EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7489
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Novel Generalized Fractional Calderón–Zygmund Operators with Rough Spatial Kernels and Applications to Two-Dimensional Delayed Parabolic Equations

Waqar Afzal[1], Mujahid Abbas[2,3], Jorge E. Macías-Díaz[4,5,*], Ernesto Urenda-Cázares[6], Mehreen Shehzadi Khan[7], Hafeez Sultana[7], Omniat Omer Yousif Karrar[7], Leema Aliyarukunju[7], Daniel Breaz[8], Luminiţa-Ioana Cotîrlă[9]

[1] *Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan*
[2] *Department of Mechanical Engineering Sciences, Faculty of Engineering and the Built Environment, Doornfontein Campus, University of Johannesburg, South Africa*
[3] *Department of Medical Research, China Medical University, Taichung 40402, Taiwan*
[4] *Department of Mathematics and Didactics of Mathematics, Tallinn University, Tallinn 10120, Estonia*
[5] *Department of Mathematics and Physics, Autonomous University of Aguascalientes, Aguascalientes 20100, Mexico*
[6] *University Center for Exact Sciences and Engineering, University of Guadalajara, Mexico*
[7] *Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia*
[8] *Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, 510009, Alba Iulia, Romania*
[9] *Department of Mathematics, Technical University of Cluj-Napoca, 400114, Cluj-Napoca, Romania*

Abstract. In this paper, we investigate the boundedness properties of a novel class of generalized fractional Calderón–Zygmund operators with spatially rough kernels. By employing refined analytical techniques together with appropriate structural assumptions, we derive sharp estimates and recover several known results as special cases. As an application, these results are applied to study the regularity and well-posedness of a nonlinear delayed parabolic equation.

2020 Mathematics Subject Classifications: 11B73, 11B83

Key Words and Phrases: Herz spaces, calderón–Zygmund operators, spatial roughness, well-posedness.

---

*Corresponding Author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7489

Email addresses: `jorge.maciasdiaz@edu.uaa.mx` (J. E. Macías-Díaz), `ernesto.urenda@academics.udg.mx` (E. Urenda-Cázares)

1. Introduction

The nonlinear parabolic equations encapsulate the balance of momentum and mass conservation, providing a rigorous framework for analyzing complex flow phenomena. Although there is a fundamental importance to know the existence, uniqueness and regularity of solutions to these equations, their non-linear coupled nature makes it extremely challenging to demonstrate and shows why it has been called one of the Millennium Prize Problems. Nonlinear functional analysis was used to solve these equations because the discovery of Temam [1] was an important basis. This publication was made a must-read to any further researches in this sphere. Still, the regularity and decay of strong solutions, especially in infinite spaces, were refined in the early works of scholars such as Da Veiga [2]. These prior observations were decisively important for later interpretation. In the same effort, the development of function space theory has been significant. Especially, Triebel [3] introduced the so-called hybrid spaces, which is a clever generalization of the ideas of spatial decay and temporal smoothing as it appears in the heat kernels. This new method was extraordinarily successful, not just in the Navier-Stokes case but also in an extended form beyond this in the more general elliptic and parabolic partial differential equations.

The study of fluid mechanics has made great progress with the introduction of spaces with non-standard growth on exponents, such as $\mathrm{L}^{\mathrm{p}(\alpha)}$. The importance of these spaces is due to the fact that they provide non-standard growth conditions, which better describes the real-world fluids with their inhomogeneity. This development has led to the thrilling developments in research. For example, the effect of replacing classical Lebesgue spaces with variable exponent counterparts on the structure and solutions of the N-S system was studied in [4]. An alternative approach was presented in [5], where the equations were investigated under viscosity in interval-valued function spaces, which provide a suitable algebraic framework for problems in higher dimensions. Further progress was made in [6], where stationary N-S equations were analyzed in the variable exponent setting, leading to new results. More recently, [7] addressed fractional and nonlocal variants of the N-S equations and established global well-posedness in Fourier–Besov–Morrey spaces with variable exponents, which capture both frequency behavior and spatial localization. Ruslan Abdulkadirov and Pavel Lyakhov [8] investigated estimates of mild solutions to the Navier–Stokes equations in the framework of weak Herz-type Besov–Morrey spaces.

Let us consider $\mathbf{R}^{\mathrm{n}}$ with $\mathrm{n} \geq 2$, and a fixed time interval $(0, \mathrm{T})$ where $0 < \mathrm{T} < \infty$. The *incompressible Navier–Stokes equations* system in $\mathbf{R}^{\mathrm{n}} \times (0, \mathrm{T})$ is written as:

$$\begin{cases} \frac{\partial \mathrm{u}}{\partial \mathrm{t}} - \Delta \mathrm{u} + (\mathrm{u} \cdot \nabla)\mathrm{u} + \nabla \mathrm{p} = \mathrm{f}, & \text{in } \mathbf{R}^{\mathrm{n}} \times (0, \mathrm{T}), \\ \operatorname{div} \mathrm{u} = 0, & \text{in } \mathbf{R}^{\mathrm{n}} \times (0, \mathrm{T}), \\ \mathrm{u}(0) = \mathrm{u}_0, & \text{in } \mathbf{R}^{\mathrm{n}}. \end{cases}$$

Here, the vector field u represents the *velocity of the fluid*, and f denotes the *external forces* acting on the fluid. The scalar field p represents the *pressure*. In this context, T denotes the finite time horizon or the maximum time for which the system is being considered. The work in [9] addressed the Cauchy problem for the incompressible N-S equations on $\mathbf{R}^{\mathrm{n}}$, where the analysis was directed toward the potential blow-up of local solutions and the characterization of global uniqueness for large initial data within the homogeneous weak Herz space $\mathbf{W\dot{K}}^{\beta}_{\mathrm{p,q}}$. Key results include embedding theorems linking weak Herz and homogeneous Besov spaces, essential for establishing boundedness estimates.

In [10], the authors investigated the stability of incompressible N-S equations within generalized Herz spaces, which allow for precise localization and decay properties. Building on this framework, [11] highlighted its limitations by demonstrating three-dimensional hyperdissipative N-S system in analogous spaces, illustrating the sensitive interplay between dissipation, nonlinearity, and the choice of function space. The work in [12] investigated asymptotically self-similar solutions in weak Herz spaces, focusing on behavior near critical regularity thresholds. A different strategy was pursued in [13], where the three-dimensional N-S equations were treated as a perturbation of the two-dimensional system, providing deeper understanding of both weak and strong solution structures. Meanwhile, [14] analyzed N-S and Euler flows, emphasizing the importance of scaling-adapted techniques for addressing non-homogeneous dynamics. In [15], a Lagrangian formulation was applied to the variable-density incompressible N-S equations, revealing intricate connections between transport effects and regularity. Finally, [16] explored the necessary conditions for the smoothness threshold that ensures the global continuation of solutions.

In this study, we focus on a problem initiated by Huanzhi Ge and Feng Du, who, in their recent work [17], proved the well-posedness and analyzed the determining regularity of Navier–Stokes equations with delay. Building on their foundational contributions, we propose a new methodology that emphasizes the role of operator-theoretic boundedness in addressing the dynamics of delayed fluid systems. Specifically, we turn our attention to the incompressible Navier–Stokes equations incorporating a memory-type delay term, described by the following system:

$$\begin{cases} \frac{\partial \mathrm{u}}{\partial \mathrm{s}} - \mathrm{v}\Delta \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla \mathrm{p} = \psi(\mathrm{s}) + \Phi(\mathrm{s}, \mathrm{u}(\mathrm{s}+\theta)), & \text{in } (\sigma,\mathrm{s}) \in \Omega \times (\iota, \mathrm{T}), \\ \operatorname{div} \mathrm{u} = 0, & \text{in } (\sigma,\mathrm{s}) \in \Omega \times (\iota, \mathrm{T}), \\ \mathrm{u} = 0, & \text{on } (\sigma,\mathrm{s}) \in \partial\Omega \times (\iota, \mathrm{T}), \\ \mathrm{u}(\sigma, \iota+\theta) = \phi(\sigma,\theta), & \sigma \in \Omega, \quad \theta \in (-\infty, 0]. \end{cases}$$

In the considered system, the unknown functions $\mathrm{u}(\sigma,\mathrm{s}) = (\mathrm{u}^{(1)}(\sigma,\mathrm{s}), \mathrm{u}^{(2)}(\sigma,\mathrm{s}))$ and $\mathrm{p}(\sigma,\mathrm{s})$ denote the velocity field and pressure of the fluid, respectively. The term $\psi(\mathrm{s}) = (\psi^{(1)}(\mathrm{s}), \psi^{(2)}(\mathrm{s}))$ represents an external force varying over time, while $\mathrm{v} > 0$ stands for the viscosity coefficient. The delayed term $\mathrm{u}(\mathrm{s}+\theta)$, with $\theta \in (-\infty, 0]$, accounts for the influence of past states on the current flow. Additionally, the mapping $\Phi(\cdot,\cdot)$ introduces a continuous feedback mechanism driven by this delay. For related results concerning operator boundedness and functional inequalities, we refer the reader to [18–21] and the references therein.

