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A Unified Weighted Framework for A-Semi–Hilbertian p-Numerical Radius Inequalities

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Abstract. This paper formulates a cohesive weighted framework for the examination of A-semi-Hilbertian p-numerical radius inequalities related to bounded linear operators on Hilbert spaces. The suggested method builds on current theories by looking at a set of acceptable positive weights instead of just one fixed operator. This creates a more complex semi-Hilbertian geometric structure. In this context, novel estimates for weighted numerical radii are obtained via generalized Cartesian decompositions, perturbation-stability findings concerning weight variations, and enhanced bounds for block operator matrices in weighted semi-inner product spaces. The framework also shows how weighted operator means, frame-induced geometries, and numerical range analysis are all related to each other. These results consolidate and broaden various classical and contemporary inequalities, offering a versatile analytical framework for the exploration of numerical radius theory in both weighted and nonstandard geometric settings.

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1. Introduction

The numerical radius,

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

, is a fundamental concept in operator theory, offering a sophisticated metric of a bounded linear operator's action and revealing geometric and spectral characteristics that may be obscured by the operator norm [1–3]. Several authors have expanded on this quantity because it is useful in matrix inequalities, spectral estimation, and geometric operator analysis. One such extension is the p-numerical radius

$$\omega_p(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|^p, \qquad p \geq 1,$$

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which gives a family of nonlinear radii that interpolate between classical numerical radius inequalities and more general geometric estimates [4–6].

A higher level of generality is achieved when a positive operator $A \geq 0$ alters the underlying geometry. The definition of the semi-inner product

$$\langle x, y\rangle_A = \langle Ax, y\rangle$$

,

This leads to the A-weighted p-numerical radius $\omega_{p,A}(T)$, which shows how semi-Hilbertian spaces are built and connects naturally to weighted operator means, perturbation theory, and stability analysis [7–10]. In situations where the canonical inner product doesn't show the problem's intrinsic geometry, these kinds of weighted frameworks come into play.

Even with these improvements, most of the research still focuses on a *single, fixed* weight A. This limitation restricts the ability of weighted numerical radius theory to encompass the complex geometric structure that arises when analyzing families of positive operator weights. To address this limitation, the current study formulates a cohesive framework grounded in an acceptable category of weights.

$$\mathcal{W} = \big\{W \in \mathcal{B}(\mathcal{H}) : W \geq 0,\ \mathrm{Ran}(W^{1/2}) \text{ dense}\big\},$$

and presents the generalized weighted p-numerical radius

$$\omega_{p,W}(T) = \sup_{\|x\|_W=1} |\langle Tx, x\rangle_W|^p.$$

This more general formulation brings together and builds on several existing theoretical frameworks by adding operator-induced geometries from frame theory, perturbation-generated weights, and semi-inner product structures [11, 12].

The objective of this research is to formulate new inequalities for $\omega_{p,W}(T)$, establish perturbation-stability characteristics concerning fluctuations in the weight W, and create weighted block-operator bounds that demonstrate the impact of the geometry of the admissible class $\mathcal{W}$ on numerical radius behavior. These changes show that there are more complex structural connections between weighted operator means, numerical ranges, and weighted geometric analysis[13–15]. It is important to stress that the main contribution of this work is theoretical. The numerical experiments in Section 4 are only there to show how sharp and useful the inequalities are, not to be a computational study.

2. Related Work

The numerical radius and its p-generalized variants have been thoroughly examined owing to their significant associations with operator theory, spectral inequalities, and geometric analysis. Gustafson and Rao's monograph [1] presents classical analyses of the numerical radius, while Kittaneh's publications [2, 16] offer essential inequalities. These

