):***

  $$T_1 = \{E_1, E_2\}, \qquad T_2 = \{E_3, E_4\}, \qquad T_3 = \{E_5, E_6\}.$$

- ***Level 2 (departments):***

  $$D_A = \{T_1, T_2\}, \qquad D_B = \{T_2, T_3\}.$$

- ***Level 3 (divisions):***

  $$\Delta_1 = \{D_A\}, \qquad \Delta_2 = \{D_A, D_B\}.$$

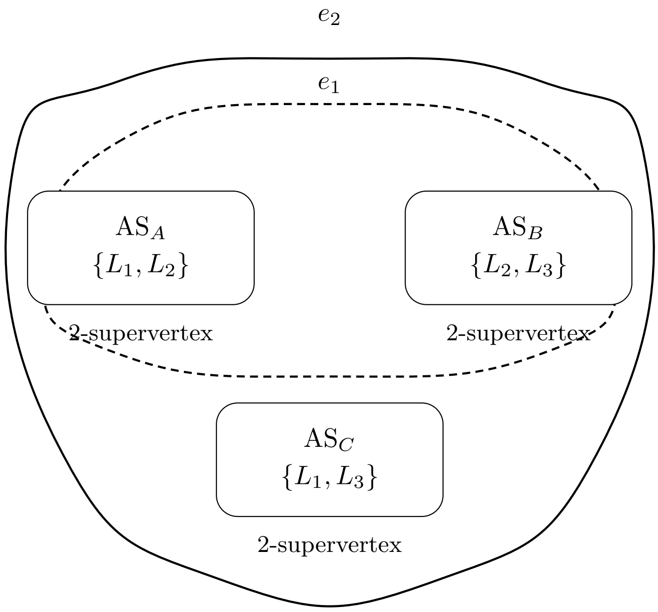

Figure 2: A 2-Super-hypergraph for Internet autonomous systems. Each 2-supervertex represents an autonomous system viewed as a set of LANs, and each 2-superedge represents a peering group among autonomous systems.

Define

$$V = \{\Delta_1, \Delta_2\} \subseteq \mathrm{PSET}^3(V_0).$$

Let

$$e = \{\Delta_1, \Delta_2\},$$

and define

$$E = \{e\} \subseteq \mathrm{PSET}(V) \setminus \{\varnothing\}.$$

Then

$$\mathrm{SuHG}^{(3)} = (V, E)$$

is a 3*-Super-hypergraph. Here each* 3*-supervertex* Δ_i *is a division, represented as a set of departments, and the unique* 3*-superedge* e *represents a cross-divisional strategic alliance.*

Thus, $\mathrm{SuHG}^{(3)} = (V, E)$ *models the corporate organizational structure at three hierarchical levels, while also encoding an inter-divisional relationship at the top level.*

Figure 3 illustrates this 3*-Super-hypergraph model.*

2.2. Fuzzy n-Super-hypergraphs

A *fuzzy set* on a universe U assigns to each element $x \in U$ a membership grade $\mu(x) \in [0, 1]$, representing the degree of partial inclusion [5]. Extending this concept, a *fuzzy hypergraph* assigns membership values in $[0, 1]$ to both vertices and hyperedges of a crisp hypergraph $H = (V, E)$ [46]. As a further generalization of these ideas, the notion of a *fuzzy n-Super-hypergraph* has been introduced (cf. [3]). The definition is presented below.

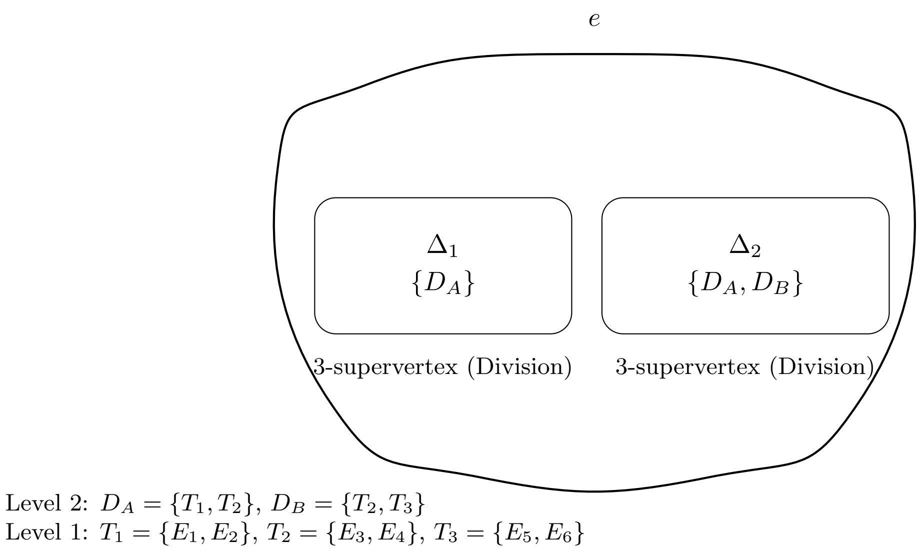

Figure 3: A 3-Super-hypergraph for corporate organizational structure. Each 3-supervertex represents a division viewed as a set of departments, and the unique 3-superedge represents a cross-divisional strategic alliance.

Definition 5 (Fuzzy n-Super-hypergraph)**.** *[3] Let* $\mathrm{SuHG}^{(n)} = (V, E)$ *be an n-Super-hypergraph, where each $e \in E$ is a subset of V. A* fuzzy n-Super-hypergraph *is a quadruple*

$$\big(V,\, E,\, \sigma,\, \mu\big),$$

with

$$\sigma : V \to [0,1], \quad \mu : E \to [0,1],$$

subject to the consistency condition

$$\mu(e) \;\leq\; \min_{v \in e} \sigma(v) \quad \textit{for all } e \in E.$$

That is, each n-superedge's membership degree does not exceed the smallest membership among its incident n-supervertices.

Example 4 (Fuzzy 2-Super-hypergraph for Supply Chain Resilience)**.** *Consider a simple two-tier supply chain network. Let the base set of production sites be*

$$V_0 = \{F_A,\, F_B,\, F_C\},$$

where F_X denotes factory X. The first powerset $\mathrm{PSET}^1(V_0)$ *yields regional clusters:*

$$R_1 = \{F_A,\, F_B\}, \quad R_2 = \{F_B,\, F_C\}, \quad R_3 = \{F_A,\, F_C\}.$$

The second powerset $\mathrm{PSET}^2(V_0)$ *groups these regions into distribution coalitions:*

$$D_1 = \{R_1,\, R_2\}, \quad D_2 = \{R_2,\, R_3\}, \quad D_3 = \{R_1,\, R_3\}.$$

Set

$$V = \{D_1,\, D_2,\, D_3\} \;\subseteq\; \mathrm{PSET}^2(V_0),$$

$$E = \big\{\, e_1 = \{D_1,\, D_2\},\; e_2 = \{D_2,\, D_3\},\; e_3 = \{D_1,\, D_2,\, D_3\} \big\} \;\subseteq\; \mathrm{PSET}^2(V_0).$$

Define the reliability function $\sigma : V \to [0,1]$ *by*

$$\sigma(D_1) = 0.92, \quad \sigma(D_2) = 0.85, \quad \sigma(D_3) = 0.78,$$

where $\sigma(D_i)$ *represents the resilience score of coalition* D_i*. Then choose the coalition-collaboration strength* $\mu : E \to [0,1]$ *so that*

$$\mu(e_1) = 0.85, \quad \mu(e_2) = 0.78, \quad \mu(e_3) = 0.78,$$

which satisfy $\mu(e_1) \le \min\{0.92, 0.85\} = 0.85$*,* $\mu(e_2) \le \min\{0.85, 0.78\} = 0.78$*, and* $\mu(e_3) \le \min\{0.92, 0.85, 0.78\} = 0.78$*. Thus* $\big(V,\, E,\, \sigma,\, \mu\big)$ *is a fuzzy 2-Super-hypergraph modeling the resilience and cooperative strength of distribution coalitions in a supply chain.*