Operator boundedness, including that of singular integrals [22], heat semigroups [23], and fractional Laplacians [24], plays a central role in establishing the regularity, existence, and uniqueness of solutions to physical models such as the N-S systems. For example, recent studies have investigated pullback attractors and delay effects in two-dimensional flows [25], and the temporal and spatial decay have been analyzed by deriving upper and lower bounds for the classical Navier–Stokes equations [26]. Along the way, the introduction of functional spaces like the Herz-type, Morrey-type, and variable exponent spaces has given ways of analysing nonlinear PDE efficiently. The smoothness of solutions in such spaces, together with operator bound estimates and embedding, are critical to the extension of classical results. Drihem [27] investigated composition operators and composition of these to semilinear parabolic equations in the Herz-type Triebel-Lizorkin spaces. Generalized Lebesgue and Sobolev spaces have been a subject of other works in the search of how they can be used in the treatment of various types of PDEs [28]. More has been done concerning the regularity theory on variable exponent Herz spaces [29], embedding between Morrey-type and Herz spaces [30], and the structure of Herz-type Hardy spaces with variable smoothness and integrability [31]. Recently also, use of parabolic and anisotropic mixed-norm Herz spaces has attracted interest in boundary value and elliptic problems [32, 33]. Besides that, there is also more depth to both Hardy spaces related to generalized Herz spaces [34] and estimates of operators in weighted Morrey spaces [35]. Finally, the higher-order commutator and parametric integral operator theory on Morrey-Herz spaces of variable exponent [36] is an idea of further development of the aforesaid area. For additional relevant results concerning regularity criteria for weak solutions of various classes of equations, we refer the reader to the following [37–41] references.

The primary motivation of this work is to build upon and extend the results presented in Afzal et al. [42], where the Riesz potential with damping in the kernel was introduced and its boundedness properties were analyzed in the setting of Lebesgue spaces. Inspired by this approach, we focus on the Bessel–Riesz operator, incorporating an additional layer of spatial roughness into the kernel, which introduces new analytical challenges and richer structural behavior. In particular, we extend the scope of investigation from the classical Lebesgue spaces to Herz spaces, which provide both local and global features of functions and have proven to be highly effective in harmonic analysis. Furthermore, by establishing new boundedness results for this generalized operator, we are able to apply the developed theory to a nonlinear model of fluid mechanics. Specifically, we examine the regularity of the two-dimensional delayed Navier–Stokes system, where the coefficients are taken from the VMO (vanishing mean oscillation) class, a setting that plays a crucial role in ensuring stability and robustness of the system. This combination of operator-theoretic techniques with applications to fluid dynamics highlights both the theoretical depth and the practical significance of our contributions.

The structure of the article is arranged as follows. Section 1 presents a comprehensive introduction and highlights the primary motivation behind this study, placing our contribution within the broader framework of operator boundedness in harmonic analysis. In Section 2, we recall several essential lemmas, definitions, and auxiliary results that will be repeatedly utilized throughout the paper. Section 3 is devoted to establishing the boundedness properties of fractional operators along with their classical counterparts. In Section 4, we demonstrate an application of these results to a nonlinear parabolic-type equation. Finally, Section 5 summarizes our main findings and discusses potential directions for future research.

.

2. Preliminary Concepts

In this section, we compile the fundamental definitions and preliminary results that form the analytical framework for the developments presented in this article. To supplement the exposition and provide additional theoretical background, we refer the reader to authoritative monographs in the literature, together with the reference [43]. Moreover, we systematically establish the notation and conventions that will be employed throughout the subsequent sections to ensure clarity and consistency.

Notational Conventions

Throughout the paper, we use the following notations. The Euclidean space is denoted by $\mathbf{R}^{\mathrm{n}}$, with $\mathbf{R}^{+}$ for positive reals, $\mathbf{B}(\varpi, r)$ for an open ball, $|\Omega|$ for Lebesgue measure, χ_{E} for the characteristic function of a set, and $\operatorname{supp}\psi$ for the support of a function; "a.e." means almost everywhere. For function spaces, $\mathbf{P}_0^{\log}(\Omega)$ and $\mathbf{P}(\Omega)$ denote variable exponent classes, while $\mathrm{L}_{\mathrm{loc}}^{\mathrm{q}(\cdot)}(\Omega)$ and $\mathrm{W}^{1,\mathrm{p}}(\mathbf{R}^{\mathrm{n}})$ are the local variable exponent Lebesgue and Sobolev spaces. We write $\psi \lesssim \phi$ for domination up to a constant, $\Omega_1 \hookrightarrow \Omega_2$ for continuous embedding, and $\psi * \phi$ for convolution. Finally, a standard dyadic decomposition of $\mathbf{R}^{\mathrm{n}}$ into balls and annuli is employed.

2.1. Variable Exponent Spaces

Variable exponent Lebesgue spaces extend classical L^{p} spaces by allowing the integrability exponent to vary pointwise. Consider a measurable subset Ω of the Euclidean space $\mathbf{R}^{\mathrm{n}}$. We associate to each point of Ω a positive real number by means of a function $\iota : \Omega \to (0, \infty)$, often referred to in the literature as a *variable exponent*. We define

$$\iota^{-} := \operatorname*{ess\,inf}_{\xi \in \Omega} \iota(\xi), \qquad \iota^{+} := \operatorname*{ess\,sup}_{\xi \in \Omega} \iota(\xi).$$

The pointwise Hölder conjugate $\iota'(\cdot)$ is defined so that

$$\frac{1}{\iota(\xi)} + \frac{1}{\iota'(\xi)} = 1, \quad \text{for a.e. } \xi \in \Omega,$$

recovering the usual relation when ι is constant.

For each $\iota \in \mathbf{P}_0(\Omega) := \{\iota : \Omega \to (0, \infty) \mid \iota^{-} > 0\}$, the associated modular functional is

$$\vartheta_{\iota(\cdot)}(\varphi) := \int_{\Omega} \Phi_{\iota(\xi)}(|\varphi(\xi)|)\, d\xi,$$

where

$$\Phi_{\iota}(\tau) := \begin{cases} \tau^{\iota}, & 0 < \iota < \infty, \\ 0, & \tau \le 1,\ \iota = \infty, \\ \infty, & \tau > 1,\ \iota = \infty. \end{cases}$$

Definition 1 ([43]). *Let Ω be a measurable subset of the Euclidean space $\mathbf{R}^{\mathrm{n}}$. We consider a mapping $\iota : \Omega \to (0, \infty)$ which assigns a strictly positive real value to each point in Ω; such a mapping is commonly known as a variable exponent. The corresponding Lebesgue space with variable exponent is introduced as*

$$\mathrm{L}^{\iota(\cdot)}(\Omega) := \Big\{\chi : \Omega \to \mathbf{K} \text{ measurable} \;\Big|\; \exists\, \tau > 0 \text{ such that } \phi_{\iota(\cdot)}\Big(\frac{\chi}{\tau}\Big) < \infty\Big\},$$

and it is endowed with the Luxemburg-type norm

$$\|\chi\|_{\iota(\cdot)} := \inf\Big\{\tau > 0 \;:\; \phi_{\iota(\cdot)}\Big(\frac{\chi}{\tau}\Big) \leq 1\Big\}.$$

Whenever $\iota^- > 1$, this space is complete and hence constitutes a Banach space. In addition, if the essential supremum ι^+ is finite, then the following identity is valid:

$$\chi \in \mathrm{L}^{\iota(\cdot)}(\Omega) \quad \Longleftrightarrow \quad \int_\Omega |\chi(\xi)|^{\iota(\xi)}\, d\xi < \infty.$$

In order to guarantee the boundedness of the principal operators under consideration, we impose appropriate regularity conditions on the exponent function $\delta : \Omega \to \mathbf{R}^+$. Specifically, we assume that δ satisfies the classical log-Hölder continuity conditions.

The first condition, commonly termed local log-Hölder continuity, restricts the oscillation of δ between points that are close to one another. In particular, there exists a constant $c_{\log}(\delta) > 0$ such that for every pair of points $\varpi, \varsigma \in \Omega$,

$$|\delta(\varpi) - \delta(\varsigma)| \leq \frac{c_{\log}(\delta)}{\log\Big(\mathrm{e} + \frac{1}{|\varpi - \varsigma|}\Big)}.$$

This condition prevents abrupt fluctuations of the exponent within small neighborhoods of the domain.

If the origin belongs to the domain, i.e., $0 \in \Omega$, we further assume the *log-Hölder continuity at the origin*. That is, there exist constants $c_{\log}(\delta) > 0$ and $r_0 \in (0, 1]$ such that for every $\varpi \in \Omega$ satisfying $|\varpi| < r_0$,

$$|\delta(\varpi) - \delta(0)| \leq \frac{c_{\log}(\delta)}{\log\Big(\mathrm{e} + \frac{1}{|\varpi|}\Big)}.$$

This assumption ensures that the exponent remains stable and avoids excessive oscillations in a neighborhood of the origin.

Finally, in order to regulate the asymptotic behaviour of the exponent on large scales, we additionally assume the so–called *log-Hölder continuity at infinity.* More precisely, there exist constants $\delta_\infty \in \mathbf{R}$ and $c_{\log}(\delta) > 0$ such that for every $\varpi \in \Omega$,

$$|\delta(\varpi) - \delta_\infty| \leq \frac{c_{\log}(\delta)}{\log\big(\mathrm{e} + |\varpi|\big)}.$$

This condition ensures that the exponent stabilizes and converges toward a limiting value as $|\varpi| \to \infty$.

If a function δ satisfies the three log-Hölder continuity conditions described above, then it is referred to as *globally log-Hölder continuous*, which we denote concisely by $\delta \in \mathbf{C}_{\log}(\delta)$. Consequently, the family of globally regular variable exponents is given by

$$\mathbf{P}_0^{\log}(\Omega) := \left\{ \delta \in \mathbf{P}_0(\Omega) : \tfrac{1}{\delta} \in \mathbf{C}_{\log}(\delta) \right\}.$$

Proposition 1 ([43]). *Let Θ denote a measurable set and assume that $\mathtt{p}(\cdot) \in \mathbf{P}(\Theta)$. Under this assumption, the following assertions hold true.*

First, when the essential supremum $\mathtt{p}^+$ is finite, the modular function exhibits a scaling behavior governed by the variable exponent. In particular, for every scalar $\iota \geq 1$,

$$\iota^{\mathtt{p}^-} \vartheta(\xi) \leq \vartheta(\iota \xi) \leq \iota^{\mathtt{p}^+} \vartheta(\xi),$$

while the inequalities are reversed whenever $0 < \iota < 1$. This property describes how the modular responds to scalar multiplication under varying exponents.

