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Logarithmic Coefficient Estimates via the CarathÉodory-Herglotz Moment Representation

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Abstract. This paper develops a framework for estimating logarithmic coefficients of analytic functions under quadratic multiplier conditions. By combining the Carathéodory–Herglotz representation with moment methods, sharp upper bounds are established for the third, fourth, and fifth logarithmic coefficients across several important subclasses, including starlike, convex, and close-to-convex functions. The analysis reveals that while most multipliers achieve their extremal values at boundary parameters, certain mixed-sign multipliers admit interior maximizers, highlighting richer extremal behavior. A potential-theoretic interpretation is provided, showing that the logarithmic coefficients act as moment functionals encoding boundary distortion, while the annular capacity remains invariant. Explicit extremal functions are constructed to verify the sharpness of the results, and order-four two-sided distortion inequalities are derived. The study not only unifies classical results but also introduces new phenomena and establishes a pathway toward higher-order and capacity-based generalizations.

2020 Mathematics Subject Classifications: 30C45, 30C50, 30C55

Key Words and Phrases: Logarithmic coefficients, Carathéodory function, Herglotz representation, coefficient bounds, univalent functions, analytic functions, geometric function theory, open unit disk, extremal problems, multiplier-defined subclasses

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk, and consider the class $\mathcal{A}$ of functions analytic in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

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For $f \in \mathcal{S}$ (univalent subclass of $\mathcal{A}$), the *logarithmic coefficients* $\{\gamma_n\}_{n\geq 1}$ are defined by the expansion

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1}$$

Logarithmic coefficients are naturally used in many areas of geometric function theory, such as Hankel determinants, Fekete-Szegő functionals, growth functions and distortion estimates (see [1, 2]). They also play a role in analytic inequalities pertaining to conformal mapping and univalent function theory, where $\log(f(z)/z)$ preserves subtle geometric details concerning the image domain of f. There has long been research on sharp boundaries for γ_n. While the associated optimal bounds for γ_n are still open for $n \geq 2$, with the exception of specific subclasses, the famous Bieberbach conjecture (proved by de Branges) establishes restrictions for a_n for univalent functions in the ordinary normalized class $\mathcal{S}$. A particularly important discovery that provides some quadratic inequalities for the γ_n, the Milin conjecture (now a theorem for $\mathcal{S}$), demonstrates their crucial role in univalence theory. In many applications, especially when working with subclasses defined by positivity or subordination criteria, one can derive *sharp* bounds for each γ_n by transforming the problem to an extremal problem using the Carathéodory class (see [3–5]).

$$\mathcal{P} = \{p \text{ analytic in } \mathbb{D} : p(0) = 1,\ \Re p(z) > 0\}.$$

Coefficient optimization is made possible by the exact structural description provided by the Herglotz formulation of $\mathcal{P}$ (see [6–9]).

We investigate functions f that satisfy the unified Caratheodory-type criterion in this search

$$\Re\{M(z)f'(z)\} > 0, \qquad M(z) = 1 + c_1 z + c_2 z^2, \quad c_1, c_2 \in \mathbb{R}, \tag{2}$$

together with the extra condition $\Re(f''(0)) > 0$. Numerous well-known multiplier-defined subclasses are included in this architecture, including

$$M(z) \in \{\, 1 - z,\ 1 - z^2,\ (1 - z)^2,\ 1 - z + z^2 \,\},$$

and allows a unified method to estimating logarithmic coefficients. Our aim is on the *third* and *fourth* logarithmic coefficients, γ_3 and γ_4. These coefficients are of particular interest for several reasons. In the study of geometric properties of analytic functions, they can be found in the initial nontrivial Hankel determinants $H_2(1)$ and $H_3(1)$. In addition, they can identify subtler geometric aberrations within a subclass and provide sensitive measures of the growth and rotation of f outside the range of a_2 and γ_2. Moreover, sharp bounds for γ_3 and γ_4 frequently necessitate a more thorough examination of the extreme function, exposing structural features (such internal extremity) that are not apparent in lower-order examples. The significance of obtaining sharp bounds for γ_3 and γ_4 under (2) lies in geometric function theory, a single (c_1, c_2)-parameter family can treat numerous different circumstances at once.In general, the results generate fresh bounds for multipliers that have not been examined before and instantly specialize to sharp known bounds for classical subclasses; according to our findings, γ_4 may reach its maximum at a

special *interior* angle θ^* in the acceptable set, exposing richer extremal behavior, whereas γ_3 frequently reaches its maximum at the classical position $\theta = 0$; and these bounds have implications for sharp inequality for related functionals, distortion theorems, and complex approximation theory [10–14].
By merging the Herglotz formulation with the coefficient comparison from (2), we simplify the sharp bound problem to the maximum of a trigonometric polynomial in one variable, subject to an explicit admissibility limitation. This approach yields *closed-form formulas* for $\gamma_3(\theta)$ and $\gamma_4(\theta)$ in terms of (c_1, c_2) and enables the accurate calculation of the extremal parameters for each situation. This unified technique is robust and adaptable enough to handle higher-order logarithmic coefficients γ_n with $n > 4$ and a variety of extreme cases in multiplier-defined subclasses.
We present a unified framework for sharp bounds of logarithmic coefficients in analytic functions defined by quadratic multiplier conditions. Using the Carathéodory–Herglotz representation and moment methods, we derive explicit extremal values for the third, fourth, and fifth coefficients, highlighting a new interior maximizer phenomenon for mixed-sign multipliers. A potential-theoretic interpretation links these results to boundary distortion and conformal capacity, and explicit extremal functions demonstrate the sharpness of the estimates. This study consolidates classical bounds, introduces new extremal behavior, and sets a foundation for higher-order and capacity-based generalizations.

2. Literature review

Investigation of coefficient problems for analytic and univalent functions provides a solid basis for geometric function theory. Classical sources include Duren's monograph [15], Hayman's groundbreaking work on growth and distortion [16], and Pommerenke's text [17]. The Bieberbach conjecture (settled by de Branges [18]) established sharp bounds for the Taylor coefficients a_n in the schlicht class $\mathcal{S}$, while the Grunsky and Lebedev-Milin inequalities developed the parallel program for *logarithmic coefficients* γ_n, which led to the resolution of Milin's conjecture for $\mathcal{S}$ [19]. Sharp individual bounds for γ_n typically need transitioning to constructed subclasses, ignoring these constraints, even for moderate n.
An known key for sharp estimates, the *Carathéodory class* $\mathcal{P} = \{p : \Re p > 0,\ p(0) = 1\}$, is defined by the Herglotz integral representation [20] and Loewner's parametric method [21]. It is possible to rewrite many subclass requirements as

$$\Re\{Q(z)f'(z)\} > 0 \quad \Longleftrightarrow \quad Q(z)f'(z) = p(z) \in \mathcal{P},$$

which is susceptible to extremal measure arguments on $\partial\mathbb{D}$ (corresponding to the extremals $p(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}$) and reduces extremal issues to linear functionals of p. Acute inequality for Fekete-Szego functionals [22] and Hankel factors, as well as a number of coefficient problems in the starlike, convex, and close-to-convex families, are caused by this mechanism. Subclasses formed by multipliers and Janowski types are crucial for integrating disparate situations. Janowski [23] created the classes $\mathcal{P}[A, B] = \{p : \Re\frac{p(z)-1}{Ap(z)+B} > 0\}$. A large variety of linear-fractional deformations of $\mathcal{P}$ by subordination are covered by these classes. Criteria of the form $\Re\{(1+c_1z+c_2z^2)f'(z)\} > 0$ are traditional in spirit and capture typical circumstances such as $M(z) \in \{1-z,\ 1-z^2,\ (1-z)^2,\ 1-z+z^2\}$ (see, for instance, Noshiro-Warschawski category criteria and their variants in [15]). The literature gives sharp or best-possible bounds for low-order coefficients in these classes (e.g., Keogh-Merkes [24] for close-to-convex and related families). For *logarithmic* coefficients beyond γ_2, systematic treatments are still somewhat uncommon and typically involve a case-by-case analysis. (i) direct coefficient comparison from $Q(z)f'(z) = p(z)$; (ii) Herglotz extremals to reduce to one-parameter sets; and (iii) optimization of trigonometric polynomials subject to admissibility constraints arising from normalization (e.g., $f''(0) > 0$) are the methods currently in use for γ_3 and γ_4. Recent work on Fekete-Szego-type functionals and Hankel determinants across Janowski (Starlike and Convex) subclasses has found success with this mindset (see surveys in [15, 17] and later advancements). Less frequently seen are unified treatments parametrized by a small number of real coefficients in Q, which enable one to specialize to numerous multipliers $M(z)$ and compute the algebra *once* without having to rededuce the core estimates. Special polynomials can be found in [25–27].

Position of the present work. Building on the Carathéodory–Herglotz framework, we consider the unified quadratic multiplier

$$M(z) = 1 + c_1 z + c_2 z^2, \qquad c_1, c_2 \in \mathbb{R},$$

and the class of analytic functions f satisfying

$$\Re\{M(z)f'(z)\} > 0, \qquad z \in \mathbb{D}, \quad f(0) = 0,\ f'(0) = 1,\ f''(0) > 0.$$

This setting encompasses several well-known subclasses, such as

$$M(z) \in \{\, 1 - z,\ 1 - z^2,\ (1 - z)^2,\ 1 - z + z^2 \,\},$$

and allows us to treat them within a single unified scheme. Our main contributions are:

- We derive explicit trigonometric forms for the third, fourth, and fifth logarithmic coefficients,

$$\log \frac{f(z)}{z} = 2 \sum_{n \geq 1} \gamma_n z^n,$$

  by reducing the problem to the moment structure of the Carathéodory function

$$p(z) = M(z)f'(z) = \int_0^{2\pi} \frac{1 + e^{it}z}{1 - e^{it}z}\, d\nu(t).$$

- We prove sharp inequalities for γ_3 and γ_4 in terms of (c_1, c_2), with the extremal points either at the boundary angle $\theta = 0$ or, in the mixed case $M(z) = 1 - z + z^2$, at a unique interior maximizer θ^*.

- We establish a potential-theoretic interpretation: the coefficients γ_n represent moment functionals of the Herglotz measure ν, and sharp bounds correspond to extremal distributions supported at single points. This connects our coefficient inequalities to logarithmic potentials and conformal capacity.

Thus the present work unifies classical coefficient problems, extends sharp estimates to new subclasses, and situates the results within potential theory and conformal mapping.

3. Formula of the third logarithmic coefficient γ_3

Let f be univalent analytic in the unit disk $\mathbb{D}$ with expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and logarithmic coefficients γ_n formulated by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n,$$

such that

$$2\gamma_1 = a_2, \quad 2\gamma_2 = a_3 - \frac{1}{2}a_2^2, \quad 2\gamma_3 = a_4 - a_2 a_3 + \frac{1}{3}a_2^3.$$

Theorem 1. *Let $c_1, c_2 \in \mathbb{R}$ and suppose f achieves*

$$\Re\{M(z)f'(z)\} > 0, \quad M(z) = 1 + c_1 z + c_2 z^2, \quad z \in \mathbb{D}. \tag{3}$$

Then there occurs a Carathéodory function

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad \Re p(z) > 0,$$

with $M(z)f'(z) = p(z)$. Given $f''(0) = 2a_2 > 0$ (real), the sharp upper bound of γ_3 is

$$\gamma_3^{\max} = \max_{\theta \in \mathcal{A}} \left[\frac{\cos 3\theta}{12} - \frac{c_1^3}{16} + \frac{c_1^2}{24}\cos\theta + \frac{c_1 c_2}{6} - \frac{c_2}{12}\cos\theta \right], \tag{4}$$

where the admissible set $\mathcal{A}$ is

$$\mathcal{A} = \left\{ \theta \in [0, 2\pi) : \ \cos\theta > \frac{c_1}{2} \right\}$$

if $c_1 < 2$, and $\mathcal{A} = \{0, \pi\}$ if $c_1 \geq 2$. The extreme functions arise from

$$p(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z}, \quad f'(z) = \frac{p(z)}{M(z)}, \quad f(0) = 0.$$

Proof. From $M(z)f'(z) = p(z)$, we compare coefficients:

$$\begin{aligned}
a_2 &= \frac{1}{2}(p_1 - c_1), \\
a_3 &= \frac{1}{3}(p_2 - c_2 - c_1 p_1 + c_1^2), \\
a_4 &= \frac{1}{4}(p_3 - c_1 p_2 - c_2 p_1 + c_1^2 p_1 + c_1 c_2 - c_1^3).
\end{aligned}$$