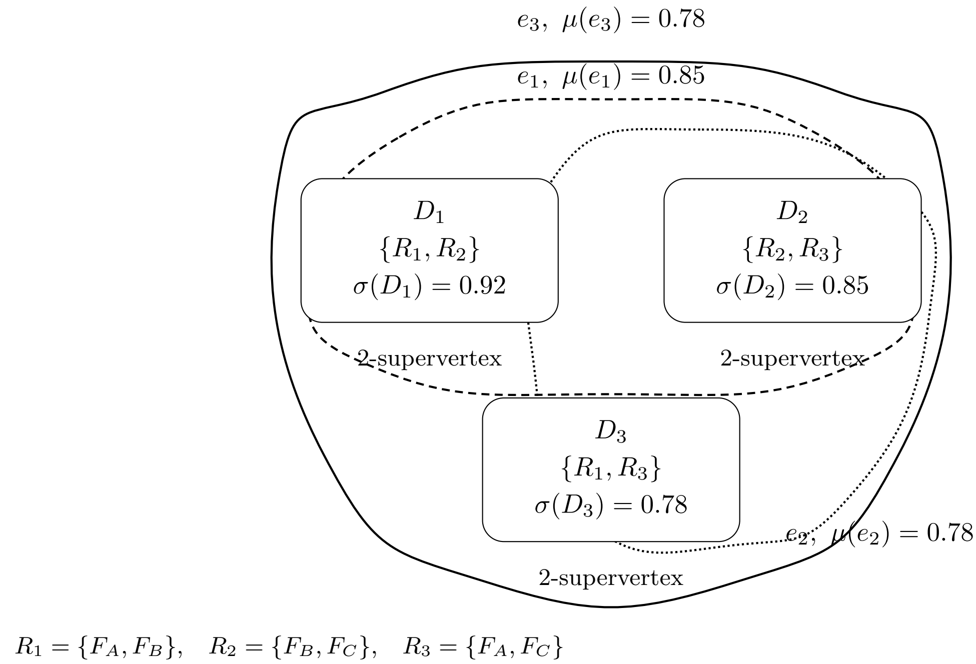

Figure 4: A fuzzy 2-Super-hypergraph for supply chain resilience. Each 2-supervertex denotes a distribution coalition represented as a set of regional clusters, with vertex-membership σ indicating resilience. Each superhyperedge denotes a collaboration group, with edge-membership μ representing coalition-collaboration strength.

2.3. Soft Super-hypergraph

Soft graphs capture families of subgraphs indexed by a parameter set, where each parameter selects a subgraph on a fixed vertex and edge base [47]. Equivalently, a soft graph realizes the concept of a soft set—a parameterized collection of subsets that models uncertainty or preference without requiring numeric membership or probability assignments [48, 49]. A *soft hypergraph* extends this notion by associating each parameter with a subset of vertices and the induced hyperedges within a hypergraph [50, 51]. Finally, a *soft n-Super-hypergraph* applies these parameterized substructures to each layer of an n-Super-hypergraph, yielding a hierarchically organized, uncertainty-aware model (cf.[52]).

Definition 6 (Soft Hypergraph)**.** *[50, 51] Let $H = (V, E)$ be a hypergraph with $E \subseteq \mathrm{PSET}(V)$, and let C be a nonempty set of parameters. A* soft hypergraph *over H with parameters C is a quadruple*

$$\big(H,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathrm{PSET}(V), \qquad B : C \longrightarrow \mathrm{PSET}(E),$$

and for each $c \in C$,

$$B(c)\ \subseteq\ \{\, e \in E \mid e \subseteq A(c)\}.$$

The pair $\big(A(c), B(c)\big)$ is called the soft subhypergraph *of H at parameter c.*

Definition 7 (Soft n-Super-hypergraph)**.** *[52] Let $\mathrm{SuHG}^{(n)} = (V, E)$ be an n-Super-hypergraph, and let C be a nonempty parameter set. A* soft n-Super-hypergraph *is a quintuple*

$$\big(V,\, E,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathrm{PSET}(V), \qquad B : C \longrightarrow \mathrm{PSET}(E),$$

such that for every $c \in C$,

$$A(c) \subseteq V, \quad B(c) \subseteq \{\, e \in E \mid e \subseteq A(c)\}.$$

Thus $\big(A(c), B(c)\big)$ is a sub-Super-hypergraph of $\mathrm{SuHG}^{(n)}$ at parameter c.

Example 5 (Soft 2-Super-hypergraph for University Department Initiatives)**.** *Let the base set of faculty be*

$$V_0 = \{p_1,\, p_2,\, p_3\},$$

where each p_i is a professor. The first powerset $\mathrm{PSET}^1(V_0)$ yields research teams:

$$T_1 = \{p_1, p_2\}, \quad T_2 = \{p_2, p_3\}, \quad T_3 = \{p_1, p_3\}.$$

The second powerset $\mathrm{PSET}^2(V_0)$ groups these teams into departments:

$$D_1 = \{T_1,\, T_2\}, \quad D_2 = \{T_2,\, T_3\}, \quad D_3 = \{T_1,\, T_3\}.$$

Set

$$V = \{D_1,\, D_2,\, D_3\}\ \subseteq\ \mathrm{PSET}^2(V_0), \quad E = \big\{\, e_1 = \{D_1, D_2\},\ e_2 = \{D_2, D_3\}\big\}.$$

Then $\mathrm{SuHG}^{(2)} = (V, E)$ *is a 2-Super-hypergraph of departments.*

Let the parameter set be two university initiatives:

$$C = \{GrantReview,\ CurriculumCommittee\}.$$

Define

$$A(GrantReview) = \{D_1,\, D_2\}, \quad B(GrantReview) = \{e_1\},$$

since those departments participate in the grant-review panel; and

$$A(CurriculumCommittee) = \{D_2,\, D_3\}, \quad B(CurriculumCommittee) = \{e_2\},$$

for the curriculum-planning committee. Each $\big(A(c), B(c)\big)$ *is thus a sub-Super-hypergraph of* $\mathrm{SuHG}^{(2)}$ *at parameter* c.

2.4. Intuitionistic Fuzzy Soft Hypergraph

An intuitionistic fuzzy set assigns to each element both a membership and a non-membership degree, thereby explicitly representing hesitation or uncertainty (cf. [53]). In recent years, the *intuitionistic fuzzy hypergraph*, an extension of intuitionistic fuzzy sets to hypergraph structures, has been actively studied (cf.[54]). An *intuitionistic fuzzy soft hypergraph* further generalizes this notion by associating parameterized intuitionistic fuzzy soft sets to assign membership and non-membership degrees to both vertices and hyperedges (cf. [55]).

Definition 8 (Intuitionistic Fuzzy Soft Hypergraph)**.** *(cf.[55]) Let* V *be a nonempty set (the vertex universe), let* $E \subseteq \mathrm{PSET}(V)$ *be a family of (crisp) hyperedges, and let* R *be a nonempty set of parameters. An* intuitionistic fuzzy soft hypergraph *over* (V, E) *with parameter set* R *is a quintuple*

$$\tilde{H} \;=\; (V,\, E,\, R,\, N,\, S),$$

where

(i) $N = (N_\mu, N_\nu)$ *is an intuitionistic fuzzy soft set on* V*; that is,*

$$N : R \longrightarrow \mathrm{IF}(V), \quad N(a) = \big(N_\mu(a),\, N_\nu(a)\big),$$

where for each $a \in R$, $N_\mu(a), N_\nu(a) : V \to [0, 1]$ *satisfy*

$$0 \;\le\; N_\mu(a)(v) \;+\; N_\nu(a)(v) \;\le\; 1 \quad \text{for all } v \in V.$$