Next, assume that $\mathtt{p}^+(\Theta \setminus \Theta_\infty) < \infty$. Then, for all $\iota \geq 1$,

$$\vartheta(\iota \xi) \leq \iota^{\mathtt{p}^+(\Theta \setminus \Theta_\infty)} \vartheta(\xi),$$

which provides a control estimate of the modular over the complement of a fixed subset $\Theta_\infty \subset \Theta$.

Moreover, if $\xi \in \mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)$ and its Luxemburg norm satisfies $\|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)} > 0$, then the normalized function obeys

$$\vartheta_{\mathtt{p}(\cdot)} \left(\frac{\xi}{\|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)}} \right) \leq 1.$$

In addition, if $\mathtt{p}^+ < \infty$, this inequality becomes an equality, i.e.,

$$\vartheta_{\mathtt{p}(\cdot)} \left(\frac{\xi}{\|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)}} \right) = 1,$$

which characterizes the normalization property of the Luxemburg norm.

Finally, provided that $\mathtt{p}^+ < \infty$, the norm and modular are related by two-sided estimates. Specifically, if $\|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)} > 1$, then

$$\vartheta_{\mathtt{p}(\cdot)}(\xi)^{1/\mathtt{p}^+} \leq \|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)} \leq \vartheta_{\mathtt{p}(\cdot)}(\xi)^{1/\mathtt{p}^-}.$$

Conversely, when $0 < \|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)} \leq 1$, the inequalities reverse in the exponents:

$$\vartheta_{\mathtt{p}(\cdot)}(\xi)^{1/\mathtt{p}^-} \leq \|\xi\|_{\mathrm{L}^{\mathtt{p}(\cdot)}(\Theta)} \leq \vartheta_{\mathtt{p}(\cdot)}(\xi)^{1/\mathtt{p}^+}.$$

Definition 2 ([43]). *Let* $\varphi \in \mathtt{L}^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{n}})$. *The* Hardy–Littlewood maximal function *is defined by*

$$\mathbf{M}\varphi(\iota) := \sup_{\mathtt{Q}\ni\iota} \frac{1}{|\mathtt{Q}|} \int_{\mathtt{Q}} |\varphi(\zeta)| \, \mathrm{d}\zeta,$$

where the supremum ranges over all axis-aligned cubes $\mathtt{Q} \subset \mathbf{R}^{\mathrm{n}}$ *that contain the point* ι. *The operator satisfies the following properties:*

(i) ***Subadditivity and Homogeneity:*** *For any* $\varphi, \chi \in \mathbf{X}(\mathbf{R}^{\mathrm{n}})$ *and any scalar* $\upsilon \in \mathbf{R}$,

$$\mathbf{M}(\varphi + \chi)(\iota) \leq \mathbf{M}\varphi(\iota) + \mathbf{M}\chi(\iota), \quad \mathbf{M}(\upsilon\varphi)(\iota) = |\upsilon| \, \mathbf{M}\varphi(\iota).$$

(ii) ***Lower Bound in Measure:*** *If* $\chi \not\equiv 0$, *then for every bounded measurable region* $\Theta \subset \mathbf{R}^{\mathrm{n}}$, *there exists a constant* $\epsilon > 0$ *such that*

$$\mathbf{M}\chi(\iota) \geq \epsilon, \quad \forall \, \iota \in \Theta.$$

(iii) ***Lack of Integrability:*** *If* $\chi \neq 0$ *a.e., then* $\mathbf{M}\chi \notin \mathrm{L}^1(\mathbf{R}^{\mathrm{n}})$.

(iv) ***Preservation of Supremum Norm:*** *If* $\chi \in \mathrm{L}^\infty(\mathbf{R}^{\mathrm{n}})$, *then*

$$\|\mathbf{M}\chi\|_\infty = \|\chi\|_\infty.$$

(v) ***Weak Type Estimate:*** *Let* $\varphi \in \mathrm{L}^{\mathrm{p}}(\mathbf{R}^{\mathrm{n}})$ *for some* $1 \leq \mathtt{p} < \infty$, *and* $\mathtt{t} > 0$. *Then,*

$$|\{\iota \in \mathbf{R}^{\mathrm{n}} : \mathbf{M}\varphi(\iota) > \mathtt{t}\}| \leq \frac{3^{\mathrm{n}} 4^{\mathrm{np}}}{\mathtt{t}^{\mathrm{p}}} \int_{\mathbf{R}^{\mathrm{n}}} |\varphi(\iota)|^{\mathrm{p}} \, \mathrm{d}\iota.$$

(vi) ***Strong Type Bound:*** *If* $1 < \mathtt{p} \leq \infty$, *then*

$$\|\mathbf{M}\varphi\|_{\mathrm{L}^{\mathrm{p}}(\mathbf{R}^{\mathrm{n}})} \leq \mathbf{C}(\mathrm{n})(\mathrm{p}')^{1/\mathrm{p}} \|\varphi\|_{\mathrm{L}^{\mathrm{p}}(\mathbf{R}^{\mathrm{n}})}.$$

(vii) ***Boundedness in Variable Exponent Spaces:*** *Suppose* $\mathtt{p}(\cdot) \in \mathbf{P}(\mathbf{R}^{\mathrm{n}})$ *with* $1/\mathtt{p}(\cdot) \in \mathbf{P}_0^{\log}(\mathbf{R}^{\mathrm{n}})$. *Then, for any* $\mathtt{t} > 0$ *and measurable* φ,

$$\left\| \mathtt{t} \, \chi_{\{\iota : \mathbf{M}\varphi(\iota) > \mathtt{t}\}} \right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathrm{n}})} \leq \mathbf{C} \|\varphi\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathrm{n}})}.$$

Moreover, if $\mathtt{p}_- > 1$, *it follows that*

$$\|\mathbf{M}\varphi\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathrm{n}})} \leq \mathbf{C} \|\varphi\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathrm{n}})}.$$

Definition 3 ([44]). *Let* $\mathrm{p}, \mathrm{q} \in \mathbf{P}_0(\mathbf{R}^{\mathrm{n}})$ *be given variable exponent functions, and consider a measurable function* $\alpha : \mathbf{R}^{\mathrm{n}} \to \mathbf{R}$ *such that* $\alpha \in \mathrm{L}^\infty(\mathbf{R}^{\mathrm{n}})$.

The inhomogeneous Herz space with variable exponents, denoted by $\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$*, consists of those measurable functions* $\psi \in L^{p(\cdot)}_{loc}(\mathbf{R}^n)$ *for which the quasi-norm*

$$\|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\psi\,\chi_{\mathbf{B}_0}\|_{p(\cdot)} + \left\|\left(2^{k\,\alpha(\cdot)}\,\psi\,\chi_k\right)_{k\geq 1}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}$$

is finite.

Intuitively, this structure measures the local behavior of ψ *near the origin together with its weighted decay across dyadic annuli, adjusted to the spatial variability of the exponents.*

Similarly, the homogeneous Herz space with variable exponents $\dot{\mathbf{K}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ *consists of all functions* $\psi \in L^{p(\cdot)}_{loc}(\mathbf{R}^n \setminus \{0\})$ *for which*

$$\|\psi\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\|\left(2^{k\,\alpha(\cdot)}\,\psi\,\chi_k\right)_{k\in\mathbf{Z}}\right\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}$$

is finite. In this homogeneous setting, the origin is excluded, and the scaling sequence extends symmetrically over all dyadic levels $k \in \mathbf{Z}$*, thus encoding the global scale-invariant structure of the function* ψ*.*

Remark 1. *The generalized definition of Herz-type spaces given in Definition 3 naturally encompasses several classical settings. If the weight parameter is constant, that is,* $\alpha(\cdot) \equiv \alpha$*, while both exponents* $p(\cdot)$ *and* $q(\cdot)$ *are fixed constants, then one recovers the standard Herz spaces in their classical form [45]. When* $\alpha(\cdot) \equiv \alpha$ *and* $q(\cdot)$ *is constant but* $p(\cdot)$ *varies with position, the construction reduces to the variable exponent Herz spaces [46]. In particular, if* $\alpha(\cdot)$*,* $p(\cdot)$*, and* $q(\cdot)$ *are all taken to be constant, the formulation collapses to the original Herz space structure [47]. Hence, the generalized approach unifies the traditional Herz spaces, providing a broader setting that interpolates naturally between these cases.*

Lemma 1 ([48])**.** *Let* $\beta \in L^\infty(\mathbf{R}^n)$ *and assume that* $t_1 > 0$*. If* β *fulfills the log-Hölder continuity condition in a neighborhood of the origin as well as at infinity, then*

$$t_1^{\beta(\varepsilon)} \lesssim t_2^{\beta(\gamma)} \times \begin{cases} \left(\dfrac{t_1}{t_2}\right)^{\beta^+} & \text{if } 0 < t_2 \leq \dfrac{t_1}{2}, \\ 1 & \text{if } \dfrac{t_1}{2} < t_2 \leq 2t_1, \\ \left(\dfrac{t_1}{t_2}\right)^{\beta^-} & \text{if } t_2 > 2t_1, \end{cases}$$

for any $\varepsilon \in \mathbf{B}(0,t_1) \setminus \mathbf{B}\left(0, \dfrac{t_1}{2}\right)$ *and* $\gamma \in \mathbf{B}(0,t_2) \setminus \mathbf{B}\left(0, \dfrac{t_2}{2}\right)$*, with a constant implicit in the estimates that is independent of* ε*,* γ*,* t_1*, and* t_2*.*

Lemma 2 ([49]). *Let $\gamma \in \mathbf{P}^{\log}(\mathbf{R}^{\mathrm{n}})$. Then, for each pair of cubes $\mathbf{Q} \subset \mathbf{R} \subset \mathbf{R}^{\mathrm{n}}$, there exists a constant $\mathrm{C} > 0$, independent of $\mathbf{Q}$ and $\mathbf{R}$, such that the following inequality holds:*

$$\mathrm{A}\left(\frac{|\mathbf{R}|}{|\mathbf{Q}|}\right)^{1/\gamma^{+}} \leqslant \frac{\left\|\chi_{\mathbf{R}}\right\|_{\gamma(\cdot)}}{\left\|\chi_{\mathbf{Q}}\right\|_{\gamma(\cdot)}} \leqslant \mathrm{B}\left(\frac{|\mathbf{R}|}{|\mathbf{Q}|}\right)^{1/\gamma^{-}},$$

where the constants $\mathrm{A}, \mathrm{B} > 0$ are independent of the measures $|\mathbf{R}|$ and $|\mathbf{Q}|$.