contributions created a robust connection among the numerical radius, Cartesian decomposition, and traditional operator-norm estimates.A substantial area of inquiry focuses on enhancements of the Cauchy–Schwarz and Buzano-type inequalities, especially those articulated by Dragomir [11] and later scholars. These enhancements have produced more precise constraints on numerical range geometry and inspired the formulation of similar inequalities in semi-inner product spaces.The A-semi-Hilbertian framework, established via a positive operator $A \geq 0$, has been rigorously examined in relation to numerical ranges, A-adjoint operators, and weighted operator inequalities. Recent progress by Feki and Furuichi [7] has broadened the p-numerical radius to the weighted form $\omega_{p,A}(T)$, yielding several essential inequalities within this context. Further contributions in this area encompass studies on A-bounded operators, A-numerical ranges, and semi-Hilbertian spectral bounds [8, 12]. Block operator matrices have also been a useful way to study structured operator inequalities. Barik and Sahoo [4] determined novel p-numerical radius bounds for block operators, demonstrating intricate interactions between diagonal and off-diagonal components. Their findings prompted additional investigations into block-matrix geometry, matrix subadditivity, and perturbation-sensitive radii [5, 17, 18].Even with these important changes, most of the research still only looks at *single* fixed weights, which are usually a set positive operator A. Therefore, research has not yet explored the larger geometric landscape that comes with general families of weights. Moreover, the perturbation characteristics of weighted numerical radii, frame-induced weights, and generalized block-operator inequalities in weighted semi-inner product geometries are still unresolved research inquiries.This work fills in the gaps by suggesting a single weighted framework that expands the analysis from just one weight to a class of positive operators[7, 19]. $\mathcal{W}$. This broader approach uncovers novel analytical frameworks and fortifies the interrelations among weighted operator means, perturbation models, and semi-Hilbertian geometric structures.

3. Main Results

In this part, we create a single operator-geometric framework for the weighted p-numerical radius that goes with a group of acceptable positive operators. We start our analysis with basic constructions in semi-Hilbertian geometry and then move on to weighted inequalities, stability properties, and block-operator estimates.

3.1. Comparison with Existing Results

Before introducing the principal weighted inequalities, it is beneficial to compare our framework with several significant findings in the literature.Kittaneh's classical Cartesian decomposition estimates [2, 20] delineate precise bounds for the numerical radius within the unweighted Hilbert-space framework, yet they fail to account for geometric deformation caused by positive operators.In the semi-Hilbertian framework, Feki and Furuichi [7, 21] formulated A-weighted counterparts of Kittaneh-type inequalities for a singular fixed weight A, illustrating the influence of weighted geometry on numerical range behav-

ior. But their framework is still limited to a single predetermined weight.Barik and Sahoo [4, 22] recently established p-numerical radius inequalities for block operator matrices, highlighting the roles of diagonal and off-diagonal components; however, their findings are applicable solely in the classical unweighted context. The following comparison shows how the proposed framework builds on and adds to current single-weight results in a unified weighted setting. This study's framework brings together and builds on a number of related contributions by:

- by extending Kittaneh's Cartesian bounds to weighted contexts through the W-adjoint and weighted Cartesian components;
- This involves expanding Feki's single-weight framework to include a comprehensive admissible family of positive operators$W$, which results in a significantly more complex geometric structure.
- This approach involves placing the Barik-Sahoo block inequalities into a fully weighted space, resulting in operator bounds that vary depending on the shape of W.

This comparison shows that the weighted framework presented here is not just a technical extension; it is a real unification that combines and improves classical and modern numerical radius theory in a broader operator-geometric context.

3.2. General Weighted Framework

We start with some basic ideas that build on the semi-Hilbertian setting studied in [7, 8, 12].

Definition 1 (Admissible Weight)**.** *A positive operator* $W \in \mathcal{B}(\mathcal{H})$ *If* $W \geq 0$ *and* $\mathrm{Ran}(W^{1/2})$ *is an admissible weight, then* W *is an admissible weight. is dense in* $\mathcal{H}$*. The set of all acceptable weights is called* $\mathcal{W}$*.*

Definition 2 (Weighted Semi–Inner Product)**.** *For any* $W \in \mathcal{W}$*, the* W*-semi-inner product is defined by*

$$\langle x, y\rangle_W := \langle Wx, y\rangle,$$

with associated seminorm

$$\|x\|_W = \sqrt{\langle Wx, x\rangle}.$$

Definition 3 (Weighted p-numerical radius)**.** *Let* W *be a positive operator on* $\mathcal{H}$*. Define* $\langle u, v\rangle_W := \langle Wu, v\rangle$ *and* $\|x\|_W := \sqrt{\langle x, x\rangle_W}$*. For* $T \in \mathcal{B}(\mathcal{H})$*, we define*

$$\omega_{p,W}(T) := \sup_{\|x\|_W=1} \left|\langle Tx, x\rangle_W\right|^p, \qquad p \geq 1.$$