(ii) $S = (S_\mu, S_\nu)$ *is an intuitionistic fuzzy soft relation on the incidence set* $V \times E$*; that is,*

$$S : R \longrightarrow \mathrm{IF}(V \times E), \quad S(a) = \big(S_\mu(a),\, S_\nu(a)\big),$$

where for each $a \in R$, $S_\mu(a), S_\nu(a) : V \times E \to [0, 1]$ *satisfy*

$$0 \;\le\; S_\mu(a)(v, e) \;+\; S_\nu(a)(v, e) \;\le\; 1 \quad \text{for all } (v, e) \in V \times E.$$

Moreover, for each parameter $a \in R$, the pair $(N(a), S(a))$ must form an intuitionistic fuzzy sub-hypergraph of (V, E). Equivalently, one requires

$$S_\mu(a)(v,e) \ \leq \ N_\mu(a)(v) \quad \textit{and} \quad S_\nu(a)(v,e) \ \leq \ N_\nu(a)(v) \quad \textit{whenever } e \in E,\ v \in e.$$

Example 6 (Intuitionistic Fuzzy Soft Hypergraph for Sensor Network Reliability)**.** *Consider a small sensor network monitoring two overlapping regions. Let*

$$V = \{S_1, S_2, S_3\}, \quad E = \big\{e_1 = \{S_1, S_2\},\ e_2 = \{S_2, S_3\}\big\}, \quad R = \{\text{HighHumidity, HighTemp}\}.$$

Define the intuitionistic fuzzy soft vertex-set $N = (N_\mu, N_\nu)$ by

$$\begin{aligned} N_\mu(\text{HighHumidity})(S_1) &= 0.80, & N_\nu(\text{HighHumidity})(S_1) &= 0.10,\\ N_\mu(\text{HighHumidity})(S_2) &= 0.90, & N_\nu(\text{HighHumidity})(S_2) &= 0.05,\\ N_\mu(\text{HighHumidity})(S_3) &= 0.60, & N_\nu(\text{HighHumidity})(S_3) &= 0.20, \end{aligned}$$

$$\begin{aligned} N_\mu(\text{HighTemp})(S_1) &= 0.70, & N_\nu(\text{HighTemp})(S_1) &= 0.20,\\ N_\mu(\text{HighTemp})(S_2) &= 0.60, & N_\nu(\text{HighTemp})(S_2) &= 0.30,\\ N_\mu(\text{HighTemp})(S_3) &= 0.85, & N_\nu(\text{HighTemp})(S_3) &= 0.10. \end{aligned}$$

Next, define the intuitionistic fuzzy soft incidence relation $S = (S_\mu, S_\nu)$ by

$$\begin{aligned} S_\mu(\text{HighHumidity})(S_1, e_1) &= 0.80, & S_\nu(\text{HighHumidity})(S_1, e_1) &= 0.10,\\ S_\mu(\text{HighHumidity})(S_2, e_1) &= 0.90, & S_\nu(\text{HighHumidity})(S_2, e_1) &= 0.05,\\ S_\mu(\text{HighHumidity})(S_2, e_2) &= 0.90, & S_\nu(\text{HighHumidity})(S_2, e_2) &= 0.05,\\ S_\mu(\text{HighHumidity})(S_3, e_2) &= 0.60, & S_\nu(\text{HighHumidity})(S_3, e_2) &= 0.20, \end{aligned}$$

$$\begin{aligned} S_\mu(\text{HighTemp})(S_1, e_1) &= 0.70, & S_\nu(\text{HighTemp})(S_1, e_1) &= 0.20,\\ S_\mu(\text{HighTemp})(S_2, e_1) &= 0.60, & S_\nu(\text{HighTemp})(S_2, e_1) &= 0.30,\\ S_\mu(\text{HighTemp})(S_2, e_2) &= 0.60, & S_\nu(\text{HighTemp})(S_2, e_2) &= 0.30,\\ S_\mu(\text{HighTemp})(S_3, e_2) &= 0.85, & S_\nu(\text{HighTemp})(S_3, e_2) &= 0.10. \end{aligned}$$

Then $\tilde{H} = (V, E, R, N, S)$ is an intuitionistic fuzzy soft hypergraph modeling how sensor reliability and region-connectivity vary under high humidity and high temperature conditions.

3. Main Results of this Paper

This section presents the main results of this paper.

3.1. Intuitionistic Fuzzy Soft n-Super-hypergraphs

In this subsection, we introduce an incidence-based notion of an intuitionistic fuzzy soft n-Super-hypergraph and establish several basic structural properties.

For any nonempty set X, let

$$\mathrm{IF}(X) := \left\{ (\mu,\nu) \;\middle|\; \mu,\nu : X \to [0,1],\;\; 0 \le \mu(x) + \nu(x) \le 1 \text{ for all } x \in X \right\}$$

denote the family of all intuitionistic fuzzy sets on X.

If $\mathrm{SuHG}^{(n)} = (V, E)$ is an n-Super-hypergraph, we write

$$I(V,E) := \{(v,e) \in V \times E : \; v \in e\}$$

for its incidence set.

Definition 9 (Intuitionistic fuzzy soft n-Super-hypergraph)**.** *Let* $\mathrm{SuHG}^{(n)} = (V, E)$ *be an n-Super-hypergraph, and let R be a nonempty parameter set. An* intuitionistic fuzzy soft n-Super-hypergraph *over* $\mathrm{SuHG}^{(n)}$ *with parameter set R is a quintuple*

$$\widetilde{\mathrm{SuHG}} = (V, E, R, N, S),$$

where

$$N = (N_\mu, N_\nu) : R \longrightarrow \mathrm{IF}(V), \qquad S = (S_\mu, S_\nu) : R \longrightarrow \mathrm{IF}(I(V,E)),$$

such that, for every $a \in R$, the following conditions hold:

(i) for every $v \in V$,

$$0 \le N_\mu(a)(v) + N_\nu(a)(v) \le 1;$$

(ii) for every $(v,e) \in I(V,E)$,

$$0 \le S_\mu(a)(v,e) + S_\nu(a)(v,e) \le 1;$$

(iii) for every $(v,e) \in I(V,E)$,

$$S_\mu(a)(v,e) \le N_\mu(a)(v), \qquad S_\nu(a)(v,e) \le N_\nu(a)(v).$$

The quantity

$$\pi_N(a)(v) := 1 - N_\mu(a)(v) - N_\nu(a)(v)$$

is called the vertex hesitation degree *of v at parameter a, and*

$$\pi_S(a)(v,e) := 1 - S_\mu(a)(v,e) - S_\nu(a)(v,e)$$

is called the incidence hesitation degree *of (v,e) at parameter a.*

Remark 1 (Why classical hypergraphs and ordinary fuzzy hypergraphs may be insufficient)**.** *The usefulness of the above model becomes apparent in decision environments involving, simultaneously,*

(i) hierarchical organization of entities,

(ii) parameter-dependent assessments, and

(iii) separate positive and negative evidence.

A classical hypergraph can model higher-order relations among entities, but it does not encode parameter dependence, graded uncertainty, or hesitation. An ordinary fuzzy hypergraph improves this by attaching a single membership degree, but it still cannot distinguish between situations such as

$$(\mu, \nu) = (0.45, 0.10) \qquad \text{and} \qquad (\mu, \nu) = (0.45, 0.40).$$

In both cases the membership degree is 0.45*, yet the economic interpretation is very different: the first configuration indicates moderate support with weak opposition, whereas the second indicates moderate support together with strong negative evidence.*

By contrast, an intuitionistic fuzzy soft n*-Super-hypergraph can simultaneously encode*

- *the multi-level hierarchical structure through the* n*-Super-hypergraph layer,*
- *parameter dependence through the soft parameter set* R*, and*
- *support, opposition, and hesitation through the intuitionistic fuzzy pairs* (N_μ, N_ν) *and* (S_μ, S_ν)*.*

Hence it is strictly richer than both a classical hypergraph and an ordinary fuzzy hypergraph in such settings.