Proposition 2 ([48]). *Let $\beta \in \mathrm{L}^{\infty}(\mathbf{R}^{\mathrm{n}})$, $\rho \in \mathbf{P}(\mathbf{R}^{\mathrm{n}})$, and $\delta \in (0, \infty]$. Assume that the function $\beta(\cdot)$ satisfies the logarithmic Hölder continuity condition as $|\varepsilon| \to \infty$. Then the non-standard inhomogeneous Herz space with smoothness coincides with the corresponding Herz space of constant smoothness determined by the asymptotic limit β_{∞}; namely,*

$$\mathbf{K}^{\beta(\cdot)}_{\rho(\cdot),\delta}(\mathbf{R}^{\mathrm{n}}) = \mathbf{K}^{\beta_{\infty}}_{\rho(\cdot),\delta}(\mathbf{R}^{\mathrm{n}}).$$

In addition, if $\beta(\cdot)$ also possesses logarithmic regularity in a neighborhood of the origin, then the associated norm in quasi sense can be equivalently represented as

$$\|\phi\|_{\dot{\mathbf{K}}^{\beta(\cdot)}_{\rho(\cdot),\delta}} \approx \left\|\left\{2^{\mathrm{k}\beta(0)}\, \phi\, \chi_{\mathrm{k}}\right\}\right\|_{\ell^{\delta}_{<}(\mathrm{L}^{\rho(\cdot)})} + \left\|\left\{2^{\mathrm{k}\beta_{\infty}}\, \phi\, \chi_{\mathrm{k}}\right\}\right\|_{\ell^{\delta}_{>}(\mathrm{L}^{\rho(\cdot)})}.$$

Within this setting, the quantity β_{∞} captures the asymptotic behaviour of $\beta(\varepsilon)$ at infinity, whereas $\beta(0)$ reflects its limiting value in a neighbourhood of the origin.

Here, $\ell^{\delta}_{<}$ corresponds to the discrete quasi-norm taken over negative indices, whereas $\ell^{\delta}_{>}$ is computed over non-negative indices. Such a distinction allows one to separate the local part of the Herz norm from its asymptotic portion.

Lemma 3 ([48]). *Let $0 < \mu < 1$ and $0 < \delta \leqslant \infty$. Consider a sequence of positive reals $\{\varepsilon_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}$ such that*

$$\left\|\{\varepsilon_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}\right\|_{\ell^{\delta}} = \mathbf{J} < \infty.$$

Define two new sequences by

$$\zeta_{\mathrm{k}} := \sum_{\mathrm{j}\leqslant\mathrm{k}} \mu^{\mathrm{k}-\mathrm{j}} \varepsilon_{\mathrm{j}}, \qquad \xi_{\mathrm{k}} := \sum_{\mathrm{j}\geqslant\mathrm{k}} \mu^{\mathrm{j}-\mathrm{k}} \varepsilon_{\mathrm{j}}, \quad \mathrm{k} \in \mathbf{Z}.$$

Then both $\{\zeta_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}$ and $\{\xi_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}$ belong to ℓ^{δ}, and one has

$$\left\|\{\zeta_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}\right\|_{\ell^{\delta}} + \left\|\{\xi_{\mathrm{k}}\}_{\mathrm{k}\in\mathbf{Z}}\right\|_{\ell^{\delta}} \leqslant \mathtt{C}\,\mathbf{J},$$

where $\mathtt{C} > 0$ is a constant depending solely on μ and δ.

Definition 4 ([50]). *The space of functions with bounded mean oscillation, denoted by $\mathrm{BMO}(\mathbf{R}^{\mathrm{n}})$, is defined as*

$$\mathrm{BMO}(\mathbf{R}^{\mathrm{n}}) := \left\{\mathrm{b} \in \mathrm{L}^{1}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{n}}) : \|\mathrm{b}\|_{\mathrm{BMO}} < \infty\right\},$$

where

$$\|\mathrm{b}\|_{\mathrm{BMO}} := \sup_{\mathrm{B}\subseteq\mathbf{R}^{\mathrm{n}}} \frac{1}{|\mathrm{B}|}\int_{\mathrm{B}} |\mathrm{b}(\varpi) - \mathrm{b}_{\mathrm{B}}|\, \mathtt{d}\varpi,$$

with B *denoting a generic ball in* $\mathbf{R}^{\mathrm{n}}$*, and* $\mathrm{b}_{\mathrm{B}} := \frac{1}{|\mathrm{B}|}\int_{\mathrm{B}} \mathrm{b}(\varpi)\, \mathtt{d}\varpi$ *representing the average of* b *over the ball* B*.*

Definition 5 ([50])**.** *The space of functions with vanishing mean oscillation, denoted by* $\mathrm{VMO}(\mathbf{R}^{\mathrm{n}})$*, is a proper subspace of* $\mathrm{BMO}(\mathbf{R}^{\mathrm{n}})$*. It is defined by*

$$\mathrm{VMO}(\mathbf{R}^{\mathrm{n}}) := \left\{ \mathrm{b} \in \mathrm{BMO}(\mathbf{R}^{\mathrm{n}}) : \lim_{\sigma\to 0^+} \gamma_{\mathrm{b}}(\sigma) = 0 \right\},$$

where the function $\gamma_{\mathrm{b}}(\sigma)$*, referred to as the VMO modulus of* b*, is given by*

$$\gamma_{\mathrm{b}}(\sigma) := \sup_{\rho\leq\sigma} \sup_{\mathrm{B}_\rho} \frac{1}{|\mathrm{B}_\rho|}\int_{\mathrm{B}_\rho} \left|\mathrm{b}(\varpi) - \mathrm{b}_{\mathrm{B}_\rho}\right|\, d\varpi,$$

and B_ρ *ranges over all balls in* $\mathbf{R}^{\mathrm{n}}$ *of radius* ρ*.*

Lemma 4 ([48])**.** *Let* $\rho \in \mathbf{P}^{\log}_{\infty}(\mathbf{R}^{\mathrm{n}})$*, and set*

$$\mathbf{S} := \mathbf{B}(0, \rho_0) \setminus \mathbf{B}\big(0, \tfrac{\rho_0}{2}\big)\,.$$

If $|\mathbf{S}| \geqslant 2^{-\nu}$*, then the following norm equivalences are valid:*

$$\left\|\chi_{\mathbf{S}}\right\|_{\mathrm{L}^{\rho(\cdot)}} \;\approx\; |\mathbf{S}|^{1/\rho(\varepsilon)} \;\approx\; |\mathbf{S}|^{1/\rho_\infty},$$

where the implied constants are uniform with respect to both the radius ρ_0 *and any point* $\varepsilon \in \mathbf{S}$*.*

3. The Major results

The purpose of this section is to undertake a detailed investigation of the boundedness properties associated with a newly formulated integral operator of Bessel–Riesz type. This operator is distinguished by the simultaneous incorporation of an exponential damping factor and a rough kernel, features that substantially enrich its analytical flexibility. By allowing additional structural parameters, this framework unifies and extends several well-known operators, each of which may be recovered as a special case under appropriate parameter restrictions. Consequently, the formulation provides a versatile tool for addressing difficult integral estimates where standard operators either fail to apply or provide insufficient control.

To set the stage, we introduce the operator explicitly:

$$\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\psi(\varpi) := \int_{\mathbb{R}^{\mathrm{n}}} \frac{|\varpi-\varsigma|^{\alpha-\mathrm{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\, \Omega\!\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right) \mathrm{e}^{-\upsilon|\varpi-\varsigma|}\, \psi(\varsigma)\, \mathrm{d}\varsigma,$$

here, $\alpha \in (0, \mathfrak{n})$ characterizes the order of the kernel's singular behaviour, whereas $\gamma \geq 0$ dictates how rapidly it decreases as one moves towards infinity; furthermore, the measurable function

$$\Omega : \mathbf{R}^{\mathfrak{n}} \setminus \{0\} \to \mathbf{C}$$

is permitted to be singular or oscillatory, thereby admitting a broad class of rough kernels. The exponential term $\mathrm{e}^{-\upsilon|\varpi-\varsigma|}$ plays the role of a damping factor, ensuring rapid decay at large distances.

For points ϖ located outside the support of ψ, the operator satisfies the fundamental size estimate

$$\left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}(\psi)(\varpi)\right| \leq \int_{\mathbb{R}^{\mathfrak{n}}} \frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}} \left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| \mathrm{e}^{-\upsilon|\varpi-\varsigma|} \left|\psi(\varsigma)\right| \mathrm{d}\varsigma. \tag{3.1}$$

Inequality (3.1) illustrates how the interplay of parameters shapes the kernel:

- the parameter α introduces a fractional-type singularity;
- the term $(1+|\varpi-\varsigma|)^{-\gamma}$ governs polynomial decay at infinity;
- the exponential damping factor produces a rapid vanishing effect, which is particularly effective in enhancing integrability and mitigating roughness in Ω.