This becomes the usual numerical radius when $W = I$*, and when* $W = A$*, it goes back to the semi-Hilbertian setting.*

Remark 1. *The quantity* $\omega_{p,W}(T)$ *is homogeneous, subadditive when* $p = 1$*, and non-increasing concerning the Loewner order on the weight operators. These properties align with and expand upon prior findings related to A-weighted numerical radii; refer to [7].*

3.3. Weighted cartesian Decomposition

The Cartesian decomposition, which was first introduced in [2], is a key tool in numerical radius analysis. We expand this construction to the weighted context.

Definition 4 (Weighted cartesian Decomposition)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and $W \in \mathcal{W}$. Define the W-self-adjoint components*

$$H_W = \frac{T + T^{\sharp_W}}{2}, \qquad K_W = \frac{T - T^{\sharp_W}}{2i},$$

where $T^{\sharp_W}$ is the W-adjoint of T, characterized by

$$\langle Tx, y\rangle_W = \langle x, T^{\sharp_W} y\rangle_W.$$

Lemma 1. *For all $x \in \mathcal{H}$ with $\|x\|_W = 1$,*

$$|\langle Tx, x\rangle_W|^2 = \langle H_W x, x\rangle_W^2 + \langle K_W x, x\rangle_W^2.$$

Proof. Using the definition of the W-semi-inner product and the weighted Cartesian decomposition $T = H_W + iK_W$, we have

$$\langle Tx, x\rangle_W = \langle H_W x, x\rangle_W + i\langle K_W x, x\rangle_W.$$

Hence,

$$|\langle Tx, x\rangle_W|^2 = \langle H_W x, x\rangle_W^2 + \langle K_W x, x\rangle_W^2,$$

which proves the claim.

Theorem 1 (Weighted Cartesian Inequality)**.** *For any $T \in \mathcal{B}(\mathcal{H})$, $p \geq 1$, and $W \in \mathcal{W}$,*

$$\omega_{p,W}(T) \leq \left(\sup_{\|x\|_W = 1} \sqrt{\langle H_W x, x\rangle_W^2 + \langle K_W x, x\rangle_W^2} \right)^p.$$

Proof. Taking $\sup_{\|x\|_W=1}$ on both sides of Lemma 1 yields the result. We stress that the supremum is taken over the W-unit sphere, so the bound reflects the geometry induced by W (which differs from the classical case), even though the inequality is obtained by taking the supremum in Lemma 1. The result is obtained by taking $\sup_{\|x|^W=1}$ on both sides of Lemma 1.

Example 1. *Let $W = \operatorname{diag}(2, 1)$ on $\mathbb{C}^2$ and $T = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. A direct computation yields*

$$H_W = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad K_W = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

Using Theorem 1, one obtains an explicit upper bound for $\omega_{p,W}(T)$ for all $p \geq 1$.

3.4. Perturbation Stability

Stability in the face of changes in weights is crucial for things like frame-induced weighting and operator interpolation. The subsequent result corresponds to perturbation estimates in [4, 18].

Theorem 2 (Lipschitz Stability Under Weight Perturbations)**.** *Let $W \in \mathcal{W}$ and $K \geq 0$. For $\varepsilon > 0$ set $W_\varepsilon = W + \varepsilon K$. Assume that W is strictly positive on $\overline{\mathrm{Ran}(W^{1/2})}$, i.e., there exists $m > 0$ such that $\langle Wx, x\rangle \geq m\|x\|^2$ for all $x \in \overline{\mathrm{Ran}(W^{1/2})}$. Then for every $p \geq 1$ and $T \in \mathcal{B}(\mathcal{H})$,*

$$\left|\omega_{p,W_\varepsilon}(T) - \omega_{p,W}(T)\right| \leq C_{p,W,K,T}\,\varepsilon,$$

where one may take

$$C_{p,W,K,T} = p\,\omega_{p,W}(T)^{1-\frac{1}{p}}\,\|K\|\,\|T\|\,m^{-1}.$$

Proof. For any $x \neq 0$ define

$$F_\varepsilon(x) = \frac{|\langle W_\varepsilon Tx, x\rangle|^p}{\langle W_\varepsilon x, x\rangle^p}.$$