Example 7 (Intuitionistic fuzzy soft 2-Super-hypergraph for urban transit reliability)**.** *Consider a city transit system with two hierarchical grouping levels.*

- ***Level 0 (stops):***
$$V_0 = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}.$$
- ***Level 1 (routes):***
$$\mathcal{R}_1 = \{s_1, s_2, s_3\}, \qquad \mathcal{R}_2 = \{s_4, s_5\}, \qquad \mathcal{R}_3 = \{s_6, s_7, s_8\}.$$
- ***Level 2 (bus lines):***
$$L_A = \{\mathcal{R}_1, \mathcal{R}_2\}, \qquad L_B = \{\mathcal{R}_2, \mathcal{R}_3\}.$$

Define

$$V = \{L_A, L_B\}, \qquad e = \{L_A, L_B\}, \qquad E = \{e\}.$$

Then $\mathrm{SuHG}^{(2)} = (V, E)$ *is a* 2*-Super-hypergraph, and*

$$I(V, E) = \{(L_A, e), (L_B, e)\}.$$

Let the parameter set be

$$R = \{\mathrm{RushHour}, \mathrm{Maintenance}\}.$$

Define $N = (N_\mu, N_\nu) : R \to \mathrm{IF}(V)$ *by*

$$N_\mu(\mathrm{RushHour}) : \begin{cases} L_A \mapsto 0.70, \\ L_B \mapsto 0.60, \end{cases} \qquad N_\nu(\mathrm{RushHour}) : \begin{cases} L_A \mapsto 0.20, \\ L_B \mapsto 0.30, \end{cases}$$

and

$$N_\mu(\mathrm{Maintenance}) : \begin{cases} L_A \mapsto 0.80, \\ L_B \mapsto 0.50, \end{cases} \qquad N_\nu(\mathrm{Maintenance}) : \begin{cases} L_A \mapsto 0.10, \\ L_B \mapsto 0.40. \end{cases}$$

Next define $S = (S_\mu, S_\nu) : R \to \mathrm{IF}(I(V, E))$ *by*

$$S_\mu(\mathrm{RushHour}) : \begin{cases} (L_A, e) \mapsto 0.70, \\ (L_B, e) \mapsto 0.60, \end{cases} \qquad S_\nu(\mathrm{RushHour}) : \begin{cases} (L_A, e) \mapsto 0.20, \\ (L_B, e) \mapsto 0.30, \end{cases}$$

and

$$S_\mu(\mathrm{Maintenance}) : \begin{cases} (L_A, e) \mapsto 0.80, \\ (L_B, e) \mapsto 0.50, \end{cases} \qquad S_\nu(\mathrm{Maintenance}) : \begin{cases} (L_A, e) \mapsto 0.10, \\ (L_B, e) \mapsto 0.40. \end{cases}$$

For each parameter $a \in R$*, one checks that*

$$0 \le N_\mu(a)(L_i) + N_\nu(a)(L_i) \le 1 \qquad (L_i \in V),$$

$$0 \le S_\mu(a)(L_i, e) + S_\nu(a)(L_i, e) \le 1 \qquad ((L_i, e) \in I(V, E)),$$

and

$$S_\mu(a)(L_i, e) \le N_\mu(a)(L_i), \qquad S_\nu(a)(L_i, e) \le N_\nu(a)(L_i).$$

Therefore,

$$\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$$

is an intuitionistic fuzzy soft 2*-Super-hypergraph describing the reliability of the two bus lines under rush-hour and maintenance conditions.*

Example 8 (Intuitionistic fuzzy soft 2-Super-hypergraph for regional investment alliances)**.** *Consider an economic decision problem in which a development agency evaluates regional investment alliances under different macroeconomic scenarios.*

Let the base set of projects be

$$V_0 = \{p_1, p_2, p_3, p_4, p_5, p_6\},$$

where

$$p_1 = \textit{logistics hub}, \quad p_2 = \textit{cold-chain facility}, \quad p_3 = \textit{solar plant},$$

$$p_4 = \textit{battery storage unit}, \quad p_5 = \textit{digital trade platform}, \quad p_6 = \textit{smart warehouse}.$$

Define sectoral clusters by

$$C_1 = \{p_1, p_2\}, \qquad C_2 = \{p_3, p_4\}, \qquad C_3 = \{p_5, p_6\}.$$

At the next level, define alliances by

$$A_1 = \{C_1, C_2\}, \qquad A_2 = \{C_2, C_3\}, \qquad A_3 = \{C_1, C_3\}.$$

Set

$$V = \{A_1, A_2, A_3\}.$$

Let

$$e_1 = \{A_1, A_2\}, \qquad e_2 = \{A_2, A_3\}, \qquad e_3 = \{A_1, A_2, A_3\},$$

and set

$$E = \{e_1, e_2, e_3\}.$$

Then $\mathrm{SuHG}^{(2)} = (V, E)$ *is a* 2*-Super-hypergraph.*

Let the parameter set be

$$R = \{\mathrm{HighDemand}, \mathrm{PolicyStability}\}.$$

Define $N = (N_\mu, N_\nu) : R \to \mathrm{IF}(V)$ *by*

$$N_\mu(\mathrm{HighDemand}) : \begin{cases} A_1 \mapsto 0.82, \\ A_2 \mapsto 0.90, \\ A_3 \mapsto 0.74, \end{cases} \qquad N_\nu(\mathrm{HighDemand}) : \begin{cases} A_1 \mapsto 0.10, \\ A_2 \mapsto 0.05, \\ A_3 \mapsto 0.18, \end{cases}$$

and

$$N_\mu(\mathrm{PolicyStability}) : \begin{cases} A_1 \mapsto 0.88, \\ A_2 \mapsto 0.76, \\ A_3 \mapsto 0.80, \end{cases} \qquad N_\nu(\mathrm{PolicyStability}) : \begin{cases} A_1 \mapsto 0.06, \\ A_2 \mapsto 0.14, \\ A_3 \mapsto 0.12. \end{cases}$$

The incidence set is

$$I(V, E) = \{(A_1, e_1), (A_2, e_1), (A_2, e_2), (A_3, e_2), (A_1, e_3), (A_2, e_3), (A_3, e_3)\}.$$

Define $S = (S_\mu, S_\nu) : R \to \mathrm{IF}(I(V, E))$ *by the following values.*

Under HighDemand***:***

$$
\begin{aligned}
&S_\mu(\text{HighDemand})(A_1, e_1) = 0.80, && S_\nu(\text{HighDemand})(A_1, e_1) = 0.10,\\
&S_\mu(\text{HighDemand})(A_2, e_1) = 0.88, && S_\nu(\text{HighDemand})(A_2, e_1) = 0.05,\\
&S_\mu(\text{HighDemand})(A_2, e_2) = 0.90, && S_\nu(\text{HighDemand})(A_2, e_2) = 0.05,\\
&S_\mu(\text{HighDemand})(A_3, e_2) = 0.72, && S_\nu(\text{HighDemand})(A_3, e_2) = 0.18,\\
&S_\mu(\text{HighDemand})(A_1, e_3) = 0.78, && S_\nu(\text{HighDemand})(A_1, e_3) = 0.10,\\
&S_\mu(\text{HighDemand})(A_2, e_3) = 0.86, && S_\nu(\text{HighDemand})(A_2, e_3) = 0.05,\\
&S_\mu(\text{HighDemand})(A_3, e_3) = 0.70, && S_\nu(\text{HighDemand})(A_3, e_3) = 0.18.
\end{aligned}
$$