Remark 2. *The generalized Bessel–Riesz operator with exponential damping and rough kernel is defined by*

$$\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\psi(\varpi) := \int_{\mathbb{R}^{\mathfrak{n}}} \frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}} \Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right) \mathrm{e}^{-\upsilon|\varpi-\varsigma|} \psi(\varsigma)\, \mathrm{d}\varsigma.$$

This single framework includes several well-known operators, which can now be written unambiguously by specifying the corresponding parameter choices:

(i) ***Riesz potential:*** *For $\upsilon = 0$, $\gamma = 0$, and $\Omega \equiv 1$, we denote*

$$\mathbf{T}^{0}_{(0,0,1)}\psi(\varpi) = \int_{\mathbb{R}^{\mathfrak{n}}} \frac{\psi(\varsigma)}{|\varpi-\varsigma|^{\mathfrak{n}-\alpha}}\, \mathrm{d}\varsigma.$$

(ii) ***Rough fractional integral:*** *For $\upsilon = 0$, $\gamma = 0$, and Ω homogeneous of degree zero,*

$$\mathbf{T}^{0}_{(0,0,\Omega)}\psi(\varpi) = \int_{\mathbb{R}^{\mathfrak{n}}} \frac{\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)}{|\varpi-\varsigma|^{\mathfrak{n}-\alpha}} \psi(\varsigma)\, \mathrm{d}\varsigma.$$

(iii) ***Exponentially damped Riesz potential:*** *For $\gamma = 0$ and $\Omega \equiv 1$,*

$$\mathbf{T}^{\upsilon}_{(0,0,1)}\psi(\varpi) = \int_{\mathbb{R}^{\mathfrak{n}}} \frac{1}{|\varpi-\varsigma|^{\mathfrak{n}-\alpha}} \mathrm{e}^{-\upsilon|\varpi-\varsigma|} \psi(\varsigma)\, \mathrm{d}\varsigma.$$

(iv) ***Sublinear operator:*** *When* $\alpha = 0$, $\upsilon = 0$, $\gamma = 0$, $\Omega \equiv 1$,

$$\mathbf{T}^{0}_{(0,0,1)}\psi(\varpi) = \int_{\mathbb{R}^{\mathfrak{n}}} \frac{\psi(\varsigma)}{|\varpi - \varsigma|^{\mathfrak{n}}}\, \mathrm{d}\varsigma.$$

(v) ***Bessel–Riesz operator:*** *For* $\upsilon = 0$ *and* $\Omega \equiv 1$,

$$\mathbf{T}^{0}_{(\alpha,\gamma,1)}\psi(\varpi) = \int_{\mathbb{R}^{\mathfrak{n}}} \frac{|\varpi - \varsigma|^{\alpha-\mathfrak{n}}}{(1 + |\varpi - \varsigma|)^{\gamma}}\psi(\varsigma)\, \mathrm{d}\varsigma.$$

Thus, each operator is now clearly represented as a specific case of $\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}$*, avoiding any ambiguity in notation as highlighted by the reviewer.*

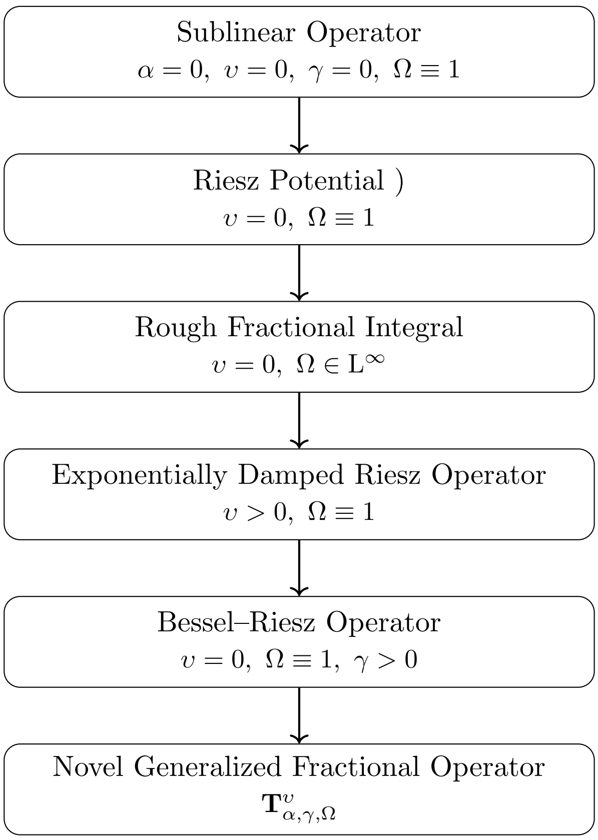

Figure 1: Structural evolution from classical integral operators to the generalized Bessel–Riesz operator with exponential damping and rough kernel.

Remark 3 (Novelty and Significance)**.** *The proposed Bessel–Riesz operator with rough kernel and exponential damping advances beyond classical Riesz and rough fractional operators. The additional decay term* $(1+|\varpi-\varsigma|)^{-\gamma}$ *enhances both integrability and decay at infinity, ensuring boundedness in broader function spaces even when* Ω *is only measurable.*

Theorem 1. *Let* $0 < \mathrm{q} \leq \infty$. *We assume that the variable exponent function* $\mathrm{p} : \mathbf{R}^{\mathtt{n}} \to (1, \infty)$ *belongs to the log-Hölder class at infinity, that is,*

$$\mathrm{p} \in \mathbf{P}^{\log}_{\infty}(\mathbf{R}^{\mathtt{n}}),$$

meaning that there exists a finite limit

$$\mathrm{p}_\infty := \lim_{|\varpi| \to \infty} \mathrm{p}(\varpi)$$

and some constant $C_\infty > 0$ *such that for all* $\varpi \in \mathbf{R}^{\mathtt{n}}$,

$$|\mathrm{p}(\varpi) - \mathrm{p}_\infty| \leq \frac{C_\infty}{\log(e + |\varpi|)}.$$

Assume further that

$$1 < \mathrm{p}^- \leq \mathrm{p}^+ < \infty, \qquad \mathtt{n} \in \mathbf{N}.$$

Let $\alpha \in \mathrm{L}^\infty(\mathbf{R}^{\mathtt{n}})$ *be log-Hölder continuous at infinity and satisfying the limiting condition*

$$-\frac{\mathtt{n}}{\mathrm{p}_\infty} < \alpha_\infty < \frac{\mathtt{n}}{\mathrm{p}'_\infty},$$

where $\alpha_\infty := \lim_{|\varpi| \to \infty} \alpha(\varpi)$.

Define the exponentially damped operator with rough kernel

$$\mathbf{T}^{\upsilon}_{\Omega}(\phi)(\varpi) := \int_{\mathbf{R}^{\mathtt{n}}} \frac{\Omega\left(\frac{\varpi - \varsigma}{|\varpi - \varsigma|}\right)}{|\varpi - \varsigma|^{\mathtt{n}}} \, \mathrm{e}^{-\upsilon|\varpi - \varsigma|} \, \phi(\varsigma) \, \mathrm{d}\varsigma,$$

where $\Omega \in \mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathtt{n}-1})$, $1 < \mathrm{s} < \infty$, *is homogeneous of degree zero, and* $\upsilon > 0$.

If $\mathbf{T}^{\upsilon}_{\Omega}$ *is bounded on* $\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathtt{n}})$, *then it extends continuously to* $\mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^{\mathtt{n}})$. *Consequently, for any* $\psi \in \mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^{\mathtt{n}}) \cap \mathrm{L}^{\mathrm{p}(\cdot)}$, *we have*

$$\|\mathbf{T}^{\upsilon}_{\Omega}(\psi)\|_{\mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^{\mathtt{n}})} \leq \mathrm{C} \, \|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^{\mathtt{n}})},$$

where the constant $\mathrm{C} > 0$ *depends on* $\mathtt{n}$, $\mathrm{p}(\cdot)$, q, $\alpha(\cdot)$, υ, *and* $\|\Omega\|_{\mathrm{L}^{\mathrm{s}}}$.

Proof. Let $\mathbf{T}^{\upsilon}_{\Omega}$ be a sublinear operator, and suppose that it is bounded on the variable exponent Lebesgue space $\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathtt{n}})$. Assume further that $\alpha \in \mathrm{L}^\infty(\mathbf{R}^{\mathtt{n}})$ is log-Hölder continuous at infinity. With these hypotheses in place, the operator $\mathbf{T}^{\upsilon}_{\Omega}$ extends continuously to the variable exponent Herz space $\mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^{\mathtt{n}})$.

To analyze this boundedness, we employ a dyadic decomposition of the underlying space and estimate the corresponding Herz quasi-norm of $\mathbf{T}^{\upsilon}_{\Omega}(\psi)$ by separately considering contributions from near-field and far-field regions:

$$|\mathbf{T}^{\upsilon}_{\Omega}\psi(\varpi)| \leq \sum_{\mathrm{i}=1}^{4} \left|\mathbf{T}^{\upsilon}_{\Omega}\left(\psi \, \chi_{\mathbf{R}_{\mathrm{i}}}\right)(\varpi)\right|,$$

where the spatial regions are specified by:

$$\begin{aligned}
\mathbf{R}_1 &:= \mathbf{B}_{-2},\\
\mathbf{R}_2 &:= \mathbf{B}_{k-2} \setminus \mathbf{B}_{-2},\\
\mathbf{R}_3 &:= \left\{ \varpi \in \mathbf{R}^{n} : 2^{k-2} \le |\varpi| < 2^{k+2} \right\},\\
\mathbf{R}_4 &:= \mathbf{R}^{n} \setminus \mathbf{B}_{k+2}.
\end{aligned}$$

Using the exponentially damped kernel estimate:

$$|\mathbf{T}_\Omega^{\upsilon}(\phi)(\varpi)| \le \int_{\mathbf{R}^n} \frac{\left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|}{|\varpi-\varsigma|^{n}} e^{-\upsilon|\varpi-\varsigma|} |\phi(\varsigma)| \, d\varsigma,$$

and the boundedness of $\mathbf{T}_\Omega^{\upsilon}$ on $L^{p(\cdot)}$, each part $\mathbf{T}_\Omega^{\upsilon}(\psi \cdot \chi_{\mathbf{R}_i})$ is estimated.
Collecting all the preceding bounds and estimates, we arrive at the final conclusion:

$$\|\mathbf{T}_\Omega^{\upsilon}(\psi)\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbf{R}^n)} \le C \, \|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbf{R}^n)},$$

where the constant $C > 0$ depends only on $\|\Omega\|_{L^s}$, υ, and the parameters in the assumptions.