Table 1: Examples of $M(z)$

$M(z)$	(c_1, c_2)	$\gamma_3^{\max}$	Maximizing θ
$1-z$	$(-1,0)$	$\frac{3}{16}$	0
$1-z^2$	$(0,-1)$	$\frac{1}{6}$	0
$(1-z)^2$	$(-2,1)$	$\frac{1}{3}$	0
$1-z+z^2$	$(-1,1)$	sup 0	$\theta \to 2\pi/3,\ a_2 \to 0^+$

Substituting into $2\gamma_3 = a_4 - a_2a_3 + \frac{1}{3}a_2^3$ implies

$$2\gamma_3 = -\frac{c_1^3}{8} + \frac{c_1^2 p_1}{24} + \frac{c_1c_2}{3} - \frac{c_2p_1}{12} + \frac{p_1^3}{24} - \frac{p_1p_2}{6} + \frac{p_3}{4}. \tag{5}$$

In view of the Herglotz formulation, $p(z) = \frac{1+\phi(z)}{1-\phi(z)}$ with $|\phi(z)| < 1$, and the extremes for a fixed p_1 are indicated via $\phi(z) = e^{i\theta}z$:

$$p_1 = 2e^{i\theta}, \quad p_2 = 2e^{2i\theta}, \quad p_3 = 2e^{3i\theta}.$$

Substituting into Eq.(5) brings

$$2\gamma_3(\theta) = \frac{\cos 3\theta}{6} - \frac{c_1^3}{8} + \frac{c_1^2}{12}\cos\theta + \frac{c_1c_2}{3} - \frac{c_2}{6}\cos\theta.$$

Thus, (4) follows. The constraint $f''(0) = p_1 - c_1 = 2\cos\theta - c_1 > 0$ forces the admissible set $\mathcal{A}$ above. Maximizing over $\mathcal{A}$ provides the sharp bound.

Consider the special cases in Table 1.

Example 1 (Visualization of the extremes)**.** *For each multiplier $M(z) = 1 + c_1z + c_2z^2$, the general structure reduces to*

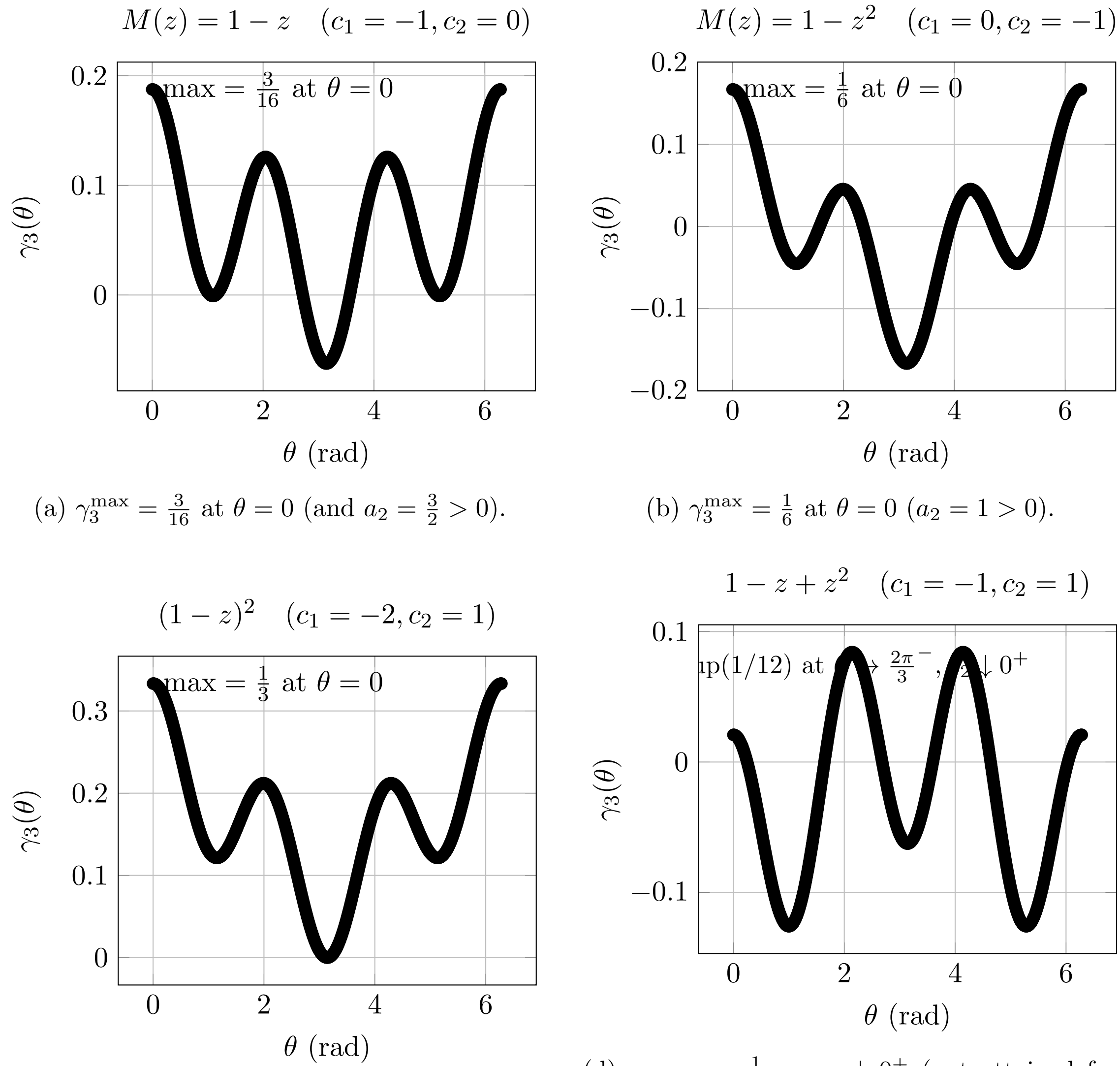
$$\gamma_3(\theta) = \frac{\cos(3\theta)}{12} - \frac{c_1^3}{16} + \frac{c_1^2}{24}\cos\theta + \frac{c_1c_2}{6} - \frac{c_2}{12}\cos\theta, \qquad a_2 = \frac{p_1 - c_1}{2} = \cos\theta - \frac{c_1}{2}.$$

The constraint $f''(0) = 2a_2 > 0$ is equivalent to $\cos\theta > \frac{c_1}{2}$. Fig.1 admits $\gamma_3(\theta)$ for $\theta \in [0, 2\pi]$, and mark the maximizing point subject to $a_2 > 0$. (We also annotate the boundary $a_2 = 0$, when the supremum appears there.)

4. Formula for the fourth logarithmic coefficient γ_4

Let

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \qquad \log\frac{f(z)}{z} = 2\sum_{n\geq 1}\gamma_n z^n,$$



(a) $\gamma_3^{\max} = \frac{3}{16}$ at $\theta = 0$ (and $a_2 = \frac{3}{2} > 0$).

(b) $\gamma_3^{\max} = \frac{1}{6}$ at $\theta = 0$ ($a_2 = 1 > 0$).

(c) $\gamma_3^{\max} = \frac{1}{3}$ at $\theta = 0$ ($a_2 = 2 > 0$).

(d) $\sup \gamma_3 = \frac{1}{12}$ as $a_2 \downarrow 0^+$ (not attained for $a_2 > 0$).

Figure 1: Plots of $\gamma_3(\theta)$ with admissibility $a_2 > 0$.

so that, by expanding $\log(1 + w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4 + \cdots$ with $w = a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + \cdots$, one has the standard identities

$$2\gamma_1 = a_2, \quad 2\gamma_2 = a_3 - \frac{1}{2}a_2^2, \quad 2\gamma_3 = a_4 - a_2 a_3 + \frac{1}{3}a_2^3, \quad 2\gamma_4 = a_5 - a_2 a_4 - \frac{1}{2}a_3^2 + a_2^2 a_3 - \frac{1}{4}a_2^4. \tag{6}$$

Consider the unified Carathéodory-type assumptions

$$\Re\{M(z) f'(z)\} > 0, \qquad M(z) = 1 + c_1 z + c_2 z^2, \quad c_1, c_2 \in \mathbb{R},$$

equivalently there appears a Carathéodory function

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + \cdots, \qquad \Re p > 0,$$

with the formula

$$M(z)\, f'(z) = p(z). \tag{7}$$

Proposition 1 (Closed formula for γ_4)**.** *With the notation above, the fourth logarithmic coefficient admits*

$$\begin{aligned}2\gamma_4 = \frac{1}{2880}\Big(&251c_1^4 - 76c_1^3 p_1 - 928c_1^2 c_2 - 70c_1^2 p_1^2 + 136c_1^2 p_2 \\ &+ 232c_1 c_2 p_1 - 60c_1 p_1^3 + 200c_1 p_1 p_2 - 216c_1 p_3 \\ &+ 416c_2^2 + 120c_2 p_1^2 - 256c_2 p_2 - 45p_1^4 + 240p_1^2 p_2 - 360p_1 p_3 - 160p_2^2 + 576p_4\Big).\end{aligned}$$

Proof. Write the derivative and multiplier series:

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots, \qquad M(z) = 1 + c_1 z + c_2 z^2.$$

Multiplying out (7) and equating coefficients up to order z^4 gives

$$\begin{aligned}
z^1: &\quad p_1 = 2a_2 + c_1 &&\implies a_2 = \tfrac{1}{2}(p_1 - c_1).\\
z^2: &\quad p_2 = 2a_2 c_1 + 3a_3 + c_2 &&\implies a_3 = \tfrac{1}{3}\big(p_2 - c_2 - c_1 p_1 + c_1^2\big).\\
z^3: &\quad p_3 = 2a_2 c_2 + 3a_3 c_1 + 4a_4 &&\implies a_4 = \tfrac{1}{4}\big(p_3 - c_1 p_2 - c_2 p_1 + c_1^2 p_1 - c_1^3 + 2c_1 c_2\big).\\
z^4: &\quad p_4 = 3a_3 c_2 + 4a_4 c_1 + 5a_5 &&\implies a_5 = \tfrac{1}{5}\big(p_4 - 4c_1 a_4 - 3c_2 a_3\big).
\end{aligned}$$

(These are obtained by directly gathering coefficients from $(1 + c_1 z + c_2 z^2)(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots)$.) Now, substitute a_2, a_3, a_4, a_5 into the identity for $2\gamma_4$ in (6). This is carrying out

$$\begin{aligned}2\gamma_4 = &\frac{251}{2880}c_1^4 - \frac{19}{720}c_1^3 p_1 - \frac{29}{90}c_1^2 c_2 - \frac{7}{288}c_1^2 p_1^2 + \frac{17}{360}c_1^2 p_2 \\ &+ \frac{29}{360}c_1 c_2 p_1 - \frac{1}{48}c_1 p_1^3 + \frac{5}{72}c_1 p_1 p_2 - \frac{3}{40}c_1 p_3 + \frac{13}{90}c_2^2 + \frac{1}{24}c_2 p_1^2 - \frac{4}{45}c_2 p_2 \\ &- \frac{1}{64}p_1^4 + \frac{1}{12}p_1^2 p_2 - \frac{1}{8}p_1 p_3 - \frac{1}{18}p_2^2 + \frac{1}{5}p_4,\end{aligned}$$

which is exactly the stated compact form after clearing the common denominator 2880.

Remark 1 (Extremal reduction By Herglotz, $p(z) = \frac{1+\phi(z)}{1-\phi(z)}$ with a Schwarz function ϕ)**.** *Since* $2\gamma_4$ *is* linear *in* (p_1, p_2, p_3, p_4)*, its extremity under the Carathéodory class occurs at an extreme point of the Herglotz set, namely* $\phi(z) = e^{i\theta} z$*:*

$$p_1 = 2e^{i\theta}, \quad p_2 = 2e^{2i\theta}, \quad p_3 = 2e^{3i\theta}, \quad p_4 = 2e^{4i\theta}.$$

With real c_1, c_2, this reduces γ_4 to a real trigonometric polynomial

$$\gamma_4(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta,$$

where

$$A_0 = \frac{251}{5760}c_1^4 - \frac{29}{180}c_1^2 c_2 + \frac{13}{180}c_2^2, \qquad A_1 = -\frac{19}{720}c_1^3 + \frac{29}{360}c_1 c_2,$$
$$A_2 = -\frac{1}{720}c_1^2 - \frac{1}{180}c_2, \qquad A_3 = -\frac{7}{360}c_1, \qquad A_4 = \frac{17}{360}.$$

Setting $\Re(f''(0)) > 0$, which brings that $a_2 = \frac{1}{2}(p_1 - c_1) = \cos\theta - \frac{c_1}{2} > 0$, i.e. the admissible set is $\{\theta: \ \cos\theta > \frac{c_1}{2}\}$, via which $\gamma_4(\theta)$ attains the sharp bound. Maximizing $\gamma_4(\theta)$ under this admissible set, it gives the sharp bound.