Under PolicyStability***:***

$$
\begin{aligned}
&S_\mu(\text{PolicyStability})(A_1, e_1) = 0.86, && S_\nu(\text{PolicyStability})(A_1, e_1) = 0.06,\\
&S_\mu(\text{PolicyStability})(A_2, e_1) = 0.72, && S_\nu(\text{PolicyStability})(A_2, e_1) = 0.14,\\
&S_\mu(\text{PolicyStability})(A_2, e_2) = 0.74, && S_\nu(\text{PolicyStability})(A_2, e_2) = 0.14,\\
&S_\mu(\text{PolicyStability})(A_3, e_2) = 0.78, && S_\nu(\text{PolicyStability})(A_3, e_2) = 0.12,\\
&S_\mu(\text{PolicyStability})(A_1, e_3) = 0.84, && S_\nu(\text{PolicyStability})(A_1, e_3) = 0.06,\\
&S_\mu(\text{PolicyStability})(A_2, e_3) = 0.70, && S_\nu(\text{PolicyStability})(A_2, e_3) = 0.14,\\
&S_\mu(\text{PolicyStability})(A_3, e_3) = 0.76, && S_\nu(\text{PolicyStability})(A_3, e_3) = 0.12.
\end{aligned}
$$

For every $a \in R$*, every* $A_i \in V$*, and every* $(A_i, e_j) \in I(V, E)$*, we have*

$$0 \le N_\mu(a)(A_i) + N_\nu(a)(A_i) \le 1,$$

$$0 \le S_\mu(a)(A_i, e_j) + S_\nu(a)(A_i, e_j) \le 1,$$

and

$$S_\mu(a)(A_i, e_j) \le N_\mu(a)(A_i), \qquad S_\nu(a)(A_i, e_j) \le N_\nu(a)(A_i).$$

Therefore,

$$\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$$

is an intuitionistic fuzzy soft 2*-Super-hypergraph.*

Economically, the membership values represent support for an alliance under a given macroeconomic scenario, the non-membership values represent reservations or risk, and the parameter set distinguishes the surrounding decision environment.

Example 9 (Scalable intuitionistic fuzzy soft 2-Super-hypergraph for a large-scale economic decision)**.** *Consider a national economic support problem with* 100 *candidate entities:*

$$V_0 = \{p_1, p_2, \dots, p_{100}\}.$$

Let the parameter set be

$$R = \{\text{HighDemand}, \text{PolicyStability}, \text{CreditTightening}\}.$$

Level 1: sectoral clusters. *For $j = 1, 2, \dots, 10$, define*

$$C_j = \{p_{10j-9}, p_{10j-8}, \dots, p_{10j}\}.$$

Level 2: regional alliances. *Define*

$$A_1 = \{C_1, C_2\}, \qquad A_2 = \{C_3, C_4\}, \qquad A_3 = \{C_5, C_6\},$$
$$A_4 = \{C_7, C_8\}, \qquad A_5 = \{C_9, C_{10}\},$$

and set

$$V = \{A_1, A_2, A_3, A_4, A_5\}.$$

Candidate support packages. *Let*

$$e_1 = \{A_1, A_2\}, \qquad e_2 = \{A_2, A_3, A_4\}, \qquad e_3 = \{A_4, A_5\},$$
$$e_4 = \{A_1, A_3, A_5\}, \qquad e_5 = \{A_1, A_2, A_3, A_4, A_5\},$$

and set

$$E = \{e_1, e_2, e_3, e_4, e_5\}.$$

Then $\mathrm{SuHG}^{(2)} = (V, E)$ *is a* 2*-Super-hypergraph.*

Step 1: project-level intuitionistic fuzzy values. *For each $a \in R$ and each $p_i \in V_0$, let*

$$(m_a(p_i), n_a(p_i)) \in [0, 1]^2$$

satisfy

$$0 \le m_a(p_i) + n_a(p_i) \le 1.$$

Step 2: cluster aggregation. *For each cluster C_j and each $a \in R$, define*

$$\alpha_\mu(a)(C_j) := \frac{1}{10} \sum_{p \in C_j} m_a(p), \qquad \alpha_\nu(a)(C_j) := \frac{1}{10} \sum_{p \in C_j} n_a(p).$$

Then

$$0 \le \alpha_\mu(a)(C_j) + \alpha_\nu(a)(C_j) \le 1.$$

Step 3: alliance-level vertex values. *For each $A_k \in V$ and $a \in R$, define*

$$N_\mu(a)(A_k) := \frac{1}{|A_k|}\sum_{C\in A_k}\alpha_\mu(a)(C), \qquad N_\nu(a)(A_k) := \frac{1}{|A_k|}\sum_{C\in A_k}\alpha_\nu(a)(C).$$

Then

$$0 \le N_\mu(a)(A_k) + N_\nu(a)(A_k) \le 1 \qquad (A_k \in V,\ a \in R).$$

Step 4: incidence values. *For each parameter $a \in R$ and each $e \in E$, choose a coherence coefficient*

$$\rho_a(e) \in [0,1].$$

For each incidence $(A_k, e) \in I(V,E)$, define

$$S_\mu(a)(A_k,e) := \rho_a(e)\,N_\mu(a)(A_k), \qquad S_\nu(a)(A_k,e) := \rho_a(e)\,N_\nu(a)(A_k).$$

Then

$$S_\mu(a)(A_k,e) \le N_\mu(a)(A_k), \qquad S_\nu(a)(A_k,e) \le N_\nu(a)(A_k),$$

and also

$$0 \le S_\mu(a)(A_k,e) + S_\nu(a)(A_k,e) = \rho_a(e)\big(N_\mu(a)(A_k) + N_\nu(a)(A_k)\big) \le 1.$$

Hence

$$\widetilde{\mathrm{SuHG}} = (V,E,R,N,S)$$

is an intuitionistic fuzzy soft 2-Super-hypergraph.

As a concrete instance, suppose that under HighDemand*,*

$$N_\mu(\mathrm{HighDemand}) : \begin{cases} A_1 \mapsto 0.81, \\ A_2 \mapsto 0.76, \\ A_3 \mapsto 0.84, \\ A_4 \mapsto 0.72, \\ A_5 \mapsto 0.79, \end{cases} \qquad N_\nu(\mathrm{HighDemand}) : \begin{cases} A_1 \mapsto 0.11, \\ A_2 \mapsto 0.14, \\ A_3 \mapsto 0.08, \\ A_4 \mapsto 0.18, \\ A_5 \mapsto 0.12. \end{cases}$$

Choose

$$\rho_{\mathrm{HighDemand}}(e_1) = 0.90, \quad \rho_{\mathrm{HighDemand}}(e_2) = 0.85, \quad \rho_{\mathrm{HighDemand}}(e_3) = 0.88,$$
$$\rho_{\mathrm{HighDemand}}(e_4) = 0.80, \quad \rho_{\mathrm{HighDemand}}(e_5) = 0.75.$$

Then, for example,

$$S_\mu(\mathrm{HighDemand})(A_2,e_1) = 0.90\times 0.76 = 0.684, \qquad S_\nu(\mathrm{HighDemand})(A_2,e_1) = 0.90\times 0.14 = 0.126,$$

$$S_\mu(\mathrm{HighDemand})(A_3,e_2) = 0.85\times 0.84 = 0.714, \qquad S_\nu(\mathrm{HighDemand})(A_3,e_2) = 0.85\times 0.08 = 0.068.$$

Thus the axioms are satisfied in this large-scale setting as well.

Theorem 1 (Basic reductions)**.** *Let* $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ *be an intuitionistic fuzzy soft* n*-Super-hypergraph as in Definition 9. Then:*

(a) If $n = 0$*, then* $\widetilde{\mathrm{SuHG}}$ *is an intuitionistic fuzzy soft hypergraph on the ordinary hypergraph* (V, E)*.*

(b) If $N_\nu(a) \equiv 0$ *and* $S_\nu(a) \equiv 0$ *for all* $a \in R$*, then for each fixed* $a \in R$*, the pair*

$$\sigma_a(v) := N_\mu(a)(v) \qquad (v \in V),$$

$$\mu_a(e) := \min_{v \in e} S_\mu(a)(v, e) \qquad (e \in E)$$

defines a fuzzy n*-Super-hypergraph* (V, E, σ_a, μ_a)*.*

(c) Suppose that for every $a \in R$*, all values of* $N_\mu(a)$ *and* $S_\mu(a)$ *belong to* $\{0, 1\}$*, and*

$$N_\nu(a) = 1 - N_\mu(a), \qquad S_\nu(a) = 1 - S_\mu(a).$$

Define

$$A(a) := \{v \in V :\ N_\mu(a)(v) = 1\},$$

$$B(a) := \{e \in E :\ S_\mu(a)(v, e) = 1 \text{ for all } v \in e\}.$$

Then (V, E, R, A, B) *is a soft* n*-Super-hypergraph.*

Proof. **(a)** If $n = 0$, then $V \subseteq \mathrm{PSET}^0(V_0) = V_0$, while

$$E \subseteq \mathrm{PSET}(V) \setminus \{\varnothing\}.$$

Hence (V, E) is an ordinary hypergraph. Since S is defined on the incidence set $I(V, E)$, Definition 9 becomes exactly the usual incidence-based definition of an intuitionistic fuzzy soft hypergraph.