Next, we analyze the part arising from $\mathbf{T}_\Omega^{\upsilon}\big(\psi\, \chi_{\mathbf{B}_{-2}}\big)$ when evaluated at points $\varpi \in \mathbf{R}_k$ for all indices $k \ge 1$. Noting that for every $\varsigma \in \mathbf{B}_{-2}$, the following property holds:

$$|\varpi - \varsigma| \ge |\varpi| - |\varsigma| > 2^{k-1} - 2^{-2} \gtrsim 2^{k},$$

the kernel estimate of $\mathbf{T}_\Omega^{\upsilon}$ yields

$$\left|\mathbf{T}_\Omega^{\upsilon}\left(\psi\, \chi_{\mathbf{B}_{-2}}\right)(\varpi)\right| \le \int_{\mathbf{B}_{-2}} \frac{\left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|}{|\varpi-\varsigma|^{n}} e^{-\upsilon|\varpi-\varsigma|} |\psi(\varsigma)| \, d\varsigma.$$

Using the pointwise estimates

$$|\varpi-\varsigma|^{-n} \lesssim 2^{-kn}, \qquad e^{-\upsilon|\varpi-\varsigma|} \lesssim e^{-\upsilon c\cdot 2^{k}},$$

We estimate the local part for $\varpi \in \mathbf{R}_k$, $k \ge 1$. From the definition of $\mathbf{T}_\Omega^{\upsilon}$, we have

$$\left|\mathbf{T}_\Omega^{\upsilon}\left(\psi\, \chi_{\mathbf{B}_{-2}}\right)(\varpi)\right| \lesssim 2^{-kn} e^{-\upsilon c\cdot 2^{k}} \int_{\mathbf{B}_{-2}} \left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| |\psi(\varsigma)| \, d\varsigma.$$

To estimate the integral, we apply the generalized variable Hölder inequality:

$$\int_{\mathbf{B}_{-2}} \left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| |\psi(\varsigma)| \, d\varsigma \le \left\|\Omega(\varpi - \cdot)\chi_{\mathbf{B}_{-2}}\right\|_{L^{p'(\cdot)}(\mathbf{B}_{-2})} \left\|\psi\, \chi_{\mathbf{B}_{-2}}\right\|_{L^{p(\cdot)}(\mathbf{B}_{-2})}.$$

Observing that $s > p'(\cdot)^{-}$ alone does not guarantee that $\widetilde{p}(\cdot) > 1$. In order for the decomposition

$$\frac{1}{p'(\varsigma)} = \frac{1}{\widetilde{p}(\varsigma)} + \frac{1}{s}$$

to be meaningful, we additionally require

$$\frac{1}{p'(\varsigma)} - \frac{1}{s} > 0 \qquad \Longleftrightarrow \qquad p'(\varsigma) < s \quad \text{for all } \varsigma \in \mathbf{R}^{\mathbf{n}}.$$

Equivalently,

$$s > p'^{+} := \operatorname*{ess\,sup}_{\varsigma \in \mathbf{R}^{\mathbf{n}}} p'(\varsigma),$$

which ensures that $\widetilde{p}(\varsigma) > 1$. Under this additional assumption, the relation

$$\frac{1}{p'(\varsigma)} = \frac{1}{\widetilde{p}(\varsigma)} + \frac{1}{s} \tag{3.2}$$

holds, and hence, using Lemma 2.1.4 of [51], we obtain

$$\left\|\Omega(\varpi - \cdot)\, \chi_{\mathbf{B}_{-2}}\right\|_{L^{p'(\cdot)}(\mathbf{B}_{-2})} \leq \left\|\Omega(\varpi - \cdot)\, \chi_{\mathbf{B}_{-2}}\right\|_{L^{s}(\mathbf{B}_{-2})} \left\|\chi_{\mathbf{B}_{-2}}\right\|_{L^{\widetilde{p}(\cdot)}(\mathbf{B}_{-2})}.$$

Making the change of variables $\omega = \frac{\varpi - \varsigma}{|\varpi - \varsigma|}$ and using polar coordinates, we deduce

$$\int_{\mathbf{B}_{-2}} \left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|^{s} d\varsigma \lesssim \int_{0}^{2^{-2}} \int_{\mathbf{S}^{\mathbf{n}-1}} |\Omega(\omega)|^{s} \rho^{\mathbf{n}-1}\, d\sigma(\omega)\, d\rho \lesssim \|\Omega\|^{s}_{L^{s}(\mathbf{S}^{\mathbf{n}-1})}.$$

Hence,

$$\left|\mathbf{T}^{\upsilon}_{\Omega}\left(\psi\, \chi_{\mathbf{B}_{-2}}\right)(\varpi)\right| \lesssim 2^{-k\mathbf{n}} e^{-\upsilon c 2^{k}} \|\Omega\|_{L^{s}(\mathbf{S}^{\mathbf{n}-1})} \|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{\widetilde{p}(\cdot)}}.$$

Since $\varsigma \in \mathbf{B}_{-2}$ implies $1 < p^{-} \leq p(\varsigma) \leq p^{+} < \infty$, the exponent functions $p(\cdot)$ and $\widetilde{p}(\cdot)$ are bounded and measurable on the compact set $\mathbf{B}_{-2}$. It follows from the local equivalence property of variable Lebesgue spaces that if two exponents $r(\cdot)$ and $s(\cdot)$ satisfy

$$0 < \inf_{\mathbf{B}_{-2}} \frac{r(\varsigma)}{s(\varsigma)} \leq \sup_{\mathbf{B}_{-2}} \frac{r(\varsigma)}{s(\varsigma)} < \infty,$$

then the corresponding norms $\|\cdot\|_{L^{r(\cdot)}(\mathbf{B}_{-2})}$ and $\|\cdot\|_{L^{s(\cdot)}(\mathbf{B}_{-2})}$ are equivalent. Therefore, the norms $L^{\widetilde{p}(\cdot)}(\mathbf{B}_{-2})$ and $L^{p(\cdot)}(\mathbf{B}_{-2})$ are equivalent that is

$$\|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{\widetilde{p}(\cdot)}} \lesssim \|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{p(\cdot)}}.$$

Consequently,

$$\left|\mathbf{T}^{\upsilon}_{\Omega}\left(\psi\, \chi_{\mathbf{B}_{-2}}\right)(\varpi)\right| \lesssim 2^{-k\mathbf{n}} e^{-\upsilon c 2^{k}} \|\Omega\|_{L^{s}(\mathbf{S}^{\mathbf{n}-1})} \|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{p(\cdot)}}.$$

Taking the $L^{p(\cdot)}$-norm and applying Lemma 4, we get

$$2^{k\alpha_\infty} \left\|\chi_k \mathbf{T}_\Omega^\upsilon(\psi\, \chi_{\mathbf{B}_{-2}})\right\|_{L^{p(\cdot)}} \lesssim 2^{k\left(\alpha_\infty - \mathtt{n} + \frac{\mathtt{n}}{p_\infty}\right)} e^{-\upsilon c\, 2^k} \|\Omega\|_{L^s(\mathbf{S}^{\mathtt{n}-1})} \|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{p(\cdot)}}.$$

Summing over $k \geq 1$ in ℓ^q, we conclude

$$\left(\sum_{k\geq 1} 2^{k\alpha_\infty q} \left\|\chi_k \mathbf{T}_\Omega^\upsilon(\psi\, \chi_{\mathbf{B}_{-2}})\right\|_{L^{p(\cdot)}}^q\right)^{1/q} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{\mathtt{n}-1})} \|\psi\, \chi_{\mathbf{B}_{-2}}\|_{L^{p(\cdot)}} \leq C\|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbf{R}^{\mathtt{n}})}.$$

We now estimate the term

$$\mathbf{T}_\Omega^\upsilon\big(\psi\, \chi_{\mathbf{B}_{k-2}\setminus \mathbf{B}_{-2}}\big)\,(\varpi).$$

Let $\varpi \in \mathbf{R}_k$ and $\varsigma \in \mathbf{B}_{k-2} \setminus \mathbf{B}_{-2} \subset \mathbf{R}_j$ for some $-1 \leq j \leq k-2$. Then

$$\left|\mathbf{T}_\Omega^\upsilon\left(\psi\, \chi_{\mathbf{B}_{k-2}\setminus \mathbf{B}_{-2}}\right)(\varpi)\right| \leq \sum_{-1\leq j\leq k-2} \int_{\mathbf{R}_j} \frac{\left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|}{|\varpi-\varsigma|^{\mathtt{n}}}\, e^{-\upsilon|\varpi-\varsigma|}\, |\psi(\varsigma)|\, d\varsigma.$$

Since $|\varpi - \varsigma| \gtrsim 2^k$, we have

$$|\varpi-\varsigma|^{-\mathtt{n}} \lesssim 2^{-k\mathtt{n}}, \qquad e^{-\upsilon|\varpi-\varsigma|} \lesssim e^{-\upsilon\, c\, 2^k}.$$