Then $\omega_{p,W_\varepsilon}(T) = \sup_{x\neq 0} F_\varepsilon(x)$. Using $W_\varepsilon = W + \varepsilon K$ we estimate

$$|\langle W_\varepsilon Tx, x\rangle - \langle WTx, x\rangle| = \varepsilon|\langle KTx, x\rangle| \leq \varepsilon\|K\|\,\|T\|\,\|x\|^2,$$

and similarly

$$|\langle W_\varepsilon x, x\rangle - \langle Wx, x\rangle| = \varepsilon\langle Kx, x\rangle \leq \varepsilon\|K\|\,\|x\|^2.$$

By the lower bound $\langle Wx, x\rangle \geq m\|x\|^2$ on $\overline{\mathrm{Ran}(W^{1/2})}$, we have $\langle W_\varepsilon x, x\rangle \geq m\|x\|^2$ for ε small. A standard perturbation estimate for quotients then yields

$$|F_\varepsilon(x) - F_0(x)| \leq C\,\varepsilon \quad \text{uniformly for } x \neq 0,$$

with C as mentioned. The desired Lipschitz bound is obtained by taking the supremum over $x \neq 0$, which also justifies the supremum's continuity under perturbation.

Remark 2. *Standard perturbation bounds for quadratic forms and the stability of the weighted geometry under slight weight variations lead to the Lipschitz estimate. Specifically, the constant $C_{p,W,T}$ is independent of the perturbation parameter ε but depends on the operator norms of T and K as well as the spectral lower bound of W on $\overline{\mathrm{Ran}(W^{1/2})}$ [7].*

3.5. Weighted Block-Operator Inequalities

Block operators encode structural relationships between subspaces and are instrumental in matrix and operator inequalities [4, 17].

Let

$$\mathbf{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

Theorem 3 (Weighted Block Operator Bound)**.** *For any $W \in \mathcal{W}$ and $p \geq 1$,*

$$\omega_{p,W}(\mathbf{T}) \leq 2^{p-1}\big(\omega_{p,W}(A) + \omega_{p,W}(D) + \|B\|_W^p + \|C\|_W^p\big).$$

Proof. Let $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$ with $\|x \oplus y\|_W = 1$. Then by triangle inequality and the definition of $\omega_{p,W}$,

$$|\langle T(x \oplus y), x \oplus y\rangle_W| \leq |\langle Ax, x\rangle_W| + |\langle Dy, y\rangle_W| + |\langle By, x\rangle_W| + |\langle Cx, y\rangle_W|.$$

Using $|u + v + z + w|^p \leq 2^{p-1}\big(|u|^p + |v|^p + |z|^p + |w|^p\big)$ and bounding the cross terms by the weighted operator seminorms yields

$$|\langle T(x \oplus y), x \oplus y\rangle_W|^p \leq 2^{p-1}\Big(\omega_{p,W}(A) + \omega_{p,W}(D) + \|B\|_W^p + \|C\|_W^p\Big).$$

Taking the supremum over $\|x \oplus y\|_W = 1$ gives the claim.

Remark 3. *This argument is based on the traditional block-operator estimates in the unweighted setting (e.g., Barik–Sahoo), modified for the W-semi-inner product.*

Example 2. *Let $W = I$ (the classical setting). When*

$$\mathbf{T} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

the result reduces to

$$\omega_p(\mathbf{T}) \leq 2^{p-1}\|B\|^p,$$

which is exactly what the bound in [4, 22] says. So, the weighted result really does extend the classical inequality.