(b) Fix $a \in R$. Since $N_\nu(a) \equiv 0$ and $S_\nu(a) \equiv 0$, the maps $N_\mu(a)$ and $S_\mu(a)$ are ordinary fuzzy membership functions. Clearly,

$$\sigma_a(v) \in [0, 1], \qquad \mu_a(e) \in [0, 1].$$

For each $e \in E$ and each $v \in e$,

$$\mu_a(e) = \min_{u \in e} S_\mu(a)(u, e) \leq S_\mu(a)(v, e) \leq N_\mu(a)(v) = \sigma_a(v).$$

Therefore

$$\mu_a(e) \leq \min_{v \in e} \sigma_a(v),$$

so (V, E, σ_a, μ_a) is a fuzzy n-Super-hypergraph.

(c) By definition, $A(a) \subseteq V$ and $B(a) \subseteq E$ for each $a \in R$. If $e \in B(a)$, then $S_\mu(a)(v, e) = 1$ for every $v \in e$. By the consistency axiom,

$$1 = S_\mu(a)(v, e) \leq N_\mu(a)(v) \leq 1,$$

hence $N_\mu(a)(v) = 1$, so $v \in A(a)$. Therefore $e \subseteq A(a)$. Thus every edge in $B(a)$ is a nonempty subset of $A(a)$, and $(A(a), B(a))$ is a crisp sub-n-Super-hypergraph of (V, E). Since this is true for all $a \in R$, (V, E, R, A, B) is a soft n-Super-hypergraph.

Theorem 2 (Parameter restriction). *Let* $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ *be an intuitionistic fuzzy soft n-Super-hypergraph, and let* $R' \subseteq R$ *be nonempty. Then*

$$\widetilde{\mathrm{SuHG}}\big|_{R'} := (V, E, R', N|_{R'}, S|_{R'})$$

is also an intuitionistic fuzzy soft n-Super-hypergraph.

Proof. For every $a \in R'$, the restrictions $N|_{R'}(a)$ and $S|_{R'}(a)$ satisfy exactly the same inequalities as in Definition 9, because $R' \subseteq R$. Nothing else changes. Hence all axioms remain valid.

Theorem 3 (Disjoint parameter union). *Let*

$$\widetilde{\mathrm{SuHG}}_1 = (V, E, R_1, N^1, S^1), \qquad \widetilde{\mathrm{SuHG}}_2 = (V, E, R_2, N^2, S^2)$$

be two intuitionistic fuzzy soft n-Super-hypergraphs on the same underlying (V, E)*, and assume that* $R_1 \cap R_2 = \varnothing$*. Define*

$$R := R_1 \cup R_2,$$

$$N(a) := \begin{cases} N^1(a), & a \in R_1, \\ N^2(a), & a \in R_2, \end{cases} \qquad S(a) := \begin{cases} S^1(a), & a \in R_1, \\ S^2(a), & a \in R_2. \end{cases}$$

Then $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ *is an intuitionistic fuzzy soft n-Super-hypergraph.*

Proof. Each parameter $a \in R$ belongs to exactly one of R_1 or R_2. Thus the corresponding data are inherited from one valid intuitionistic fuzzy soft n-Super-hypergraph, and all axioms hold parameterwise.

Theorem 4 (Pointwise intersection). *Let*

$$\widetilde{\mathrm{SuHG}}_1 = (V, E, R, N^1, S^1), \qquad \widetilde{\mathrm{SuHG}}_2 = (V, E, R, N^2, S^2)$$

be two intuitionistic fuzzy soft n-Super-hypergraphs on the same (V, E, R)*. Define*

$$\widehat{N}_\mu(a)(v) := \min\{N^1_\mu(a)(v), N^2_\mu(a)(v)\},$$

$$\widehat{N}_\nu(a)(v) := \max\{N^1_\nu(a)(v), N^2_\nu(a)(v)\},$$

and, for each $(v, e) \in I(V, E)$*,*

$$\widehat{S}_\mu(a)(v, e) := \min\{S^1_\mu(a)(v, e), S^2_\mu(a)(v, e)\},$$

$$\widehat{S}_\nu(a)(v, e) := \max\{S^1_\nu(a)(v, e), S^2_\nu(a)(v, e)\}.$$

Then

$$\widehat{\mathrm{SuHG}} = (V, E, R, \widehat{N}, \widehat{S})$$

is an intuitionistic fuzzy soft n-Super-hypergraph.

Proof. Fix $a \in R$ and $v \in V$. If $N_\nu^1(a)(v) \geq N_\nu^2(a)(v)$, then

$$\widehat{N}_\nu(a)(v) = N_\nu^1(a)(v), \qquad \widehat{N}_\mu(a)(v) \leq N_\mu^1(a)(v),$$

hence

$$\widehat{N}_\mu(a)(v) + \widehat{N}_\nu(a)(v) \leq N_\mu^1(a)(v) + N_\nu^1(a)(v) \leq 1.$$

If $N_\nu^2(a)(v) \geq N_\nu^1(a)(v)$, the same argument with index 2 gives the result. Therefore

$$0 \leq \widehat{N}_\mu(a)(v) + \widehat{N}_\nu(a)(v) \leq 1.$$

The same case analysis shows that

$$0 \leq \widehat{S}_\mu(a)(v,e) + \widehat{S}_\nu(a)(v,e) \leq 1 \qquad ((v,e) \in I(V,E)).$$

For consistency, if $(v,e) \in I(V,E)$, then

$$\widehat{S}_\mu(a)(v,e) = \min\{S_\mu^1(a)(v,e), S_\mu^2(a)(v,e)\} \leq \min\{N_\mu^1(a)(v), N_\mu^2(a)(v)\} = \widehat{N}_\mu(a)(v),$$

and

$$\widehat{S}_\nu(a)(v,e) = \max\{S_\nu^1(a)(v,e), S_\nu^2(a)(v,e)\} \leq \max\{N_\nu^1(a)(v), N_\nu^2(a)(v)\} = \widehat{N}_\nu(a)(v).$$

Hence all axioms of Definition 9 are satisfied.