By the same argument as in the previous estimate, and after applying the variable Hölder inequality in conjunction with relation (3.2), we obtain

$$\left|\mathbf{T}_\Omega^\upsilon\left(\psi\, \chi_{\mathbf{B}_{k-2}\setminus \mathbf{B}_{-2}}\right)(\varpi)\right| \lesssim \|\Omega\|_{L^s(\mathbf{S}^{\mathtt{n}-1})} \sum_{-1\leq j\leq k-2} 2^{-k\mathtt{n}}\, e^{-\upsilon\, c\, 2^k} \|\psi\, \chi_j\|_{L^{p(\cdot)}} \|\chi_j\|_{L^{p'(\cdot)}}.$$

Taking the $L^{p(\cdot)}$-norm on $\mathbf{R}_k$ and multiplying by $2^{k\alpha_\infty}$,

$$2^{k\alpha_\infty} \left\|\chi_k \mathbf{T}_\Omega^\upsilon\left(\psi\, \chi_{\mathbf{B}_{k-2}\setminus \mathbf{B}_{-2}}\right)\right\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{\mathtt{n}-1})} 2^{k\alpha_\infty} 2^{-k\mathtt{n}} e^{-\upsilon\, c\, 2^k} \|\chi_k\|_{L^{p(\cdot)}} \sum_{-1\leq j\leq k-2} \|\psi\, \chi_j\|_{L^{p(\cdot)}} \|\chi_j\|_{L^{p'(\cdot)}}.$$

Using Lemma 4,

$$\|\chi_j\|_{L^{p'(\cdot)}} \approx 2^{\frac{j\mathtt{n}}{p'_\infty}}, \qquad \|\chi_k\|_{L^{p(\cdot)}} \approx 2^{\frac{k\mathtt{n}}{p_\infty}}.$$

Thus,

$$2^{k\alpha_\infty} \left\|\chi_k \mathbf{T}_\Omega^\upsilon\left(\psi\, \chi_{\mathbf{B}_{k-2}\setminus \mathbf{B}_{-2}}\right)\right\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{\mathtt{n}-1})} 2^{k(\alpha_\infty - \mathtt{n} + \frac{\mathtt{n}}{p_\infty})} e^{-\upsilon\, c\, 2^k} \sum_{-1\leq j\leq k-2} 2^{\frac{j\mathtt{n}}{p'_\infty}} \|\psi\, \chi_j\|_{L^{p(\cdot)}}.$$

Let

$$\delta' := \alpha_\infty - \mathtt{n} + \frac{\mathtt{n}}{p_\infty}.$$

Then

$$2^{k\alpha_\infty}\left\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{B}_{k-2}\setminus\mathbf{B}_{-2}}\right)\right\|_{L^{p(\cdot)}}\lesssim\|\Omega\|_{L^s(\mathbf{S}^{n-1})}e^{-\upsilon\,c\,2^k}\sum_{-1\le j\le k-2}2^{(k-j)\delta'}2^{j\alpha_\infty}\|\psi\,\chi_j\|_{L^{p(\cdot)}}.$$

Taking the ℓ^q-quasi norm,

$$\left\{\sum_{k\ge1}2^{k\alpha_\infty q}\left\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{B}_{k-2}\setminus\mathbf{B}_{-2}}\right)\right\|_{L^{p(\cdot)}}^q\right\}^{1/q}\lesssim\|\Omega\|_{L^s(\mathbf{S}^{n-1})}\left\{\sum_{j\ge0}2^{j\alpha_\infty q}\|\psi\,\chi_j\|_{L^{p(\cdot)}}^q\right\}^{1/q}.$$

Therefore,

$$\left\{\sum_{k\ge1}2^{k\alpha_\infty q}\left\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{B}_{k-2}\setminus\mathbf{B}_{-2}}\right)\right\|_{L^{p(\cdot)}}^q\right\}^{1/q}\lesssim\|\Omega\|_{L^s(\mathbf{S}^{n-1})}\|\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}(\mathbf{R}^n)}\lesssim C\,\|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbf{R}^n)}.$$

Estimation of $\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\widetilde{\mathbf{R}}_k}\right)$:
Let

$$\widetilde{\mathbf{R}}_k:=\left\{\zeta\in\mathbf{R}^n:2^{k-2}\le|\zeta|<2^{k+2}\right\},\quad\text{so that}\quad\chi_{\widetilde{\mathbf{R}}_k}=\sum_{j=-1}^{2}\chi_{k+j}.$$

Since $\mathbf{T}_\Omega^{\upsilon}$ is bounded on $L^{p(\cdot)}$, we deduce that

$$\left\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\widetilde{\mathbf{R}}_k}\right)\right\|_{L^{p(\cdot)}}\lesssim\|\Omega\|_{L^s(\mathbf{S}^{n-1})}\sum_{j=-1}^{2}\|\psi\,\chi_{k+j}\|_{L^{p(\cdot)}}.$$

Taking the ℓ^q-quasinorm and applying the quasi-triangle inequality, we obtain:

$$\begin{aligned}\left\{\sum_{k\ge1}2^{k\alpha_\infty q}\left\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\widetilde{\mathbf{R}}_k}\right)\right\|_{L^{p(\cdot)}}^q\right\}^{\frac1q}&\lesssim\|\Omega\|_{L^s(\mathbf{S}^{n-1})}\left\{\sum_{\ell\ge0}2^{\ell\alpha_\infty q}\|\psi\,\chi_\ell\|_{L^{p(\cdot)}}^q\right\}^{\frac1q}\\&\le C\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}(\mathbf{R}^n)}.\end{aligned}$$

Estimation of $\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}}\right)$:
Let $\varpi\in\mathbf{R}_k$, and note that for $\varsigma\in\mathbf{R}^n\setminus\mathbf{B}_{k+2}$, one has $\varsigma\in\mathbf{R}_j$ with $j\ge k+3$. Then

$$\left|\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}}\right)(\varpi)\right|\lesssim\sum_{j\geqslant k+3}\int_{\mathbf{R}_j}\frac{|\Omega(\frac{\varpi-\varsigma}{|\varpi-\varsigma|})|}{|\varpi-\varsigma|^n}e^{-\upsilon|\varpi-\varsigma|}|\psi(\varsigma)|\,d\varsigma.$$

Since $|\varpi-\varsigma|\gtrsim2^j$, we have

$$|\varpi-\varsigma|^{-n}\lesssim2^{-jn},\quad e^{-\upsilon|\varpi-\varsigma|}\lesssim e^{-\upsilon\,c\,2^j}.$$

Using Lemma 4 together with Hölder's inequality, it follows that:

$$\int_{\mathbf{R}_j} |\psi(\varsigma)|\,d\varsigma \lesssim 2^{\frac{jn}{p'_\infty}} \|\psi\,\chi_j\|_{L^{p(\cdot)}},$$

we get

$$\left|\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}}\right)(\varpi)\right| \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})} \sum_{j\geqslant k+3} 2^{-j\frac{n}{p_\infty}} e^{-\upsilon\,c\,2^j} \|\psi\,\chi_j\|_{L^{p(\cdot)}}.$$

Taking the $L^{p(\cdot)}$-norm over $\varpi \in \mathbf{R}_k$ and applying Lemma 4:

$$2^{k\alpha_\infty}\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}})\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})} \sum_{j\geqslant k+3} 2^{(k-j)\left(\alpha_\infty+\frac{n}{p_\infty}\right)} 2^{j\alpha_\infty} e^{-\upsilon\,c\,2^j} \|\psi\,\chi_j\|_{L^{p(\cdot)}}.$$

Finally, taking the ℓ^q-quasinorm over $k \geq 1$ and using Lemma 3:

$$\left\{\sum_{k\geqslant 1} 2^{k\alpha_\infty q}\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}})\|_{L^{p(\cdot)}}^q\right\}^{1/q} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})} \left\{\sum_{j\geqslant 1} 2^{j\alpha_\infty q}\|\psi\,\chi_j\|_{L^{p(\cdot)}}^q\right\}^{1/q}.$$

Hence,

$$\left\{\sum_{k\geqslant 1} 2^{k\alpha_\infty q}\|\chi_k\,\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_{k+2}})\|_{L^{p(\cdot)}}^q\right\}^{1/q} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}}.$$

Final estimate: By leveraging the boundedness of $\mathbf{T}_\Omega^{\upsilon}$ together with the localized nature of ψ, it follows that

$$\|(\mathbf{T}_\Omega^{\upsilon}\psi)\,\chi_{\mathbf{B}_0}\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}} \leq C\,\|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbf{R}^n)}.$$

Decompose

$$\psi = \psi\,\chi_{\mathbf{B}_0} + \psi\,\chi_{\mathbf{R}_1} + \psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_1},$$

and estimate pointwise for $\varpi \in \mathbf{B}_0$ via subadditivity:

$$|\mathbf{T}_\Omega^{\upsilon}\psi(\varpi)| \leq |\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{B}_0})(\varpi)| + |\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{R}_1})(\varpi)| + \left|\mathbf{T}_\Omega^{\upsilon}\left(\psi\,\chi_{\mathbf{R}^n\setminus\mathbf{B}_1}\right)(\varpi)\right|.$$

Contribution from the near-field: To estimate the part of the operator arising from points in the vicinity of ϖ, we apply the known boundedness of $\mathbf{T}_\Omega^{\upsilon}$ on $L^{p(\cdot)}$,

$$\|\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{B}_0})\,\chi_{\mathbf{B}_0}\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\,\chi_{\mathbf{B}_0}\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}}.$$

Near shell: Similarly,

$$\|\mathbf{T}_\Omega^{\upsilon}(\psi\,\chi_{\mathbf{R}_1})\,\chi_{\mathbf{B}_0}\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\,\chi_{\mathbf{R}_1}\|_{L^{p(\cdot)}} \lesssim \|\Omega\|_{L^s(\mathbf{S}^{n-1})}\,\|\psi\|_{\mathbf{K}^{\alpha_\infty}_{p(\cdot),q}}.$$