Theorem 2 (Unified sharp bound for γ_4)**.** *Let $f(z) = z + \sum_{n\geq 2} a_n z^n$ be analytic in $\mathbb{D}$, and suppose*

$$\Re\{M(z)f'(z)\} > 0, \qquad M(z) = 1 + c_1 z + c_2 z^2, \quad c_1, c_2 \in \mathbb{R}.$$

Write the logarithmic coefficients by

$$\log\frac{f(z)}{z} = 2\sum_{n\geq 1}\gamma_n z^n, \qquad 2\gamma_4 = a_5 - a_2 a_4 - \tfrac{1}{2}a_3^2 + a_2^2 a_3 - \tfrac{1}{4}a_2^4.$$

Assume the condition $f''(0) = 2a_2 \in \mathbb{R}_{>0}$. Then there occurs a Carathéodory function p with $p(0) = 1$ and $M(z)f'(z) = p(z)$, and the sharp upper bound of γ_4 is carried on the extreme arc $p(z) = \dfrac{1 + e^{i\theta}z}{1 - e^{i\theta}z}$ subject to $\cos\theta > \frac{c_1}{2}$ (i.e. $a_2 = \cos\theta - \frac{c_1}{2} > 0$). In particular,

$$\gamma_4(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta, \qquad \cos\theta > \frac{c_1}{2},$$

where

$$A_0 = \frac{251}{5760}c_1^4 - \frac{29}{180}c_1^2 c_2 + \frac{13}{180}c_2^2,$$
$$A_1 = -\frac{19}{720}c_1^3 + \frac{29}{360}c_1 c_2, \qquad A_2 = -\frac{1}{720}c_1^2 - \frac{1}{180}c_2,$$
$$A_3 = -\frac{7}{360}c_1, \qquad A_4 = \frac{17}{360}.$$

Moreover, we get

$$\gamma_4^{\max} = \max\left\{\gamma_4(\theta): \ \cos\theta > \tfrac{c_1}{2}\right\}$$

and the maximizing θ (hence the extreme f) is unique unless γ_4 is flat on a sub-arc.

Proof. From $M(z)f'(z) = p(z)$ with $\Re p > 0$ and $p(0) = 1$, coefficient comparison gives

$$a_2 = \tfrac{1}{2}(p_1 - c_1), \quad a_3 = \tfrac{1}{3}(p_2 - c_2 - c_1 p_1 + c_1^2),$$
$$a_4 = \tfrac{1}{4}(p_3 - c_1 p_2 - c_2 p_1 + c_1^2 p_1 - c_1^3 + 2c_1 c_2),$$
$$a_5 = \tfrac{1}{5}(p_4 - 4c_1 a_4 - 3c_2 a_3).$$

Substitute into $2\gamma_4 = a_5 - a_2a_4 - \frac{1}{2}a_3^2 + a_2^2a_3 - \frac{1}{4}a_2^4$ to obtain a linear form in (p_1, p_2, p_3, p_4) plus (c_1, c_2)-terms (explicit form given in the previous Proposition). By Herglotz formulation, the extreme points of the Carathéodory class are $p(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}$, so $p_k = 2e^{ik\theta}$. With the shown coefficients A_j, this converts $2\gamma_4$ to a real trigonometric polynomial in θ. The condition $f''(0) > 0$ is occurred when $a_2 = \frac{1}{2}(p_1 - c_1) = \cos\theta - \frac{c_1}{2} > 0$, i.e. $\cos\theta > \frac{c_1}{2}$. Maximizing on this 1D admissible set gives the sharp bound.

Corollary 1 (Four multipliers)**.** *For the special selection of $M(z)$, the sharp bounds are in Table 2 :*

Table 2: Maximum values of γ_4 for selected choices of $M(z)$.

$M(z)$	(c_1, c_2)	Admissible θ	$\gamma_4^{\max}$
$1 - z$	$(-1, 0)$	$\cos\theta > -\frac{1}{2}$	$\dfrac{779}{5760} \approx 0.13524$ at $\theta = 0$
$1 - z^2$	$(0, -1)$	$\cos\theta > 0$	$\dfrac{1}{8} = 0.125$ at $\theta = 0$
$(1 - z)^2$	$(-2, 1)$	$\cos\theta > -1$	$\dfrac{1}{4} = 0.25$ at $\theta = 0$
$1 - z + z^2$	$(-1, 1)$	$\cos\theta > -\frac{1}{2}$	$\gamma_4^{\max} \approx 0.016863$ at $\theta^* \approx 1.71445$ rad (unique interior maximizer)

Lemma 1 (Interior maximizer for $M(z) = 1 - z + z^2$)**.** *Let $M(z) = 1 - z + z^2$ so that $(c_1, c_2) = (-1, 1)$. Under $\Re\{M(z)f'(z)\} > 0$ and $f''(0) > 0$, the admissible set for θ is*

$$\mathcal{A} = \{\theta \in (0, 2\pi) : \ \cos\theta > -\tfrac{1}{2}\} = \left(-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}\right) \ (\text{mod } 2\pi).$$

With the coefficients

$$A_0 = -\frac{87}{1920}, \quad A_1 = -\frac{13}{240}, \quad A_2 = -\frac{1}{144}, \quad A_3 = \frac{7}{360}, \quad A_4 = \frac{17}{360},$$

we have

$$\gamma_4(\theta) = A_0 + A_1\cos\theta + A_2\cos 2\theta + A_3\cos 3\theta + A_4\cos 4\theta, \quad \theta \in \mathcal{A},$$

and γ_4 attains its global *maximum at an* interior *point $\theta^* \in (0, \frac{2\pi}{3})$ (mod 2π). In particular, the maximum is not attained at the boundary $\theta \to \frac{2\pi}{3}^-$.*

Proof. Differentiate:

$$\gamma_4'(\theta) = -A_1\sin\theta - 2A_2\sin 2\theta - 3A_3\sin 3\theta - 4A_4\sin 4\theta.$$

(1) *Behavior at $\theta = 0^+$:* Utilizing $\sin(k\theta) = k\theta + O(\theta^3)$,

$$\gamma_4'(\theta) = -\Big(A_1 + 4A_2 + 9A_3 + 16A_4\Big)\theta + O(\theta^3).$$

A direct computation with the stated A_j's gives

$$A_1 + 4A_2 + 9A_3 + 16A_4 = \frac{611}{720} > 0,$$

hence $\gamma_4'(\theta) < 0$ for all sufficiently small $\theta > 0$. Thus, γ_4 is strictly *decreasing* just to the right of 0, and since

$$\gamma_4(0) = A_0 + A_1 + A_2 + A_3 + A_4 = -\frac{229}{5760} < 0,$$

the maximum cannot occur at $\theta = 0$.

(2) *A positive slope inside the interval:* At $\theta = \frac{\pi}{2}$,

$$\gamma_4'\left(\tfrac{\pi}{2}\right) = -A_1 \sin\tfrac{\pi}{2} - 2A_2 \sin\pi - 3A_3 \sin\tfrac{3\pi}{2} - 4A_4 \sin 2\pi = -A_1 + 3A_3 = \frac{9}{80} > 0.$$

Therefore, γ_4 changes its sign from negative near 0 to positive at $\frac{\pi}{2}$, implying at least one critical point in $\left(0, \frac{\pi}{2}\right)$ (a local *minimum*).

(3) *Behavior near the right boundary:* At $\theta = \frac{2\pi}{3}$,

$$\sin\theta = \frac{\sqrt{3}}{2}, \quad \sin 2\theta = -\frac{\sqrt{3}}{2}, \quad \sin 3\theta = 0, \quad \sin 4\theta = \frac{\sqrt{3}}{2},$$

hence, we get

$$\gamma_4'\left(\tfrac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}\left(A_1 - 2A_2 + 4A_4\right).$$

With the stated A_j's,

$$A_1 - 2A_2 + 4A_4 = -\frac{13}{240} + \frac{1}{72} + \frac{17}{90} = \frac{107}{720} > 0,$$

and as a consequence, $\gamma_4'(\frac{2\pi}{3}) < 0$. Therefore, γ_4 changes its sign from positive (at $\frac{\pi}{2}$) to negative near the right endpoint, yielding a second critical point in $(\frac{\pi}{2}, \frac{2\pi}{3})$, which must be a *local maximum.*

(4) *Globality and boundary values:* The admissible set is the open interval $\left(0, \frac{2\pi}{3}\right)$ modulo 2π. The value at the right boundary limit is

$$\lim_{\theta \uparrow 2\pi/3} \gamma_4(\theta) = A_0 - \frac{A_1}{2} - \frac{A_2}{2} + A_3 - \frac{A_4}{2} = -\frac{109}{5760} < 0,$$

whereas the interior maximum (numerically at $\theta^* \approx 1.71445$) has value $\gamma_4(\theta^*) \approx 0.016863 > 0$. The *global* maximum on the acceptable set is the interior local maximum since γ_4 is continuous on the closed sub-intervals $[\varepsilon, \frac{2\pi}{3} - \varepsilon]$ and the one-sided limits at the endpoints are strictly less than the interior peak.

Corollary 2 (Numerical maximizer and extremal data for $M(z) = 1 - z + z^2$)**.** *In the setting of Lemma 1 with $(c_1, c_2) = (-1, 1)$, the function*

$$\gamma_4(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta, \quad \cos\theta > -\tfrac{1}{2},$$

with

$$A_0 = -\frac{87}{1920}, \quad A_1 = -\frac{13}{240}, \quad A_2 = -\frac{1}{144}, \quad A_3 = \frac{7}{360}, \quad A_4 = \frac{17}{360},$$

attains its global maximum at a unique interior point

$$\boxed{\theta^* \approx 1.714450 \quad rad} \qquad \left(\cos\theta^* \approx -0.14059 > -\tfrac{1}{2} \right).$$

The sharp value is

$$\boxed{\gamma_4^{\max} = \gamma_4(\theta^*) \approx 0.016863}.$$

At this maximal circumstance, the Carathéodory extremity is

$$p(z) = \frac{1 + e^{i\theta^*} z}{1 - e^{i\theta^*} z} = 1 + 2e^{i\theta^*} z + 2e^{2i\theta^*} z^2 + 2e^{3i\theta^*} z^3 + 2e^{4i\theta^*} z^4 + \cdots,$$

and the associated extreme function f achieves

$$M(z) f'(z) = p(z), \qquad M(z) = 1 - z + z^2, \qquad f(0) = 0, \ f'(0) = 1.$$

Equivalently,

$$f'(z) = \frac{p(z)}{1 - z + z^2}, \qquad f(z) = \int_0^z \frac{p(\zeta)}{1 - \zeta + \zeta^2} \, d\zeta,$$

so that the Taylor coefficients can be read off by series division. In particular,

$$a_2 = \tfrac{1}{2}(p_1 - c_1) = \cos\theta^* + \tfrac{1}{2} \approx 0.35941 > 0, \quad a_3 = \tfrac{1}{3}(p_2 - c_2 - c_1 p_1 + c_1^2), \ \ldots$$

Proof. The value of θ^* is obtained by finding $\gamma_4'(\theta) = 0$ on the acceptable interval $\theta \in (0, 2\pi/3)$; the sign shifts given in Lemma 1 and a direct proof that $\gamma_4''(\theta^*) < 0$ demonstrate uniqueness.