Theorem 5 (Pointwise union)**.** *Under the hypotheses of Theorem 4, define*

$$\overline{N}_\mu(a)(v) := \max\{N_\mu^1(a)(v), N_\mu^2(a)(v)\},$$

$$\overline{N}_\nu(a)(v) := \min\{N_\nu^1(a)(v), N_\nu^2(a)(v)\},$$

and, for each $(v,e) \in I(V,E)$*,*

$$\overline{S}_\mu(a)(v,e) := \max\{S_\mu^1(a)(v,e), S_\mu^2(a)(v,e)\},$$

$$\overline{S}_\nu(a)(v,e) := \min\{S_\nu^1(a)(v,e), S_\nu^2(a)(v,e)\}.$$

Then

$$\overline{\mathrm{SuHG}} = (V, E, R, \overline{N}, \overline{S})$$

is an intuitionistic fuzzy soft n*-Super-hypergraph.*

Proof. Fix $a \in R$ and $v \in V$. If $N_\mu^1(a)(v) \geq N_\mu^2(a)(v)$, then

$$\overline{N}_\mu(a)(v) = N_\mu^1(a)(v), \qquad \overline{N}_\nu(a)(v) \leq N_\nu^1(a)(v),$$

so

$$\overline{N}_\mu(a)(v) + \overline{N}_\nu(a)(v) \leq N_\mu^1(a)(v) + N_\nu^1(a)(v) \leq 1.$$

The other case is analogous. Hence

$$0 \le \overline{N}_\mu(a)(v) + \overline{N}_\nu(a)(v) \le 1.$$

Likewise,

$$0 \le \overline{S}_\mu(a)(v,e) + \overline{S}_\nu(a)(v,e) \le 1 \qquad ((v,e) \in I(V,E)).$$

For consistency,

$$\overline{S}_\mu(a)(v,e) = \max\{S^1_\mu(a)(v,e), S^2_\mu(a)(v,e)\} \le \max\{N^1_\mu(a)(v), N^2_\mu(a)(v)\} = \overline{N}_\mu(a)(v),$$

and

$$\overline{S}_\nu(a)(v,e) = \min\{S^1_\nu(a)(v,e), S^2_\nu(a)(v,e)\} \le \min\{N^1_\nu(a)(v), N^2_\nu(a)(v)\} = \overline{N}_\nu(a)(v).$$

Therefore all axioms hold.

Theorem 6 (One-cut crispification)**.** *Let $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ be an intuitionistic fuzzy soft n-Super-hypergraph, and fix $a \in R$. Define*

$$V_a^{(1)} := \{v \in V : \ N_\mu(a)(v) = 1\},$$

$$E_a^{(1)} := \{e \in E : \ S_\mu(a)(v,e) = 1 \textit{ for all } v \in e\}.$$

Then $(V_a^{(1)}, E_a^{(1)})$ is a crisp sub-n-Super-hypergraph of (V,E).

Proof. Clearly $V_a^{(1)} \subseteq V$ and $E_a^{(1)} \subseteq E$. If $e \in E_a^{(1)}$ and $v \in e$, then

$$1 = S_\mu(a)(v,e) \le N_\mu(a)(v) \le 1,$$

so $N_\mu(a)(v) = 1$, and hence $v \in V_a^{(1)}$. Therefore $e \subseteq V_a^{(1)}$, and thus $(V_a^{(1)}, E_a^{(1)})$ is a crisp sub-n-Super-hypergraph.

Theorem 7 (α-level induced fuzzy sub-n-Super-hypergraph)**.** *Let $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ be an intuitionistic fuzzy soft n-Super-hypergraph, fix $a \in R$, and let $\alpha \in (0,1]$. Define*

$$V_\alpha := \{v \in V : \ N_\mu(a)(v) \ge \alpha\},$$

$$E_\alpha := \{e \in E : \ S_\mu(a)(v,e) \ge \alpha \textit{ for all } v \in e\}.$$

For $v \in V_\alpha$ and $e \in E_\alpha$, define

$$\sigma_\alpha(v) := N_\mu(a)(v), \qquad \mu_\alpha(e) := \min_{v \in e} S_\mu(a)(v,e).$$

Then $(V_\alpha, E_\alpha, \sigma_\alpha, \mu_\alpha)$ is a fuzzy n-Super-hypergraph.

Proof. Let $e \in E_\alpha$. Then for every $v \in e$,

$$\alpha \leq S_\mu(a)(v,e) \leq N_\mu(a)(v),$$

so $N_\mu(a)(v) \geq \alpha$, that is, $v \in V_\alpha$. Hence $e \subseteq V_\alpha$, and therefore $E_\alpha \subseteq \mathrm{PSET}(V_\alpha) \setminus \{\varnothing\}$.

Now, for each $e \in E_\alpha$ and $v \in e$,

$$\mu_\alpha(e) = \min_{u \in e} S_\mu(a)(u,e) \leq S_\mu(a)(v,e) \leq N_\mu(a)(v) = \sigma_\alpha(v).$$

Thus

$$\mu_\alpha(e) \leq \min_{v \in e} \sigma_\alpha(v),$$

which is precisely the fuzzy edge condition.

Theorem 8 (Complementation). *Let $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ be an intuitionistic fuzzy soft n-Super-hypergraph. Define*

$$N'_\mu(a)(v) := N_\nu(a)(v), \qquad N'_\nu(a)(v) := N_\mu(a)(v),$$

and, for $(v,e) \in I(V,E)$,

$$S'_\mu(a)(v,e) := S_\nu(a)(v,e), \qquad S'_\nu(a)(v,e) := S_\mu(a)(v,e).$$

Then

$$\widetilde{\mathrm{SuHG}}' = (V, E, R, N', S')$$

is also an intuitionistic fuzzy soft n-Super-hypergraph.

Proof. For every $a \in R$, $v \in V$, and $(v,e) \in I(V,E)$,

$$N'_\mu(a)(v) + N'_\nu(a)(v) = N_\nu(a)(v) + N_\mu(a)(v) \leq 1,$$
$$S'_\mu(a)(v,e) + S'_\nu(a)(v,e) = S_\nu(a)(v,e) + S_\mu(a)(v,e) \leq 1.$$

Moreover,

$$S'_\mu(a)(v,e) = S_\nu(a)(v,e) \leq N_\nu(a)(v) = N'_\mu(a)(v),$$
$$S'_\nu(a)(v,e) = S_\mu(a)(v,e) \leq N_\mu(a)(v) = N'_\nu(a)(v).$$

Hence all axioms remain valid.

Theorem 9 (Scalar attenuation). *Let $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ be an intuitionistic fuzzy soft n-Super-hypergraph, and let $c \in [0,1]$. Define*

$$N^c_\mu(a)(v) := c\, N_\mu(a)(v), \qquad N^c_\nu(a)(v) := c\, N_\nu(a)(v),$$

and, for $(v,e) \in I(V,E)$,

$$S^c_\mu(a)(v,e) := c\, S_\mu(a)(v,e), \qquad S^c_\nu(a)(v,e) := c\, S_\nu(a)(v,e).$$

Then

$$\widetilde{\mathrm{SuHG}}^c = (V, E, R, N^c, S^c)$$

is an intuitionistic fuzzy soft n-Super-hypergraph.

Proof. For every $a \in R$, $v \in V$, and $(v, e) \in I(V, E)$,

$$0 \leq N_\mu^c(a)(v) + N_\nu^c(a)(v) = c\big(N_\mu(a)(v) + N_\nu(a)(v)\big) \leq c \leq 1,$$

$$0 \leq S_\mu^c(a)(v, e) + S_\nu^c(a)(v, e) = c\big(S_\mu(a)(v, e) + S_\nu(a)(v, e)\big) \leq c \leq 1.$$

Also,

$$S_\mu^c(a)(v, e) = c\, S_\mu(a)(v, e) \leq c\, N_\mu(a)(v) = N_\mu^c(a)(v),$$
$$S_\nu^c(a)(v, e) = c\, S_\nu(a)(v, e) \leq c\, N_\nu(a)(v) = N_\nu^c(a)(v).$$

Therefore all defining conditions hold.

Theorem 10 (Convex combination)**.** *Let*

$$\widetilde{\mathrm{SuHG}}^{\,1} = (V, E, R, N^1, S^1), \qquad \widetilde{\mathrm{SuHG}}^{\,2} = (V, E, R, N^2, S^2)$$

be two intuitionistic fuzzy soft n-Super-hypergraphs on the same (V, E, R), and let $\lambda \in [0, 1]$. Define

$$N_\mu := \lambda N_\mu^1 + (1-\lambda) N_\mu^2, \qquad N_\nu := \lambda N_\nu^1 + (1-\lambda) N_\nu^2,$$

and, on $I(V, E)$,

$$S_\mu := \lambda S_\mu^1 + (1-\lambda) S_\mu^2, \qquad S_\nu := \lambda S_\nu^1 + (1-\lambda) S_\nu^2,$$

all taken pointwise. Then

$$\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$$

is an intuitionistic fuzzy soft n-Super-hypergraph.