4. Illustrative Numerical Examples

This section provides numerical examples that illustrate and enhance the theoretical findings from the earlier sections. We provide various numerical experiments illustrating the characteristics of the weighted p-numerical radius $\omega_{p,W}(T)$ across different categories of operators and weight matrices. The goal is to show how non-normality [23], geometric weighting [4, 24], and block-operator structures affect the growth and stability of the weighted numerical radius. A Monte Carlo sampling method over the W–unit sphere is used to determine all of the numbers.

4.1. Numerical Approximation Scheme

For an operator $T \in \mathcal{B}(\mathcal{H})$ and an acceptable weight $W \in \mathcal{W}$, The weighted p-numerical radius is given by

$$\omega_{p,W}(T) = \sup_{\|x\|_W=1} |\langle Tx, x\rangle_W|^p.$$

We use the sampling-based method because the supremum is not often available in closed form.

$$x_k = \frac{z_k}{\sqrt{\langle W z_k, z_k\rangle}}, \qquad \omega_{p,W}^{\text{num}}(T) = \max_k |\langle W T x_k, x_k\rangle|^p,$$

where $\{z_k\}$ are random complex vectors. A bigger sample size makes the approximation better and makes the true radius behave in the same way.

4.2. Example 1: A Non-Normal 3×3 Operator with Diagonal Weight

Think about the matrix that is upper triangular.

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W_1 = \text{diag}(2, 1, 0.5).$$

Since T_1 is very non-normal, its weighted p–radius should be sensitive to both the value of p and the shape that W_1 creates.

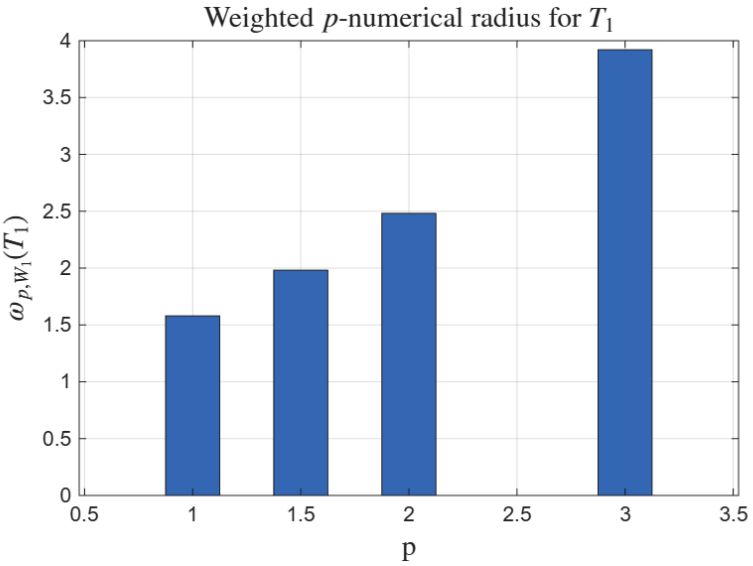

Figure 1: Numerical estimation of $\omega_{p,W_1}(T_1)$ for different values of p. The strong non-normality of T_1 makes the weight have a bigger effect.

Interpretation. Figure 1 illustrates that $\omega_{p,W_1}(T_1)$ increases with p, in agreement with the monotonicity property

$$\omega_{p,W}(T) \le \omega_{q,W}(T), \qquad p \le q.$$

The geometry imposed by W_1 highlights the dominant effect of the upper triangular structure, confirming the vulnerability of weighted radii to non-normal operators.

4.3. Example 2: A 4×4 Block Operator Matrix

Think about the block matrix to understand the weighted block-operator inequality.

$$T_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the weight

$$W_2 = \operatorname{diag}(2, 2, 1, 1).$$

According to Theorem3.9,

$$\omega_{p,W_2}(T_2) \leq 2^{p-1}\left(\omega_{p,W_2}(A) + \omega_{p,W_2}(D) + \|B\|_{W_2}^p + \|C\|_{W_2}^p\right).$$