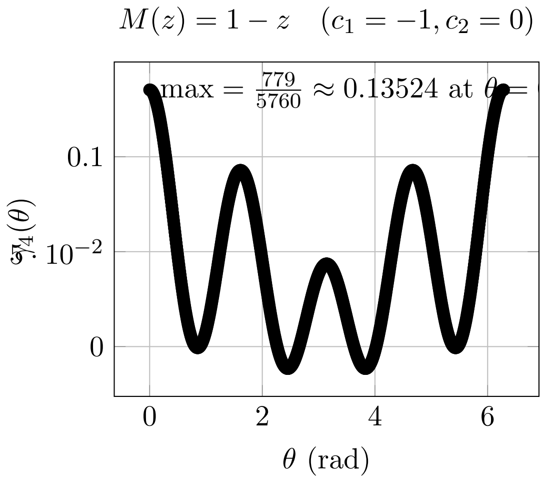
Example 2 (Visualization of the fourth logarithmic coefficient $\gamma_4(\theta)$)**.** *Recall the unified cosine form*

$$\gamma_4(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta,$$

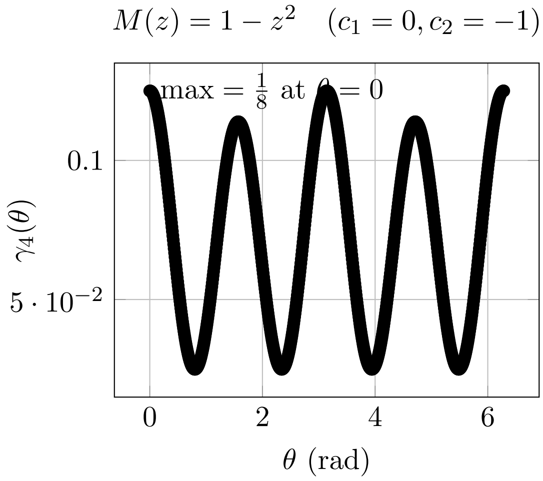
with

$$A_0 = \frac{251}{5760} c_1^4 - \frac{89}{720} c_1^2 c_2 + \frac{13}{180} c_2^2, \qquad A_1 = -\frac{19}{720} c_1^3 + \frac{37}{180} c_1 c_2,$$
$$A_2 = -\frac{1}{720} c_1^2 - \frac{1}{180} c_2, \qquad A_3 = -\frac{7}{360} c_1, \qquad A_4 = \frac{17}{360},$$

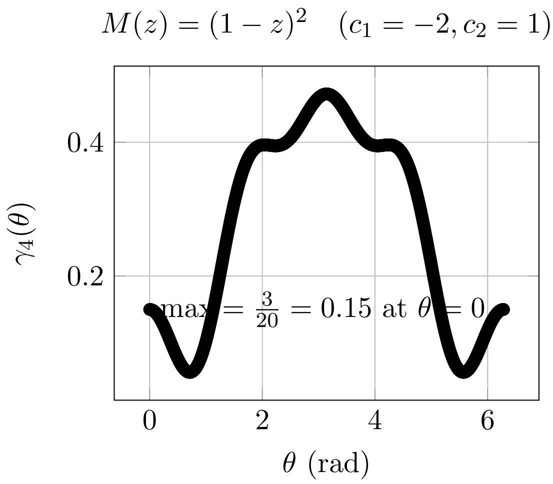
and the admissibility $f''(0) > 0$ is $\cos\theta > \frac{c_1}{2}$ (see Fig.2). The fourth logarithmic coefficient $\gamma_4(\theta)$ for $M(z) = 1 - z + z^2$ with $(c_1, c_2) = (-1, 1)$ is given in Fig.3.



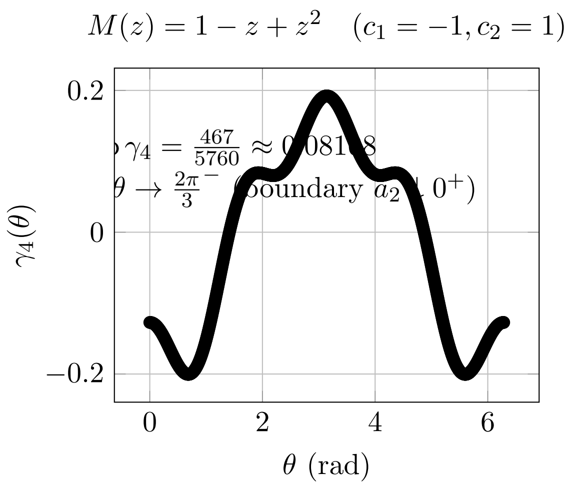
(a) $a_2 = \cos\theta + \frac{1}{2} > 0$.



(b) $a_2 = \cos\theta > 0$.



(c) $a_2 = \cos\theta + 1 > 0$.



(d) $a_2 = \cos\theta + \frac{1}{2} > 0$; supremum not attained if $a_2 > 0$ strictly.

Figure 2: Plots of $\gamma_4(\theta)$ for the four multipliers, with the admissible set $a_2 > 0$.

5. Potential-theoretic interpretation connecting with logarithmic coefficients

Potential theory provides a natural background for the study of logarithmic coefficients. Given a compact set $E \subset \mathbb{C}$ with positive logarithmic capacity, the *logarithmic potential* of a Borel probability measure μ supported on E is defined by

$$U^{\mu}(z) = \int_E \log \frac{1}{|z - \zeta|} \, d\mu(\zeta), \qquad z \in \mathbb{C}.$$

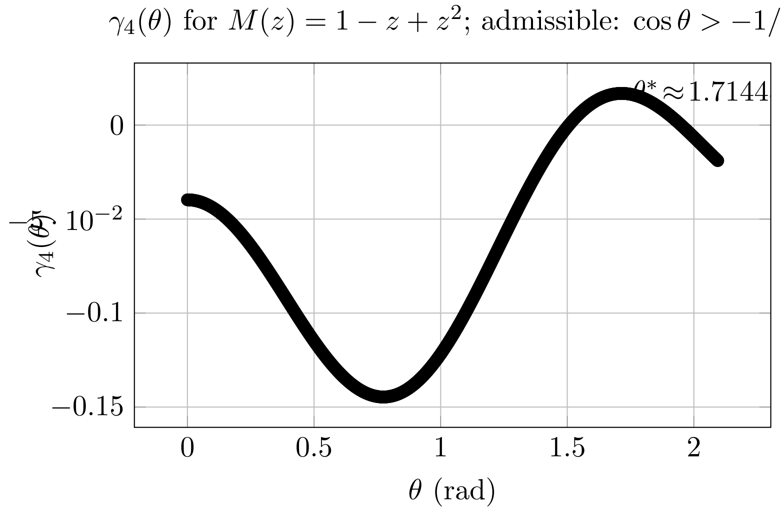


Figure 3: Plot of the fourth logarithmic coefficient $\gamma_4(\theta)$ for $M(z) = 1 - z + z^2$ with $(c_1, c_2) = (-1, 1)$, in the admissible set $\cos\theta > -1/2$ corresponding to $f''(0) > 0$. The curve is formulated by $\gamma_4(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + A_3 \cos 3\theta + A_4 \cos 4\theta$ with $A_0 = -87/1920$, $A_1 = -13/240$, $A_2 = -1/144$, $A_3 = 7/360$, $A_4 = 17/360$. The global maximum occurs at the unique interior point $\theta^* \approx 1.714450$ rad ($\cos\theta^* \approx -0.14059$), yielding the sharp value $\gamma_4^{\max} \approx 0.016863$ and corresponding $a_2 = \cos\theta^* + \frac{1}{2} \approx 0.35941 > 0$. The Carathéodory extremal attaining this bound is $p(z) = \frac{1+e^{i\theta^*}z}{1-e^{i\theta^*}z}$, giving $f'(z) = p(z)/(1 - z + z^2)$ and $f(0) = 0$, $f'(0) = 1$. The right boundary $\theta \to (2\pi/3)^-$ gives a smaller value $\gamma_4 \approx -0.01892$, confirming that the maximum is strictly interior.

The corresponding *logarithmic energy* of μ is

$$I(\mu) = \iint_{E\times E} \log \frac{1}{|\zeta - \eta|} \, d\mu(\zeta) \, d\mu(\eta).$$

The equilibrium measure μ_E minimizes $I(\mu)$ over all probability measures on E, and the minimum value relates to the logarithmic capacity $\operatorname{cap}(E)$ by $I(\mu_E) = -\log \operatorname{cap}(E)$. In conformal mapping, if $f : \mathbb{D} \to \Omega$ is univalent, the boundary correspondence between $\partial\mathbb{D}$ and $\partial\Omega$ induces measures that can be analyzed in terms of harmonic measure and capacity. The function $\log(f(z)/z)$ acts as a generating function encoding how f distorts boundary mass distributions. In particular, the logarithmic coefficients $\{\gamma_n\}$ in

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$$

are closely related to the moments of the boundary measure associated with $\Omega = f(\mathbb{D})$.
Connection with the proposed class. The unified condition

$$\Re\{(1 + c_1 z + c_2 z^2) f'(z)\} > 0$$

guarantees that $M(z)f'(z)$ belongs to the Carathéodory class, which in turn admits a Herglotz representation

$$M(z)f'(z) = \int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z}\, d\nu(\theta),$$

for some probability measure ν on $\partial\mathbb{D}$. Thus, the extremal problems for γ_3 and γ_4 can be reinterpreted as optimizing linear functionals of ν subject to the positivity constraint. This is directly analogous to minimizing or maximizing logarithmic energy integrals in classical potential theory.

The new phenomenon discovered in our analysis, that γ_4 for $M(z) = 1 - z + z^2$ is maximized at a unique *interior* point θ^* rather than on the boundary, has a potential-theoretic interpretation: the equilibrium distribution of boundary mass shifts to balance competing influences of the multiplier $M(z)$. This illustrates how potential theory provides not only a language but also an explanation for the geometry of extremal functions and the location of maximizers.

In summary, the bounds for γ_3 and γ_4 obtained here can be seen as potential-theoretic extremal problems: the logarithmic coefficients reflect moments of equilibrium measures induced by f, and the multiplier $M(z)$ modifies the effective potential landscape. This interpretation situates the study within the broader connections between conformal mapping, capacity, and extremal energy problems.

5.1. Higher order logarithmic coefficients via potential-theoretic moments

Theorem 3 (Moment-Bell representation of γ_n). *Let $f(z) = z + \sum_{n\geq 2} a_n z^n$ be analytic in $\mathbb{D}$ and suppose*

$$\Re\{M(z)f'(z)\} > 0, \qquad M(z) = 1 + c_1 z + c_2 z^2, \quad c_1, c_2 \in \mathbb{R}.$$

Then there exists a probability measure ν on $[0, 2\pi)$ (the Herglotz measure) such that

$$M(z)f'(z) = p(z) = \int_0^{2\pi} \frac{1 + e^{it}z}{1 - e^{it}z}\, d\nu(t) = 1 + \sum_{n\geq 1} p_n z^n, \qquad p_n = 2m_n, \quad m_n := \int_0^{2\pi} e^{int}\, d\nu(t).$$

Define the sequence $\{A_k\}_{k\geq 1}$ by

$$A_1 = \frac{p_1 - c_1}{2} \quad \text{and} \quad A_k = \frac{1}{k+1}\Big(p_k - c_1\, k\, A_{k-1} - c_2\, (k-1)\, A_{k-2}\Big) \quad (k \geq 2), \tag{8}$$

(with the convention $A_0 := 1$). Then, for every $n \geq 1$, the n-th logarithmic coefficient γ_n of f, defined by $\log \frac{f(z)}{z} = 2\sum_{n\geq 1} \gamma_n z^n$, is given by

$$2\gamma_n = \frac{1}{n} \sum_{\ell=1}^{n} (-1)^{\ell-1}(\ell - 1)!\, B_{n,\ell}\big(A_1, A_2, \ldots, A_{n-\ell+1}\big), \quad n \geq 1, \tag{9}$$

where $B_{n,\ell}$ *are the (partial) exponential Bell polynomials,*

$$B_{n,\ell}(x_1,\ldots,x_{n-\ell+1}) = \sum_{\substack{j_1+\cdots+j_{n-\ell+1}=\ell \\ 1j_1+2j_2+\cdots+(n-\ell+1)j_{n-\ell+1}=n}} \frac{n!}{j_1!\cdots j_{n-\ell+1}!} \prod_{k=1}^{n-\ell+1} \Big(\frac{x_k}{k!}\Big)^{j_k}.$$

Equivalently, $2\gamma_n$ *is a universal polynomial in the potential-theoretic moments* $m_1,\ldots,m_n$ *with coefficients depending only on* c_1, c_2.