Proof. For every $a \in R$ and $v \in V$,

$$N_\mu(a)(v) + N_\nu(a)(v) = \lambda\big(N_\mu^1(a)(v) + N_\nu^1(a)(v)\big) + (1-\lambda)\big(N_\mu^2(a)(v) + N_\nu^2(a)(v)\big) \leq 1.$$

Similarly, for every $(v, e) \in I(V, E)$,

$$S_\mu(a)(v, e) + S_\nu(a)(v, e) \leq 1.$$

Moreover,

$$S_\mu(a)(v, e) = \lambda S_\mu^1(a)(v, e) + (1-\lambda) S_\mu^2(a)(v, e) \leq \lambda N_\mu^1(a)(v) + (1-\lambda) N_\mu^2(a)(v) = N_\mu(a)(v),$$

and likewise

$$S_\nu(a)(v, e) \leq N_\nu(a)(v).$$

Hence the convex combination is again an intuitionistic fuzzy soft n-Super-hypergraph.

Theorem 11 (Product construction over a fixed product super-hypergraph)**.** *Let*

$$\widetilde{\mathrm{SuHG}}_1 = (V_1, E_1, R_1, N^1, S^1), \qquad \widetilde{\mathrm{SuHG}}_2 = (V_2, E_2, R_2, N^2, S^2)$$

be two intuitionistic fuzzy soft Super-hypergraphs. Assume that a product Super-hypergraph

$$\mathrm{SuHG}^{\boxtimes} = (V, E)$$

has been fixed, where

$$V := V_1 \times V_2,$$

$$E := \{\, e_1 \boxtimes e_2 : \ e_1 \in E_1, \ e_2 \in E_2 \,\},$$

and

$$e_1 \boxtimes e_2 := \{(v_1, v_2) \in V_1 \times V_2 : \ v_1 \in e_1, \ v_2 \in e_2\}.$$

Let

$$R := R_1 \times R_2.$$

For $(a_1, a_2) \in R$ *and* $(v_1, v_2) \in V$*, define*

$$N_\mu(a_1, a_2)(v_1, v_2) := \min\{N^1_\mu(a_1)(v_1), N^2_\mu(a_2)(v_2)\},$$

$$N_\nu(a_1, a_2)(v_1, v_2) := \max\{N^1_\nu(a_1)(v_1), N^2_\nu(a_2)(v_2)\}.$$

For each incidence

$$\big((v_1, v_2), \, e_1 \boxtimes e_2\big) \in I(V, E),$$

define

$$S_\mu(a_1, a_2)\big((v_1, v_2), e_1 \boxtimes e_2\big) := \min\{S^1_\mu(a_1)(v_1, e_1), S^2_\mu(a_2)(v_2, e_2)\},$$

$$S_\nu(a_1, a_2)\big((v_1, v_2), e_1 \boxtimes e_2\big) := \max\{S^1_\nu(a_1)(v_1, e_1), S^2_\nu(a_2)(v_2, e_2)\}.$$

Then

$$\widetilde{\mathrm{SuHG}}^{\boxtimes} = (V, E, R, N, S)$$

is an intuitionistic fuzzy soft Super-hypergraph over $\mathrm{SuHG}^{\boxtimes}$*.*

Proof. Fix $(a_1, a_2) \in R$ and $(v_1, v_2) \in V$. If $N^1_\nu(a_1)(v_1) \geq N^2_\nu(a_2)(v_2)$, then

$$N_\nu(a_1, a_2)(v_1, v_2) = N^1_\nu(a_1)(v_1), \qquad N_\mu(a_1, a_2)(v_1, v_2) \leq N^1_\mu(a_1)(v_1),$$

hence

$$N_\mu(a_1, a_2)(v_1, v_2) + N_\nu(a_1, a_2)(v_1, v_2) \leq 1.$$

The other case is analogous. Thus

$$0 \leq N_\mu + N_\nu \leq 1.$$

The same argument yields

$$0 \leq S_\mu + S_\nu \leq 1$$

on all incidences.

Now let $\big((v_1, v_2), e_1 \boxtimes e_2\big) \in I(V, E)$. Then $v_1 \in e_1$ and $v_2 \in e_2$. Therefore

$$S_\mu(a_1, a_2)\big((v_1, v_2), e_1 \boxtimes e_2\big) \le \min\{N^1_\mu(a_1)(v_1), N^2_\mu(a_2)(v_2)\} = N_\mu(a_1, a_2)(v_1, v_2),$$

and

$$S_\nu(a_1, a_2)\big((v_1, v_2), e_1 \boxtimes e_2\big) \le \max\{N^1_\nu(a_1)(v_1), N^2_\nu(a_2)(v_2)\} = N_\nu(a_1, a_2)(v_1, v_2).$$

Thus all axioms of Definition 9 hold.

Theorem 12 (Positive-support sub-n-Super-hypergraph). *Let* $\widetilde{\mathrm{SuHG}} = (V, E, R, N, S)$ *be an intuitionistic fuzzy soft n-Super-hypergraph, and fix* $a \in R$. *Define*

$$V^+_a := \{v \in V : \ N_\mu(a)(v) > 0\},$$

$$E^+_a := \{e \in E : \ S_\mu(a)(v, e) > 0 \textit{ for all } v \in e\}.$$

Then (V^+_a, E^+_a) *is a crisp sub-n-Super-hypergraph of* (V, E).

Proof. Clearly $V^+_a \subseteq V$ and $E^+_a \subseteq E$. If $e \in E^+_a$ and $v \in e$, then

$$0 < S_\mu(a)(v, e) \le N_\mu(a)(v),$$

hence $N_\mu(a)(v) > 0$, so $v \in V^+_a$. Therefore $e \subseteq V^+_a$, and thus (V^+_a, E^+_a) is a crisp sub-n-Super-hypergraph.

4. Conclusion and Future Works

This paper introduced the concept of the *Intuitionistic Fuzzy Soft n-Super-hypergraph*, a new structure that integrates intuitionistic fuzzy Super-hypergraphs and soft Super-hypergraphs into a unified n-th order framework. The proposed model is intended to provide a more expressive mathematical tool for representing hierarchical, parameter-dependent, and uncertainty-aware relations in complex systems. In particular, it may offer a useful foundation for studying real-world networked problems in areas such as decision-making, economic evaluation, supply-chain analysis, and operations research, where both multi-level structure and uncertain information naturally arise.

In future research, we plan to investigate extensions involving bidirected graphs [56], pseudographs [57], meta-superhypergraphs[58], recursive superhypergraphs[59], and multidirected graphs [60], in order to broaden the modeling scope of the proposed framework. We also intend to explore the development of programming tools and computational implementations that support the construction, storage, and analysis of intuitionistic fuzzy soft n-Super-hypergraphs, as well as practical applications to large-scale decision-support problems. Furthermore, the basic concepts and operations introduced in this paper, such as union, intersection, and Cartesian product, deserve deeper investigation from both mathematical and computational perspectives. In particular, we expect future work to address their algebraic properties, structural invariants, and computational complexity, especially in view of the rapid growth of n-th level powerset-based structures.

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Conflict of Interest

The authors affirm that there are no conflicts of interest associated with this manuscript.

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Author Contributions

Takaaki Fujita and Florentin Smarandache formulated the research questions. Takaaki Fujita carried out the theoretical investigation and drafted the initial manuscript. Both authors reviewed, refined, and approved the final version.

Data Availability

As this paper presents purely theoretical developments, no datasets were generated or analyzed. We welcome future empirical studies to test and build upon our theoretical framework.

Scope and Limitations

The concepts introduced here remain to be validated empirically. Although we have endeavored to ensure accuracy and proper attribution, errors may persist. Readers are encouraged to consult the cited literature and share any corrections. Our results hold under the specific assumptions stated; extending them beyond this scope is a subject for future research. The opinions and conclusions presented are those of the authors and do not necessarily reflect the views of their institutions.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

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