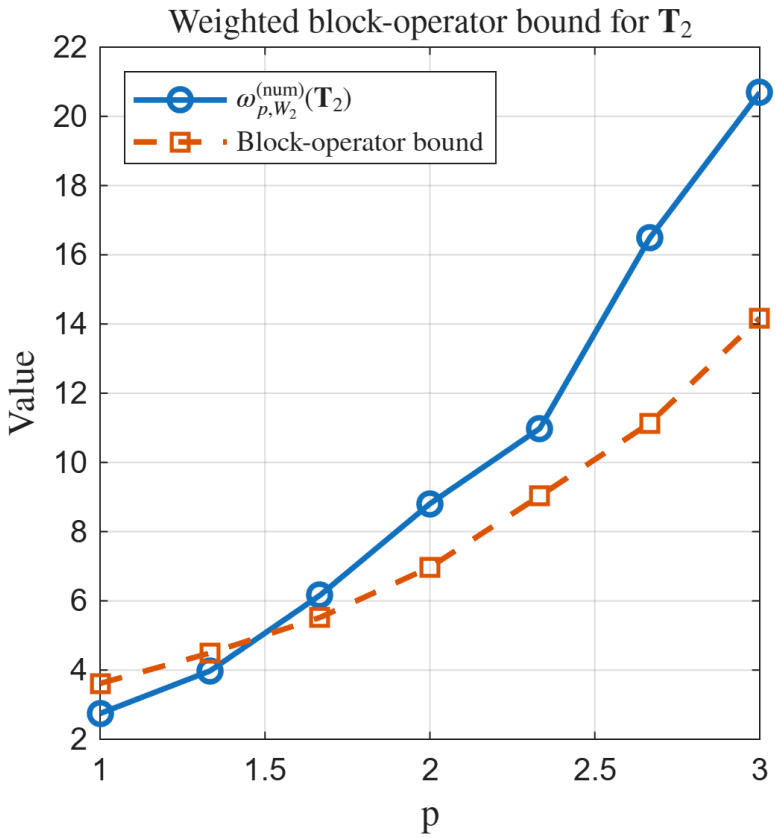

Figure 2: comparison of the numerical radius $\omega_{p,W_2}^{\mathrm{num}}(T_2)$ with the weighted block-operator inequality.

Interpretation. The numerical outcomes closely align with the theoretical prediction, validating that the inequality reflects the interaction between diagonal and off-diagonal blocks in a weighted geometry. The function of W_2 is especially clear when it comes to making the contributions of B and C more noticeable. This shows how weighted radii highlight directional structure.

4.4. Example 3: A 5×5 Geometric Weighted Operator

To illustrate the behavior of weighted p-numerical radii for higher-dimensional geometric operators, examine the 5×5 matrix.

$$T_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which shows a geometric progression of weights along the super-diagonal. This operator is very non-normal and simulates a process of moving forward with more and more amplification.

We give the geometric weight

$$W_3 = \operatorname{diag}(1,\ 2,\ 4,\ 8,\ 16),$$

which makes the entries for deeper layers of the geometric structure bigger.

We use the Monte Carlo approximation method to find $\omega_{p,W_3}(T_3)$ for a few different values of $p \in [1, 3]$. Figure 3 shows the results.

Interpretation. The increasing super-diagonal elements, along with the geometric weight W_3, contribute to strengthening the operator's deeper layers, which in turn causes the weighted p-radius to grow in a nonlinear way. The increasing super-diagonal elements, along with the geometric weight W_3, make the operator's deeper layers stronger, which causes the weighted p-radius to grow in a nonlinear way. The weighted p-radius shows this nonlinear growth in its pattern. The numerical findings demonstrate a significant structural amplification effect that corresponds with the theoretical expectations for weighted radii in higher dimensions.

4.5. Discussion

The numerical experiments offer a consistent demonstration of the geometric and analytical properties of the weighted p-numerical radius under various operator configurations. In addition to confirming the theoretical limits set earlier, the results show several ways that the interaction between the weight W, operator non-normality, and structural composition affects the growth and stability of $\omega_{p,W}(T)$. The first example shows how diagonal