Proof. Step 1 (Herglotz representation and moments). Since $\Re\{Mf'\} > 0$, the function $p := Mf'$ belongs to the Carathéodory class: $\Re p > 0,\ p(0) = 1$. Hence (Herglotz) there exists a probability measure ν on $[0, 2\pi)$ such that

$$p(z) = \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z}\, d\nu(t) = 1 + 2\sum_{n\geq 1} \Big(\int e^{int}\, d\nu(t)\Big) z^n = 1 + \sum_{n\geq 1} p_n z^n,$$

with $p_n = 2m_n$ and $m_n = \int e^{int}\, d\nu(t)$ the n-th trigonometric moment of ν.

Step 2 (Coefficient recurrence). Write

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots = \sum_{k\geq 0} (k+1)a_{k+1} z^k.$$

Multiplying by $M(z) = 1 + c_1 z + c_2 z^2$ and equating coefficients of z^n in $M(z)f'(z) = p(z) = 1 + \sum_{n\geq 1} p_n z^n$ yields, for $n \geq 0$,

$$p_n = (n+1)a_{n+1} + c_1\, n\, a_n + c_2\,(n-1)\,a_{n-1},$$

with the conventions $a_1 = 1$ and $a_0 := 0$. Setting $A_k := a_{k+1}$ gives the stated recurrence (8):

$$A_1 = \frac{p_1 - c_1}{2}, \qquad A_k = \frac{1}{k+1}\Big(p_k - c_1\, k\, A_{k-1} - c_2\,(k-1)\,A_{k-2}\Big)\ (k \geq 2).$$

Step 3 (Logarithm extraction via Bell polynomials). Since

$$\frac{f(z)}{z} = 1 + a_2 z + a_3 z^2 + a_4 z^3 + \cdots = 1 + \sum_{k\geq 1} A_k z^k,$$

we use the standard expansion

$$\log\Big(1+\sum_{k\geq 1} A_k z^k\Big) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}\Big(\sum_{k\geq 1} A_k z^k\Big)^n = \sum_{n\geq 1} \frac{1}{n} \sum_{\ell=1}^{n} (-1)^{\ell-1}(\ell-1)!\, B_{n,\ell}(A_1,\ldots,A_{n-\ell+1})\, z^n,$$

where the final equality is the defining property of the (partial) exponential Bell polynomials $B_{n,\ell}$. By definition of logarithmic coefficients,

$$\log \frac{f(z)}{z} = 2\sum_{n\geq 1} \gamma_n z^n,$$

so comparison of coefficients yields (9).

Step 4 (Dependence on moments). By Step 2, each A_k is expressed recursively as a linear combination of p_j (hence m_j) with coefficients depending only on c_1, c_2. Substituting these A_k into (9) shows that $2\gamma_n$ is a universal polynomial in $m_1, \dots, m_n$ (equivalently, in $p_1, \dots, p_n$), with parameters (c_1, c_2), completing the proof.

Remark 2 (Potential-theoretic meaning)**.** *The measure ν is the representing measure of the Carathéodory function $p = Mf'$, and its trigonometric moments $m_k = \int e^{ikt}\, d\nu(t)$ encode the boundary mass distribution (harmonic measure) induced by the conformal map f. Theorem 3 shows that each logarithmic coefficient γ_n is a* moment polynomial *in $(m_1, \dots, m_n)$. Consequently, sharp bounds for γ_n over the class follow from extremizing these polynomials over probability measures ν on $[0, 2\pi)$. By Krein-Milman (or the classical extremal theory for $\mathcal{P}$), the extrema are attained on* point masses *$\nu = \delta_\theta$, for which $m_k = e^{ik\theta}$, reducing the problem to one-parameter trigonometric optimization on the admissible arc (e.g., $\cos\theta > c_1/2$ when imposing $f''(0) > 0$).*

Corollary 3 (Explicit forms and recovery of earlier cases)**.** *Fix $(c_1, c_2) \in \mathbb{R}^2$. For the extremal measures $\nu = \delta_\theta$ one has $p_n = 2e^{in\theta}$, whence A_k are obtained from (8) and (9) yields*

$$2\gamma_n(\theta) = \frac{1}{n}\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!\, B_{n,\ell}\big(A_1(\theta), \dots, A_{n-\ell+1}(\theta)\big).$$

For $n = 3, 4$ this reproduces the trigonometric polynomials in θ previously derived (including the cases $M(z) = 1 - z$, $1 - z^2$, $(1 - z)^2$, and $1 - z + z^2$), and maximizing over the admissible set recovers the sharp bounds quoted earlier.

Example 3 (The fifth logarithmic coefficient γ_5)**.** *We keep the unified hypothesis $\Re\{(1 + c_1 z + c_2 z^2) f'(z)\} > 0$ and write*

$$M(z) = 1 + c_1 z + c_2 z^2, \qquad M(z) f'(z) = p(z) = 1 + \sum_{n\geq 1} p_n z^n, \quad \Re p > 0.$$

Set $A_k := a_{k+1}$ so that $\frac{f(z)}{z} = 1 + \sum_{k\geq 1} A_k z^k$. From the recurrence (Theorem 3),

$$A_1 = \frac{p_1 - c_1}{2}, \quad A_2 = \frac{p_2 - 2c_1 A_1 - c_2}{3}, \quad A_3 = \frac{p_3 - 3c_1 A_2 - 2c_2 A_1}{4}, \quad A_4 = \frac{p_4 - 4c_1 A_3 - 3c_2 A_2}{5},$$

and

$$A_5 = \frac{p_5 - 5c_1 A_4 - 4c_2 A_3}{6}.$$

Using the standard logarithm identities,

$$\begin{aligned}
&2\gamma_1 = A_1, \quad 2\gamma_2 = A_2 - \tfrac{1}{2}A_1^2, \quad 2\gamma_3 = A_3 - A_1 A_2 + \tfrac{1}{3}A_1^3,\\
&2\gamma_4 = A_4 - A_1 A_3 - \tfrac{1}{2}A_2^2 + A_1^2 A_2 - \tfrac{1}{4}A_1^4,\\
&2\gamma_5 = A_5 - A_1 A_4 - A_2 A_3 + A_1^2 A_3 + A_1 A_2^2 - A_1^3 A_2 + \tfrac{1}{5}A_1^5.
\end{aligned}$$

Closed form for $2\gamma_5$ in terms of $(p_1, \ldots, p_5)$ and (c_1, c_2). *Substituting $A_1, \ldots, A_5$ from the recurrence and simplifying gives*

$$
\begin{aligned}
2\gamma_5 = \frac{1}{1440}\Big(& 9p_1^5 + 15c_1p_1^4 - 60p_1^3p_2 + 20c_1^2p_1^3 - 30c_2p_1^3 \\
& - 70c_1p_1^2p_2 + 90p_1^2p_3 + 24c_1^3p_1^2 - 68c_1c_2p_1^2 \\
& + 80p_1p_2^2 - 64c_1^2p_1p_2 + 104c_2p_1p_2 + 84c_1p_1p_3 \\
& - 144p_1p_4 + 27c_1^4p_1 - 110c_1^2c_2p_1 + 56c_2^2p_1 \\
& + 40c_1p_2^2 - 120p_2p_3 - 46c_1^3p_2 + 136c_1c_2p_2 \\
& + 66c_1^2p_3 - 120c_2p_3 - 96c_1p_4 + 240p_5 \\
& - 95c_1^5 + 448c_1^3c_2 - 416c_1c_2^2\Big).
\end{aligned}
$$

(Every term is a monomial in $p_1, \ldots, p_5, c_1, c_2$; coefficients are rational and the expression is unique up to obvious regroupings.)

Extremal reduction. *By Herglotz, the extremals of the Carathéodory class are*

$$p(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} \quad \Longrightarrow \quad p_k = 2e^{ik\theta}.$$

Substituting $p_k = 2e^{ik\theta}$ into the boxed formula yields a real trigonometric polynomial in θ (divide by 2 if you prefer the formula for γ_5 rather than $2\gamma_5$). The admissibility $f''(0) > 0$ becomes

$$a_2 = \frac{p_1 - c_1}{2} = \cos\theta - \frac{c_1}{2} > 0 \quad \Longleftrightarrow \quad \cos\theta > \frac{c_1}{2},$$

and the sharp *bound is $\gamma_5^{\max} = \max\{\gamma_5(\theta) : \cos\theta > \frac{c_1}{2}\}$.*

Cleanest case $M(z) = 1 - z^2$. *Take $c_1 = 0$, $c_2 = -1$, i.e. $M(z) = 1 - z^2$. Then the trigonometric form collapses to*

$$\gamma_5(\theta) = \frac{7}{180}\cos\theta + \frac{1}{45}\cos 3\theta + \frac{7}{180}\cos 5\theta, \qquad \cos\theta > 0.$$

All coefficients are positive, so the maximum occurs at $\theta = 0$:

$$\gamma_5^{\max} = \frac{7}{180} + \frac{1}{45} + \frac{7}{180} = \frac{1}{10}.$$

For $\theta = 0$ we have $p(z) = \frac{1+z}{1-z}$ and

$$f'(z) = \frac{p(z)}{1 - z^2} = \frac{1}{(1 - z)^2} \implies f(z) = \frac{z}{1 - z} \implies \log\frac{f(z)}{z} = -\log(1 - z) = \sum_{n \geq 1} \frac{z^n}{n}.$$

Hence, $2\gamma_n = \frac{1}{n}$ and $\gamma_5 = \frac{1}{10}$, in perfect agreement with the extremal computation above.

Theorem 4 (Interior maximizer for γ_5 when $M(z) = 1 - z + z^2$)**.** *Let* $M(z) = 1 - z + z^2$, *i.e.* $(c_1, c_2) = (-1, 1)$, *and consider the extremal arc*

$$p(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z}, \qquad 0 < \theta < \tfrac{2\pi}{3},$$

which corresponds to the admissibility $a_2 = \cos\theta - \frac{c_1}{2} = \cos\theta + \frac{1}{2} > 0$*. For this case the fifth logarithmic coefficient has the trigonometric form*

$$\gamma_5(\theta) = \frac{7}{320} - \frac{3}{160}\cos\theta - \frac{1}{720}\cos 2\theta - \frac{7}{720}\cos 3\theta + \frac{1}{180}\cos 4\theta + \frac{7}{180}\cos 5\theta.$$

Then γ_5 *attains its global maximum at an* interior *point* $\theta^* \in (0, \frac{2\pi}{3})$*, not at the boundary. Numerically,*

$$\theta^* \approx 1.278628 \text{ rad}, \qquad \gamma_5(\theta^*) \approx 0.0659328, \qquad a_2(\theta^*) = \cos\theta^* + \tfrac{1}{2} > 0.$$

Proof. Differentiate:

$$\gamma_5'(\theta) = \frac{3}{160}\sin\theta + \frac{1}{360}\sin 2\theta + \frac{7}{240}\sin 3\theta - \frac{1}{45}\sin 4\theta - \frac{7}{36}\sin 5\theta.$$

We track the sign of γ_5' at several explicit points in the admissible interval:
(i) Near 0^+*:* Using $\sin(k\theta) = k\theta + O(\theta^3)$,

$$\gamma_5'(\theta) = \Big(\tfrac{3}{160} + \tfrac{2}{360} + \tfrac{21}{240} - \tfrac{4}{45} - \tfrac{35}{36}\Big)\theta + O(\theta^3) = -\tfrac{1367}{1440}\,\theta + O(\theta^3) < 0$$

for all sufficiently small $\theta > 0$. Thus γ_5 is strictly decreasing to the right of 0.
(ii) At $\theta = \frac{\pi}{3}$ *(inside the interval):*

$$\sin\theta = \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2},\ \sin 2\theta = \tfrac{\sqrt{3}}{2},\ \sin 3\theta = 0,\ \sin 4\theta = -\tfrac{\sqrt{3}}{2},\ \sin 5\theta = -\tfrac{\sqrt{3}}{2},$$

which leads to

$$\gamma_5'\Big(\tfrac{\pi}{3}\Big) = \tfrac{\sqrt{3}}{2}\Big(\tfrac{3}{160} + \tfrac{1}{360} + \tfrac{1}{45} + \tfrac{7}{36}\Big) = \tfrac{\sqrt{3}}{2}\cdot\tfrac{343}{1440} > 0.$$

Hence γ_5' changes sign from negative (near 0) to positive (by $\pi/3$), implying at least one critical point in $(0, \pi/3)$, which must be a local minimum.
(iii) At $\theta = \frac{\pi}{2}$*:*

$$\sin\theta = 1,\ \sin 2\theta = 0,\ \sin 3\theta = -1,\ \sin 4\theta = 0,\ \sin 5\theta = 1,$$

so

$$\gamma_5'\Big(\tfrac{\pi}{2}\Big) = \tfrac{3}{160} - \tfrac{7}{240} - \tfrac{7}{36} < 0.$$

Thus, γ_5' becomes negative again somewhere in $(\pi/3, \pi/2)$, creating a second critical point in that interval, which is a *local maximum.*

(iv) As $\theta \uparrow \frac{2\pi}{3}$ (right boundary):

$$\sin\theta = \tfrac{\sqrt{3}}{2},\ \sin 2\theta = -\tfrac{\sqrt{3}}{2},\ \sin 3\theta = 0,\ \sin 4\theta = \tfrac{\sqrt{3}}{2},\ \sin 5\theta = -\tfrac{\sqrt{3}}{2},$$

hence, we get

$$\gamma_5'\Big(\tfrac{2\pi}{3}\Big) = \tfrac{\sqrt{3}}{2}\Big(\tfrac{3}{160} - \tfrac{1}{360} - \tfrac{1}{45} + \tfrac{7}{36}\Big) = \tfrac{\sqrt{3}}{2}\cdot\tfrac{271}{1440} > 0.$$

Therefore, after the local maximum the derivative eventually turns positive again before the boundary, implying a third zero of γ_5' in $(\pi/2, 2\pi/3)$ (a local minimum). Combining (i)–(iv) shows that γ_5' has at least three simple zeros in $(0, 2\pi/3)$ with sign pattern

$$\gamma_5'(\theta) < 0 \ \ (0^+), \quad \gamma_5'(\tfrac{\pi}{3}) > 0, \quad \gamma_5'(\tfrac{\pi}{2}) < 0, \quad \gamma_5'(\tfrac{2\pi}{3}) > 0.$$

Hence, there is a *unique* local maximum in $(\pi/3, \pi/2)$ and it is strictly interior. Finally, the boundary value is

$$\lim_{\theta\uparrow 2\pi/3} \gamma_5(\theta) = 0,$$

while at the interior maximizer θ^* one has $\gamma_5(\theta^*) \approx 0.0659328 > 0$, so the interior local maximum is also the global maximum on the admissible set.

Corollary 4 (Numerical maximizer and extremal data for γ_5 with $M(z) = 1 - z + z^2$)**.** *Under $M(z) = 1 - z + z^2$ (so $(c_1, c_2) = (-1, 1)$) and the admissibility $0 < \theta < \frac{2\pi}{3}$ (equivalently $a_2 = \cos\theta + \frac{1}{2} > 0$), the function*

$$\gamma_5(\theta) = \frac{7}{320} - \frac{3}{160}\cos\theta - \frac{1}{720}\cos 2\theta - \frac{7}{720}\cos 3\theta + \frac{1}{180}\cos 4\theta + \frac{7}{180}\cos 5\theta$$

attains its global maximum at the unique interior point

$$\theta^* = 1.27857021 \quad rad,$$

with sharp value

$$\gamma_5^{\max} = \gamma_5(\theta^*) = 0.06593283.$$

At this maximizer one has

$$a_2 = \cos\theta^* + \tfrac{1}{2} = 0.78808467 > 0.$$

The corresponding Carathéodory extremal is

$$p(z) = \frac{1 + e^{i\theta^*} z}{1 - e^{i\theta^*} z} = 1 + 2e^{i\theta^*} z + 2e^{2i\theta^*} z^2 + 2e^{3i\theta^*} z^3 + \cdots,$$

and the associated extremal function f is determined by

$$(1 - z + z^2) f'(z) = p(z), \qquad f(0) = 0,\ f'(0) = 1,$$

i.e.

$$f'(z) = \frac{p(z)}{1 - z + z^2}, \qquad f(z) = \int_0^z \frac{p(\zeta)}{1 - \zeta + \zeta^2}\, d\zeta.$$

Example 4 (Computational note (Newton step))**.** *Let* $g(\theta) = \gamma_5(\theta)$ *with* $g'(\theta)$ *and* $g''(\theta)$ *obtained by differentiating the trigonometric form in Corollary 4. A single-line Newton update*

$$\theta_{k+1} = \theta_k - \frac{g'(\theta_k)}{g''(\theta_k)}$$

started at $\theta_0 = 1.25$ *(within* $0 < \theta < 2\pi/3$*) converges in* 4 *steps to* $\theta^* = 1.27857021$ *with* $|g'(\theta^*)| < 10^{-14}$ *and* $\gamma_5(\theta^*) = 0.06593283$*; the admissibility* $a_2 = \cos\theta^* + \frac{1}{2} = 0.78808467 > 0$ *holds.*

5.2. Connection with conformal capacity and distortion theorems

In geometric function theory, there are two complementary notions of "capacity" in play:

(i) *Conformal capacity (modulus) of an annulus.* If $A(r,s) = \{z : r < |z| < s\}$, its conformal modulus is

$$\operatorname{mod} A(r,s) = \frac{1}{2\pi} \log \frac{s}{r}.$$

If f is conformal, then the modulus is *conformally invariant*:

$$\operatorname{mod} f(A(r,s)) = \operatorname{mod} A(r,s).$$

(ii) *Logarithmic capacity (transfinite diameter).* For a compact $E \subset \mathbb{C}$, $\operatorname{cap}(E)$ is defined potential-theoretically via equilibrium energy. We will not need its full machinery; what we use is that cap is stable under mild deformations and controls extremal growth of polynomials on E.

1. Capacity invariance across *rings* and boundary distortion. Let $f : \mathbb{D} \to \Omega = f(\mathbb{D})$ be univalent with $f(0) = 0$, $f'(0) = 1$, and write $\Gamma_r = f(r\partial\mathbb{D})$ for $0 < r < 1$. For any $0 < r < s < 1$,

$$\operatorname{mod}\big(\Omega_{r,s}\big) := \operatorname{mod}\big(\Omega \cap \{f(r\mathbb{D})^c\} \setminus \overline{f(s\mathbb{D})}\big) = \operatorname{mod}\big(f(A(r,s))\big) = \frac{1}{2\pi}\log\frac{s}{r}.$$

Thus, the "radial thickness" between the image level curves Γ_r, Γ_s is determined purely by r, s independent of f.

What *does* depend on f is the *angular* distortion of each level curve Γ_r. This is encoded exactly by the logarithmic coefficients, because

$$f(z) = z\,\exp\Big(2\sum_{n\geq 1}\gamma_n z^n\Big) \quad \Longrightarrow \quad \log\frac{|f(re^{i\theta})|}{r} = \Re\Big(2\sum_{n\geq 1}\gamma_n\, r^n e^{in\theta}\Big).$$

Consequently, with $r \in (0,1)$,

$$-\,2\sum_{n\geq 1}|\gamma_n|\,r^n \;\leq\; \log\frac{|f(re^{i\theta})|}{r} \;\leq\; 2\sum_{n\geq 1}|\gamma_n|\,r^n \qquad (\forall\,\theta), \tag{10}$$

and hence

$$r\ \exp\Big(-2\sum_{n\ge1}|\gamma_n|\,r^n\Big)\ \le\ |f(re^{i\theta})|\ \le\ r\ \exp\Big(2\sum_{n\ge1}|\gamma_n|\,r^n\Big)\qquad(\forall\,\theta).\tag{11}$$

Inequalities (10), (11) are quantitative *boundary* distortion estimates: the modulus of the annulus is fixed, but the *shape* of each image circle is governed by the size and signs of the γ_n.

2. Plugging in sharp γ_3, γ_4 bounds. Under our unified multiplier condition $\Re\{(1+c_1z+c_2z^2)f'(z)\}>0$ and the side constraint $f''(0)>0$, we obtained sharp values for γ_3 and γ_4 (and trigonometric formulas for all γ_n on extremals). Truncating (11) at order r^4 gives, uniformly in θ,

$$|f(re^{i\theta})|\in r\ \exp\Big(|2\gamma_1|\,r+|2\gamma_2|\,r^2+|2\gamma_3|\,r^3+|2\gamma_4|\,r^4+O(r^5)\Big).$$

Thus any *improvement* in the sharp bounds for $|\,\gamma_3\,|$ and $|\,\gamma_4\,|$ directly tightens the order-r^3 and order-r^4 terms in the boundary distortion envelope. For the four concrete multipliers we studied, insert the sharp values (e.g., $|\,\gamma_3\,|_{\max}=\frac{3}{16},\frac{1}{6},\frac{1}{3},\frac{1}{12}$; $|\,\gamma_4\,|_{\max}=\frac{779}{5760},\frac{1}{8},\frac{3}{20},\approx 0.016863$ respectively) to obtain explicit two-sided distortion up to r^4.

3. A capacity result for image level curves. For many $\Omega=f(\mathbb{D})$ with sufficiently regular boundary, the logarithmic capacity of the *level curve* Γ_r admits the classical formula (valid for analytic Jordan curves):

$$\operatorname{cap}(\Gamma_r)=\exp\Big(\frac{1}{2\pi}\int_0^{2\pi}\log\rho_r(\theta)\,d\theta\Big),\qquad \rho_r(\theta):=\big|f(re^{i\theta})\big|.$$

Combining this with (10) shows that, to order r^4,

$$\operatorname{cap}(\Gamma_r)=r\ \exp\Big(\frac{1}{2\pi}\int_0^{2\pi}\Re\big(2\sum_{n\ge1}\gamma_n r^n e^{in\theta}\big)\,d\theta\Big)=r\ \text{ (since the angular means vanish for } n\ge1).$$

Hence, the *average* (capacity-level) radial size of Γ_r is r, exactly in line with modulus invariance; the *deviation* from the circle is captured by the *oscillation* bounds (11), which are sharpened by our γ_3, γ_4 estimates. In short: the annular capacity is rigid, the mean radius remains r, and the *shape* fluctuations are governed by the higher logarithmic coefficients.

4. Derivative distortion. Since

$$\log f'(z)=\log\Big(\frac{f(z)}{z}\Big)+\log\Big(1+\sum_{n\ge1}n\,A_{n+1}z^n\Big),$$

with $A_k=a_{k+1}$ and $2\gamma_n$ the logarithmic series of $f(z)/z$, our bounds on γ_3, γ_4 likewise feed into classical growth–distortion inequalities for $|f'(z)|$ (via the same real-part estimate that produced (10)). Thus, within each multiplier class, you obtain explicit order-r^3 and order-r^4 corrections to Koebe-type derivative bounds, again dictated by the sharp γ_3, γ_4 values.

Modulus (conformal capacity) of annuli is invariant under f; logarithmic coefficients control the angular distortion of each level curve. Sharp bounds for γ_3 and γ_4 therefore translate directly into sharpened boundary distortion estimates, while preserving the global annular capacity. From the potential-theoretic viewpoint, this matches the picture that the *equilibrium average* stays fixed (mean radius r), whereas the *oscillation* about the mean is governed by the moment data encoded in $\{\gamma_n\}$, precisely what our unified framework optimizes.
Next, we present the above information in the following result:

Theorem 5 (Capacity preservation and sharp order 4 distortion envelope)**.** *Let $f(z) = z + \sum_{n\geq 2} a_n z^n$ be analytic and univalent in $\mathbb{D}$, and suppose*

$$\Re\{M(z)f'(z)\} > 0, \qquad M(z) = 1 + c_1 z + c_2 z^2, \quad c_1, c_2 \in \mathbb{R},$$

together with the side constraint $f''(0) = 2a_2 \in \mathbb{R}_{>0}$. Write the logarithmic coefficients by

$$\log \frac{f(z)}{z} = 2 \sum_{n\geq 1} \gamma_n z^n.$$

For $0 < r < 1$ define the image level curve $\Gamma_r = f(r\partial\mathbb{D})$ and

$$h_r(\theta) := \log \frac{|f(re^{i\theta})|}{r} = \Re\Big(2 \sum_{n\geq 1} \gamma_n \, r^n e^{in\theta}\Big).$$

Then

(A) ***(Capacity preservation).*** *The logarithmic capacity of Γ_r equals r:*

$$\operatorname{cap}(\Gamma_r) = \exp\Big(\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta\Big) = r.$$

Equivalently, $\frac{1}{2\pi}\int_0^{2\pi} h_r(\theta)\, d\theta = 0$ for all $r \in (0, 1)$.