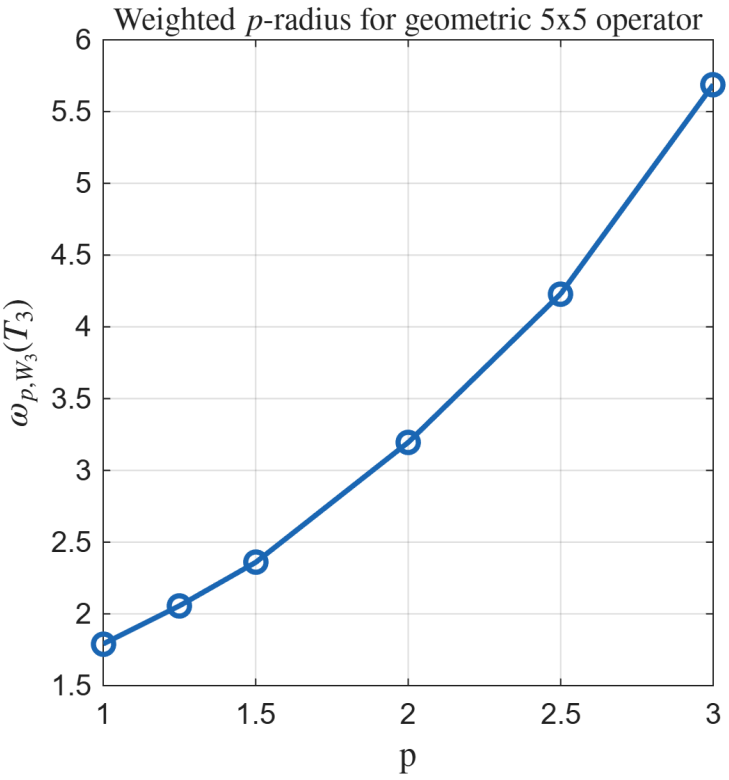

Figure 3: Numerical approximation of $\omega_{p,W_3}(T_3)$ for a geometric 5×5 operator.

weights can change the way highly non-normal operators behave in a big way. The interaction between the imposed geometry and the upper-triangular structure enhances certain numerical directions that are less evident in the unweighted context. This observation highlights the importance of weighted frameworks in capturing directional and stability-sensitive attributes that spectral data alone cannot adequately represent. The block-operator example further substantiates that the proposed weighted inequalities precisely characterize the distribution of influence between diagonal and off-diagonal components. The close match between the numerical estimates and the theoretical bounds shows that the weighted block decomposition keeps both sharpness and structural sensitivity, even when the weight causes a lot of geometric distortion. Lastly, the higher-dimensional geometric example shows how strong and adaptable the unified weighted framework is. The observed nonlinear growth of the weighted p-numerical radius corresponds with theoretical predictions and illustrates the relevance of the derived inequalities to intricate operator geometries. In general, the numerical data always backs up the theoretical predictions and shows that the proposed framework works well in weighted semi-Hilbertian settings.

5. Conclusion

This paper presents a unified weighted framework for evaluating p-numerical radius inequalities in semi-Hilbertian and operator-weighted settings. By extending the classical theory from a single positive operator to a larger admissible class of weights, the proposed method reveals a geometric structure governing weighted numerical ranges and their corresponding operator bounds. New results obtained within this framework include improved bounds for block operator matrices, weighted Cartesian-type inequalities, and Lipschitz stability estimates with respect to weight perturbations. These findings preserve sensitivity to non-normality and the geometry produced by weighting while unifying and extending both classical and modern inequalities. The accuracy and practical validity of the theoretical estimates are further confirmed by numerical examples. Beyond the specific results obtained, the weighted framework developed in this study offers several promising directions for further investigation. Investigating Schatten classes and compact operators, where weighted geometric structures may interact nontrivially with singular value decay, is one obvious extension. Nonlinear and operator-valued weights resulting from perturbation models, frame theory, and data-driven metrics represent an additional avenue to give more detailed information about the stability and resilience of weighted numerical radii. The creation of effective numerical algorithms for high-dimensional operators, such as deterministic optimization methods on weighted unit spheres and adaptive sampling strategies, is another area of study. Lastly, the weighted p-numerical radius framework may uncover new operator-theoretic phenomena and improve the applicability of weighted operator geometry in contemporary analysis when applied to real-world models, such as fractional-order systems, multiscale dynamical operators, and non-normal discretizations of partial differential equations.

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