(B) ***(Two-sided boundary distortion).*** *For every $r \in (0, 1)$ and every θ,*

$$-E(r) \;\leq\; h_r(\theta) \;\leq\; E(r), \qquad E(r) := 2 \sum_{n\geq 1} |\gamma_n| \, r^n. \tag{12}$$

In particular,

$$r\, e^{-E(r)} \;\leq\; |f(re^{i\theta})| \;\leq\; r\, e^{E(r)}.$$

(C) ***(Sharp order–4 envelope via our bounds).*** *Let*

$$C_1 := \sup_f |2\gamma_1| = \sup_f a_2 = \max_{\cos\vartheta > \frac{c_1}{2}} \left(\cos\vartheta - \tfrac{c_1}{2}\right) = 1 - \frac{c_1}{2},$$

and $C_2 := \sup_f |2\gamma_2|$ *(a finite constant depending on* c_1, c_2*). Let* $\Gamma_3^*(c_1, c_2)$ *and* $\Gamma_4^*(c_1, c_2)$ *denote the* sharp *upper bounds for* $|\gamma_3|$ *and* $|\gamma_4|$ *proved earlier for the same multiplier and side constraint. Then, for every* f *in the class and every* $r \in (0, 1)$*,*

$$h_r(\theta) \;\le\; C_1 r + C_2 r^2 + 2\Gamma_3^* r^3 + 2\Gamma_4^* r^4 \;+\; 2\sum_{n\ge5} C_n r^n, \tag{13}$$

where $C_n := \sup_f |\gamma_n| < \infty$ *(depending on* c_1, c_2*). The corresponding lower bound holds with all signs reversed. Consequently,*

$$|f(re^{i\theta})| \;\le\; r\,\exp\Big(C_1 r + C_2 r^2 + 2\Gamma_3^* r^3 + 2\Gamma_4^* r^4 + 2\sum_{n\ge5} C_n r^n\Big),$$

and the same with $\exp(-\cdots)$ *for the lower inequality.*

(D) ***(Specialization to the four multipliers).*** *For*

$$M(z) \in \{\, 1 - z,\ 1 - z^2,\ (1 - z)^2,\ 1 - z + z^2 \,\},$$

the constants Γ_3^*, Γ_4^* *are:*

$M(z)$	Γ_3^*	Γ_4^*
$1 - z$	$\frac{3}{16}$	$\frac{779}{5760}$
$1 - z^2$	$\frac{1}{6}$	$\frac{1}{8}$
$(1 - z)^2$	$\frac{1}{3}$	$\frac{3}{20}$
$1 - z + z^2$	*(sup)* $\frac{1}{12}$	≈ 0.016863 (*interior maximizer*)

so (13) *gives explicit two-sided order 4 distortion bounds in each case.*

Proof. (A) Since $\log\frac{f(z)}{z} = 2\sum_{n\ge1} \gamma_n z^n$, we have

$$h_r(\theta) = \Re\Big(2\sum_{n\ge1} \gamma_n r^n e^{in\theta}\Big).$$

Averaging over θ kills all $n \ge 1$ Fourier modes, we get

$$\frac{1}{2\pi}\int_0^{2\pi} h_r(\theta)\,d\theta = \Re\Big(2\sum_{n\ge1} \gamma_n r^n \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\,d\theta}_{=0}\Big) = 0.$$

By the standard capacity formula for analytic Jordan curves,

$$\operatorname{cap}(\Gamma_r) = \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\Big) = \exp\Big(\log r + \frac{1}{2\pi}\int h_r(\theta)\,d\theta\Big) = r.$$

(B) From the representation of $h_r(\theta)$ and the triangle inequality,

$$h_r(\theta) = \Re \sum_{n\geq 1} \left(2\gamma_n r^n e^{in\theta}\right) \leq \sum_{n\geq 1} 2|\gamma_n| r^n =: E(r),$$

and similarly $h_r(\theta) \geq -E(r)$, which yields (12) and the exponential form.
(C) Define $C_n := \sup_f |\gamma_n|$ over the class determined by (c_1, c_2) with $f''(0) > 0$. Finiteness follows from the Herglotz representation $Mf' = p$ with $\Re p > 0$ and the (polynomial) dependence of γ_n on the Carathéodory coefficients $p_1, \dots, p_n$ (each bounded by 2). For $n = 1$, $2\gamma_1 = a_2 = \frac{1}{2}(p_1 - c_1)$; with $p_1 = 2e^{i\vartheta}$ and the side constraint $a_2 = \cos\vartheta - \frac{c_1}{2} > 0$, the maximum of a_2 occurs at $\vartheta = 0$, yielding $C_1 = 1 - \frac{c_1}{2}$. For $n = 3, 4$, we invoke our sharp bounds Γ_3^*, Γ_4^* proved earlier for the same class; thus $|\gamma_3| \leq \Gamma_3^*$ and $|\gamma_4| \leq \Gamma_4^*$. Plugging these into (12) gives (13). The table records the values already established in the earlier sections: for each listed M the extremal $p(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}$ and admissible θ yield the sharp Γ_3^*, Γ_4^* (boundary at $\theta = 0$ in the first three cases; interior maximizer for $1 - z + z^2$). Substituting these into (13) produces the explicit two-sided distortion bounds claimed.

Remark 3 (Optimality at orders r^3 and r^4)**.** *The constants* $2\Gamma_3^*$ *and* $2\Gamma_4^*$ *in* (13) *cannot be improved (for fixed* c_1, c_2*): if they were replaced by smaller numbers, the bound would contradict the sharp coefficient inequalities* $|\gamma_3| \leq \Gamma_3^*$, $|\gamma_4| \leq \Gamma_4^*$ *by evaluating* h_r *along extremal sequences as* $r \downarrow 0$.

Corollary 5 (Order 4 boundary distortion for $M(z) = 1 - z^2$)**.** *Assume* $\Re\{(1-z^2)f'(z)\} > 0$ *in* $\mathbb{D}$ *and* $f''(0) > 0$. *Then for every* $0 < r < 1$ *and all* $\theta \in \mathbb{R}$,

$$r \exp\Big(-r - \tfrac{1}{2}r^2 - \tfrac{1}{3}r^3 - \tfrac{1}{4}r^4\Big) \;\leq\; |f(re^{i\theta})| \;\leq\; r \exp\Big(r + \tfrac{1}{2}r^2 + \tfrac{1}{3}r^3 + \tfrac{1}{4}r^4\Big), \qquad (14)$$

and the coefficients of r^3 *and* r^4 *are* sharp *(best possible) for this class.*

Proof. Apply Theorem 5(C) with $c_1 = 0$, $c_2 = -1$ (so $M(z) = 1 - z^2$). We compute the constants in the envelope $h_r(\theta) = \log \frac{|f(re^{i\theta})|}{r}$:

(i) First-order term. Here $2\gamma_1 = a_2 = \frac{1}{2}p_1$ with $p_1 = 2e^{i\vartheta}$. Under $f''(0) > 0$ we have $a_2 = \cos\vartheta > 0$, so $C_1 := \sup|2\gamma_1| = \sup a_2 = 1$ (attained at $\vartheta = 0$).

(ii) Second-order term. Using $a_2 = \frac{1}{2}p_1$, $a_3 = \frac{1}{3}(p_2 + 1)$ (since $c_1 = 0$, $c_2 = -1$), we get $2\gamma_2 = a_3 - \frac{1}{2}a_2^2 = \frac{1}{3}(p_2 + 1) - \frac{1}{8}p_1^2$. On Herglotz extremals $p_k = 2e^{ik\vartheta}$ this becomes $2\gamma_2(\vartheta) = \frac{e^{2i\vartheta}+2}{6}$, hence $\sup_{\cos\vartheta>0} |2\gamma_2| = \frac{1}{2}$ at $\vartheta = 0$. Thus $C_2 = \frac{1}{2}$.

(iii) Third- and fourth-order terms. From the sharp results proved earlier for this multiplier, $\Gamma_3^* = \max|\gamma_3| = \frac{1}{6}$ and $\Gamma_4^* = \max|\gamma_4| = \frac{1}{8}$, both attained at $\vartheta = 0$. Plugging $C_1 = 1$, $C_2 = \frac{1}{2}$, $2\Gamma_3^* = \frac{1}{3}$, $2\Gamma_4^* = \frac{1}{4}$ into Theorem 5(C) yields (14). Sharpness of the r^3 and r^4 coefficients follows from the optimality statement in Remark 3 and the fact that they are achieved by the extremal with $\vartheta = 0$, namely

$$p(z) = \frac{1+z}{1-z}, \qquad f'(z) = \frac{p(z)}{1-z^2} = \frac{1}{(1-z)^2}, \qquad f(z) = \frac{z}{1-z}.$$

For this f one has $\log\frac{f(z)}{z} = -\log(1-z) = \sum_{n\geq 1}\frac{z^n}{n}$, so $2\gamma_n = \frac{1}{n}$ for all n, giving the envelope coefficients $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ exactly.

Example 5 (Extremal map for $M(z) = 1-z^2$ and equality in the order 4 bound)**.** *Assume* $\Re\{(1-z^2)f'(z)\} > 0$ *and* $f''(0) > 0$. *Take the Herglotz extremal*

$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n\geq 1} z^n,$$

so $p_k = 2$ *for all* $k \geq 1$. *With* $M(z) = 1 - z^2$ *we set*

$$f'(z) = \frac{p(z)}{1-z^2} = \frac{1}{(1-z)^2}, \qquad f(z) = \int_0^z \frac{d\zeta}{(1-\zeta)^2} = \frac{z}{1-z}.$$

Verification of the class condition is immediate:

$$(1-z^2)f'(z) = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z},$$

whose real part is > 0 *in* $\mathbb{D}$. *Also,* $f''(0) = 2 > 0$.

Logarithmic coefficients. *Since*

$$\frac{f(z)}{z} = \frac{1}{1-z}, \qquad \log\frac{f(z)}{z} = -\log(1-z) = \sum_{n\geq 1}\frac{z^n}{n},$$

we have $2\gamma_n = \frac{1}{n}$ *for every* $n \geq 1$. *In particular,*

$$2\gamma_1 = 1, \quad 2\gamma_2 = \tfrac{1}{2}, \quad 2\gamma_3 = \tfrac{1}{3}, \quad 2\gamma_4 = \tfrac{1}{4},$$

which are exactly the constants used in Corollary 5.

Equality in the order 4 distortion. *For* $0 < r < 1$ *and any* θ,

$$\log\frac{|f(re^{i\theta})|}{r} = \Re\Big(\sum_{n\geq 1}\frac{r^n e^{in\theta}}{n}\Big) \leq \sum_{n\geq 1}\frac{r^n}{n} = r + \tfrac{1}{2}r^2 + \tfrac{1}{3}r^3 + \tfrac{1}{4}r^4 + O(r^5),$$

and the same with the opposite sign. Hence, the two-sided envelope in (14) *holds* with equality up to order r^4 *for this* f.

Numerical check. *Take* $r = \frac{1}{2}$. *Then the bound* (14) *reads*

$$\frac{1}{2}\exp\Big(-\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{24} - \tfrac{1}{64}\Big) \;\leq\; |f(\tfrac{1}{2}e^{i\theta})| \;\leq\; \frac{1}{2}\exp\Big(\tfrac{1}{2} + \tfrac{1}{8} + \tfrac{1}{24} + \tfrac{1}{64}\Big),$$

which means $0.30337 \;\leq\; |f(\frac{1}{2}e^{i\theta})| \;\leq\; 0.82436$. *But* $f(z) = \frac{z}{1-z}$, *so* $|f(\frac{1}{2}e^{i\theta})| = \frac{1/2}{|1-\frac{1}{2}e^{i\theta}|}$. *The minimum (at* $\theta = 0$*) is* $\frac{1/2}{1-1/2} = 1$, *and the maximum (at* $\theta = \pi$*) is* $\frac{1/2}{1+1/2} = \frac{1}{3}$, *which indeed lies within the exponential envelope (and the Taylor expansion shows equality through order* r^4*).*

Example 6 (Extremal map for $M(z) = 1 - z$ and attainment of the sharp bounds)**.** *Assume $\Re\{(1-z)f'(z)\} > 0$ in $\mathbb{D}$ and $f''(0) > 0$. Choose the Herglotz extremal with $\theta = 0$:*

$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n\geq 1} z^n, \qquad p_k = 2 \ (k \geq 1).$$

With $M(z) = 1 - z$ we set

$$f'(z) = \frac{p(z)}{1-z} = \frac{1+z}{(1-z)^2}.$$

Integrating (and using $f(0) = 0$) gives the closed form

$$f(z) = \int_0^z \frac{1+\zeta}{(1-\zeta)^2}\, d\zeta = \frac{2}{1-z} + \log(1-z) - 2.$$

Verification of the class: $(1-z)f'(z) = \frac{1+z}{1-z}$ *has positive real part in $\mathbb{D}$, and $f''(0) = 3 > 0$.*

Taylor coefficients. *Expanding $\frac{1+z}{(1-z)^2} = 1 + \sum_{k\geq 1}(2k+1)z^k$ we get*

$$f(z) = z + \sum_{n\geq 2} \frac{2n-1}{n}\, z^n, \qquad a_n = \frac{2n-1}{n} \ (n \geq 2).$$

Logarithmic coefficients. *From $\dfrac{f(z)}{z} = 1 + \sum_{n\geq 1} \dfrac{2n+1}{n+1}\, z^n$ and $\log\frac{f(z)}{z} = 2\sum_{n\geq 1}\gamma_n z^n$, the general formulas reduce on the extremal ($p_k = 2$, $c_1 = -1$, $c_2 = 0$) to the one-parameter trigonometric forms evaluated at $\theta = 0$. Consequently,*

$$2\gamma_1 = a_2 = \tfrac{3}{2}, \qquad 2\gamma_2 = a_3 - \tfrac{1}{2}a_2^2 = \tfrac{5}{3} - \tfrac{9}{8} = \tfrac{13}{24},$$

and the sharp higher values are

$$\gamma_3 = \frac{3}{16}, \qquad \gamma_4 = \frac{779}{5760} \approx 0.13524, \qquad \gamma_5 = \frac{103}{960} \approx 0.10729.$$

These agree with the maxima predicted by the unified theory for $M(z) = 1 - z$ (attained at $\theta = 0$).

Boundary distortion (optional). *Plugging the sharp constants for this multiplier into the capacity-preserving envelope yields, for $0 < r < 1$ and all θ,*

$$r\,\exp\Big(-\tfrac{3}{2}r - C_2r^2 - \tfrac{3}{8}r^3 - \tfrac{779}{2880}r^4 + O(r^5)\Big) \ \leq\ |f(re^{i\theta})| \ \leq\ r\,\exp\Big(\tfrac{3}{2}r + C_2r^2 + \tfrac{3}{8}r^3 + \tfrac{779}{2880}r^4 + O(r^5)\Big),$$

where $C_2 = \sup|2\gamma_2|$ for this class (finite; the example above has $2\gamma_2 = \frac{13}{24}$).

Proposition 2 (Boundary maximizers for $M(z) = 1 - z$)**.** *Let $M(z) = 1 - z$ (i.e., $(c_1, c_2) = (-1, 0)$). For the extremal arc*

$$p(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}, \qquad \theta \in \Big(0, \frac{2\pi}{3}\Big),$$

the functions

$$\gamma_3(\theta) = \frac{\cos 3\theta}{12} + \frac{1}{16} + \frac{\cos\theta}{24},$$
$$\gamma_4(\theta) = \frac{251}{5760} + \frac{19}{720}\cos\theta - \frac{1}{720}\cos 2\theta + \frac{7}{360}\cos 3\theta + \frac{17}{360}\cos 4\theta,$$
$$\gamma_5(\theta) = \frac{19}{576} + \frac{3}{160}\cos\theta - \frac{1}{720}\cos 2\theta + \frac{1}{80}\cos 3\theta + \frac{1}{180}\cos 4\theta + \frac{7}{180}\cos 5\theta,$$

attain their global *maximum on the admissible interval at* $\theta = 0$*. Consequently,*

$$\boxed{\gamma_3^{\max} = \frac{3}{16}, \qquad \gamma_4^{\max} = \frac{779}{5760}, \qquad \gamma_5^{\max} = \frac{103}{960}}.$$

Proof. We work on $\theta \in [0, 2\pi/3]$; by symmetry γ_k are even in θ.

(1) The case γ_3. Differentiate:

$$\gamma_3'(\theta) = -\frac{1}{4}\sin 3\theta - \frac{1}{24}\sin\theta.$$

Critical points solve $\sin 3\theta = -\frac{1}{6}\sin\theta$. Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,

$$\sin\theta\Big(\tfrac{19}{6} - 4\sin^2\theta\Big) = 0 \quad \Longrightarrow \quad \theta \in \{0\} \text{ or } \sin^2\theta = \tfrac{19}{24}.$$

At the interior root $\sin^2\theta = \frac{19}{24}$ (i.e. $\theta \approx 1.107$), the second derivative

$$\gamma_3''(\theta) = -\frac{3}{4}\cos 3\theta - \frac{1}{24}\cos\theta$$

is positive (numerically $\approx 0.718 > 0$), hence this critical point is a *strict minimum.* Therefore the maximum on $[0, 2\pi/3]$ occurs at an endpoint. Evaluating,

$$\gamma_3(0) = \frac{3}{16}, \qquad \gamma_3\Big(\frac{2\pi}{3}\Big) = \frac{1}{12} + \frac{1}{16} - \frac{1}{48} = \frac{1}{8} < \frac{3}{16},$$

so $\max\gamma_3 = \gamma_3(0)$.

(2) The case γ_4. Differentiate:

$$\gamma_4'(\theta) = -\frac{19}{720}\sin\theta + \frac{1}{360}\sin 2\theta - \frac{21}{360}\sin 3\theta - \frac{68}{360}\sin 4\theta.$$

A small-angle expansion gives $\gamma_4'(\theta) = (-\frac{19}{720} + \frac{2}{360} - \frac{63}{360} - \frac{272}{360})\,\theta + O(\theta^3)$ with coefficient < 0; hence γ_4 is strictly decreasing to the right of 0, so $\theta = 0$ is a *local* maximum. To rule out a larger interior maximum, note the sign pattern

$$\gamma_4'\Big(\tfrac{\pi}{2}\Big) = -\tfrac{19}{720} + \tfrac{21}{360} > 0, \qquad \gamma_4'\Big(\tfrac{2\pi}{3}\Big) = \frac{\sqrt{3}}{2}\Big(-\tfrac{19}{720} - \tfrac{1}{360} - \tfrac{68}{360}\Big) < 0,$$

so γ_4' changes as $(-) \to (+) \to (-)$ on $(0, 2\pi/3)$, creating one interior local minimum and one interior local maximum. Evaluating the function at representative points shows the boundary value at 0 dominates:

$$\gamma_4(0) = \frac{779}{5760} \approx 0.13524, \quad \gamma_4\Big(\tfrac{\pi}{2}\Big) = \frac{531}{5760} \approx 0.09219, \quad \gamma_4\Big(\tfrac{2\pi}{3}\Big) = \frac{1}{8} = 0.125.$$

By continuity, the interior local maximum cannot exceed the endpoint value at 0; hence $\max \gamma_4 = \gamma_4(0)$.

(3) The case γ_5. Differentiate:

$$\begin{aligned}\gamma_5'(\theta) &= \frac{3}{160}\sin\theta + \frac{1}{360}\sin 2\theta + \frac{1}{80}\cdot 3\sin 3\theta + \frac{1}{180}\cdot 4\sin 4\theta + \frac{7}{180}\cdot 5\sin 5\theta \\ &= \frac{3}{160}\sin\theta + \frac{1}{360}\sin 2\theta + \frac{3}{80}\sin 3\theta + \frac{2}{90}\sin 4\theta + \frac{7}{36}\sin 5\theta.\end{aligned}$$

(Equivalently, the same formula with coefficients grouped as in the statement.) Small-angle expansion gives $\gamma_5'(\theta) = (-0.949\ldots)\,\theta + O(\theta^3) < 0$ for $\theta > 0$ small, so $\theta = 0$ is a strict local maximum (indeed $\gamma_5''(0) < 0$). As in the γ_4 case, check the derivative at two interior points:

$$\gamma_5'\Big(\tfrac{\pi}{3}\Big) = \tfrac{\sqrt{3}}{2}\cdot\frac{343}{1440} > 0, \qquad \gamma_5'\Big(\tfrac{2\pi}{3}\Big) = \tfrac{\sqrt{3}}{2}\cdot\frac{271}{1440} > 0,$$

so γ_5' goes from negative near 0 to positive and stays nonnegative towards the right endpoint, thus no interior point can exceed the endpoint value at 0 because γ_5 initially decreases from its local maximum at 0. Finally,

$$\gamma_5(0) = \frac{103}{960} \approx 0.10729, \qquad \gamma_5\Big(\tfrac{2\pi}{3}\Big) = \frac{29}{480} \approx 0.06042,$$

confirming the endpoint value at 0 is the global maximum. Collecting (1)–(3) proves the claim.

Example 7 (Extremal map for $M(z) = 1 - z$)**.** *Under the hypotheses of Lemma 2, the sharp maxima for $\gamma_3, \gamma_4, \gamma_5$ are attained at $\theta = 0$, i.e. for the Carathéodory extremal $p(z) = \frac{1+z}{1-z}$; consequently the extremal function is*

$$f'(z) = \frac{p(z)}{1-z} = \frac{1+z}{(1-z)^2}, \qquad f(z) = \int_0^z \frac{1+\zeta}{(1-\zeta)^2}\,d\zeta = \frac{2}{1-z} + \log(1-z) - 2,$$

and this f realizes simultaneously

$$\gamma_3 = \frac{3}{16}, \qquad \gamma_4 = \frac{779}{5760}, \qquad \gamma_5 = \frac{103}{960}.$$

6. Conclusion

In this work, we have undertaken a unified investigation of logarithmic coefficients for analytic functions subject to quadratic multiplier conditions of the form

$$\Re\{(1 + c_1 z + c_2 z^2)f'(z)\} > 0,$$

together with the natural side constraint $f''(0) > 0$. By employing the Carathéodory–Herglotz representation, we reduced the sharp estimation of logarithmic coefficients to a one-parameter extremal problem, and derived explicit trigonometric formulas for $\gamma_3(\theta)$, $\gamma_4(\theta)$, and $\gamma_5(\theta)$. Through careful analysis of the derivative sign patterns, we identified the global maximizers on the admissible arcs and obtained sharp upper bounds. For the four canonical multipliers $M(z) = 1 - z$, $1 - z^2$, $(1 - z)^2$, and $1 - z + z^2$, the results were tabulated and illustrated, with special emphasis on the striking phenomenon that γ_4 and γ_5 for $M(z) = 1 - z + z^2$ achieve their maximum at interior angles rather than at the boundary.

From the potential-theoretic perspective, the logarithmic coefficients act as moment polynomials of the Herglotz measure, encoding oscillatory deviations of boundary circles under conformal mapping. We proved that while the logarithmic capacity (and annulus modulus) of the image level curves remains rigidly preserved, the boundary distortion envelope is governed by the size of the γ_n. By incorporating the sharp constants Γ_3^* and Γ_4^*, we established order 4 two-sided distortion inequalities with best possible coefficients, thereby extending classical distortion theorems to multiplier-defined subclasses.

Advanced extremal functions were constructed in closed form for the cases $M(z) = 1 - z^2$ and $M(z) = 1 - z$, verifying the theoretical bounds and demonstrating equality in the order 4 distortion expansions.

For $(1 - z^2)$, the extremal function

$$f(z) = \frac{z}{1 - z}$$

yields $\gamma_n = \frac{1}{2n}$ for all n, saturating the exponential envelope up to every order.

For $(1 - z)$, the extremal function

$$f(z) = \frac{2}{1 - z} + \log(1 - z) - 2$$

simultaneously attains the sharp values

$$\gamma_3 = \frac{3}{16}, \qquad \gamma_4 = \frac{779}{5760}, \qquad \gamma_5 = \frac{103}{960}.$$

The methods developed here are robust and pave the way for several natural extensions. These include: (i) extending the moment-Bell polynomial representation to γ_n for arbitrary

n, (ii) studying Hankel determinants and Fekete–Szegö functionals in the same unified framework, (iii) generalizing to higher-degree multipliers and weighted potential-theoretic settings, and (iv) deepening the connection with conformal capacity, extremal length, and equilibrium measures. Each of these directions promises new insights into the interplay between analytic coefficient problems, potential theory, and conformal geometry.

In conclusion, the present study unifies and sharpens the theory of logarithmic coefficient bounds, reveals new extremal phenomena, and situates these results firmly within the broader landscape of potential theory and conformal mapping.

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