Far-field contribution: Consider $\varpi \in \mathbf{B}_0$ and $\varsigma \in \mathbf{R}_\mathrm{k} \subset \mathbf{R}^\mathfrak{n} \setminus \mathbf{B}_1$ with $\mathrm{k} \geq 2$. In this case, the distance between ϖ and ς satisfies $|\varpi - \varsigma| \gtrsim 2^\mathrm{k}$. Consequently, we deduce:

$$\left|\mathbf{T}_\Omega^\upsilon \left(\psi\, \chi_{\mathbf{R}^\mathfrak{n}\setminus\mathbf{B}_1}\right)(\varpi)\right| \lesssim \|\Omega\|_{\mathrm{L}^\mathrm{s}(\mathbf{S}^{\mathfrak{n}-1})} \sum_{\mathrm{k}\geq 2} 2^{-\mathrm{k}\mathfrak{n}} e^{-\upsilon\, \mathrm{c} 2^\mathrm{k}} \int_{\mathbf{R}_\mathrm{k}} |\psi(\varsigma)|\, \mathrm{d}\varsigma.$$

Using Lemma 4 together with Hölder's inequality, it follows that:

$$\int_{\mathbf{R}_\mathrm{k}} |\psi(\varsigma)|\, \mathrm{d}\varsigma \lesssim 2^{\frac{\mathrm{k}\mathfrak{n}}{\mathrm{p}'_\infty}} \|\psi\, \chi_\mathrm{k}\|_{\mathrm{L}^{\mathrm{p}(\cdot)}},$$

so

$$\left|\mathbf{T}_\Omega^\upsilon \left(\psi\, \chi_{\mathbf{R}^\mathfrak{n}\setminus\mathbf{B}_1}\right)(\varpi)\right| \lesssim \|\Omega\|_{\mathrm{L}^\mathrm{s}(\mathbf{S}^{\mathfrak{n}-1})} \sum_{\mathrm{k}\geq 2} 2^{-\mathrm{k}\frac{\mathfrak{n}}{\mathrm{p}_\infty}} e^{-\upsilon\, \mathrm{c} 2^\mathrm{k}} \|\psi\, \chi_\mathrm{k}\|_{\mathrm{L}^{\mathrm{p}(\cdot)}}.$$

Taking the $\mathrm{L}^{\mathrm{p}(\cdot)}$-norm over $\mathbf{B}_0$ gives

$$\left\|\mathbf{T}_\Omega^\upsilon \left(\psi\, \chi_{\mathbf{R}^\mathfrak{n}\setminus\mathbf{B}_1}\right) \chi_{\mathbf{B}_0}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \lesssim \|\Omega\|_{\mathrm{L}^\mathrm{s}(\mathbf{S}^{\mathfrak{n}-1})} \|\psi\|_{\mathbf{K}^{\alpha_\infty}_{\mathrm{p}(\cdot),\mathrm{q}}}.$$

Combining all three parts, we conclude

$$\|(\mathbf{T}_\Omega^\upsilon \psi)\, \chi_{\mathbf{B}_0}\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \lesssim \|\Omega\|_{\mathrm{L}^\mathrm{s}(\mathbf{S}^{\mathfrak{n}-1})} \|\psi\|_{\mathbf{K}^{\alpha_\infty}_{\mathrm{p}(\cdot),\mathrm{q}}} \leq \mathrm{C}\, \|\psi\|_{\mathbf{K}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}}(\mathbf{R}^\mathfrak{n})}.$$

This concludes the proof.

Remark 4. *Theorem 1 unifies and extends several existing results. If $\alpha(\cdot) = \alpha$, $\upsilon = 0$, and $\Omega \equiv 1$, it reduces to Theorem 3.3 in [52]. With the further assumption $\mathrm{p}(\cdot) = \mathrm{p}$, the result coincides with Theorem 2.1 in [53]. Moreover, for a constant kernel $\Omega \equiv 1$ and $\upsilon = 0$, it refines Theorem 4.2 and related results in [48].*

Theorem 2. *Let $0 < \alpha < \mathfrak{n}$ and $\gamma \geq 0$. Consider the variable exponents*

$$\mathrm{p} \in \mathbf{P}_0^{\log}(\mathbf{R}^\mathfrak{n}) \cap \mathbf{P}_\infty^{\log}(\mathbf{R}^\mathfrak{n}), \quad \mathrm{q} \in \mathbf{P}_0(\mathbf{R}^\mathfrak{n}),$$

satisfying the growth restriction

$$1 < \mathrm{p}^- \leq \mathrm{p}^+ < \frac{\mathfrak{n}}{\alpha}.$$

Assume that the function $\mathfrak{a} \in \mathrm{L}^\infty(\mathbf{R}^\mathfrak{n})$ is log-Hölder continuous at both the origin and at infinity, with bounds

$$\alpha - \frac{\mathfrak{n}}{\mathrm{p}^+} < \mathfrak{a}^- \leq \mathfrak{a}^+ < \mathfrak{n}\left(1 - \frac{1}{\mathrm{p}^-}\right),$$

ensuring compatibility with the scaling of the operator.

Let $\Omega \in \mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathfrak{n}-1})$ *for some* $1 < \mathrm{s} < \infty$ *such that* $\mathrm{s} > \mathrm{p}^{-}$, *and assume that* Ω *is homogeneous of degree zero. We define the generalized rough Bessel–Riesz operator with exponential decay by*

$$\mathbf{T}_{\alpha,\gamma,\Omega}\psi(\varpi) := \int_{\mathbf{R}^{\mathfrak{n}}} \frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{\big(1+|\varpi-\varsigma|\big)^{\gamma}}\, \Omega\Big(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\Big)\, \psi(\varsigma)\, \mathrm{d}\varsigma.$$

Suppose that $\mathbf{T}_{\alpha,\gamma,\Omega}$ *is bounded from the variable exponent Lebesgue space*

$$\mathrm{L}^{\mathrm{p}(\cdot)}\big(\mathbf{R}^{\mathfrak{n}}\big) \quad \textit{into} \quad \mathrm{L}^{\mathrm{p}^{*}(\cdot)}\big(\mathbf{R}^{\mathfrak{n}}\big),$$

where the associated Sobolev conjugate exponent is given, for each point ϖ, *by*

$$\frac{1}{\mathrm{p}^{*}(\varpi)} := \frac{1}{\mathrm{p}(\varpi)} - \frac{\alpha}{\mathfrak{n}}.$$

With these assumptions in place, the operator admits a bounded extension between the corresponding variable exponent Herz spaces:

$$\mathbf{T}_{\alpha,\gamma,\Omega} : \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}\big(\mathbf{R}^{\mathfrak{n}}\big) \longrightarrow \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}^{*}(\cdot),\mathrm{q}(\cdot)}\big(\mathbf{R}^{\mathfrak{n}}\big).$$

Proof. Based on Proposition 2, let us fix $\varpi \in \mathbf{R}_{\iota}$ for some $\iota < 0$. To analyze the operator effectively, we decompose the function ψ according to its spatial position with respect to the origin. Accordingly, the entire space $\mathbf{R}^{\mathfrak{n}}$ is divided into three non-overlapping regions, allowing separate treatment of contributions from near-field, intermediate, and far-field zones.

$$\mathbf{R}^{\mathfrak{n}} = \mathbf{B}_{\iota-2} \cup \widetilde{\mathbf{R}}_{\iota} \cup \left(\mathbf{R}^{\mathfrak{n}} \setminus \mathbf{B}_{\iota+2}\right),$$

where $\mathbf{B}_{r} = \{\varsigma \in \mathbf{R}^{\mathfrak{n}} : |\varsigma| < 2^{r}\}$ denotes the open ball of radius 2^{r} centered at the origin, and

$$\widetilde{\mathbf{R}}_{\iota} := \left\{\varsigma \in \mathbf{R}^{\mathfrak{n}} : 2^{\iota-2} \le |\varsigma| < 2^{\iota+2}\right\}$$

is an annulus.

This decomposition allows us to write

$$\psi(\varsigma) = \psi(\varsigma)\cdot\chi_{\mathbf{B}_{\iota-2}}(\varsigma) + \psi(\varsigma)\cdot\chi_{\widetilde{\mathbf{R}}_{\iota}}(\varsigma) + \psi(\varsigma)\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}(\varsigma).$$

Applying the linear operator $\mathbf{T}_{\alpha,\gamma,\Omega}$ (as defined in Theorem 2) and then taking its absolute value, we utilize the triangle inequality:

$$\begin{aligned} |\mathbf{T}_{\alpha,\gamma,\Omega}\psi(\varpi)| &= \left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}} + \psi\cdot\chi_{\widetilde{\mathbf{R}}_{\iota}} + \psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)(\varpi)\right| \\ &\le \left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right| + \left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\widetilde{\mathbf{R}}_{\iota}}\right)(\varpi)\right| + \left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)(\varpi)\right|. \end{aligned}$$

We now estimate the first term, $\left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right|$. For this, we further decompose the ball $\mathbf{B}_{\iota-2}$ into a countable union of disjoint dyadic annuli:

$$\mathbf{B}_{\iota-2} = \bigcup_{\varepsilon=-\infty}^{\iota-2} \mathbf{R}_{\varepsilon}, \quad \text{where} \quad \mathbf{R}_{\varepsilon} := \left\{\varsigma \in \mathbf{R}^{\mathfrak{n}} : 2^{\varepsilon} \le |\varsigma| < 2^{\varepsilon+1}\right\}.$$

Using this decomposition within the integral definition of the operator, and applying the triangle inequality, we obtain:

$$\begin{aligned}
&\left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right| \\
&=\left|\int_{\mathbf{B}_{\iota-2}}\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\,\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\psi(\varsigma)\,\mathrm{d}\varsigma\right| \\
&=\left|\sum_{\varepsilon=-\infty}^{\iota-2}\int_{\mathbf{R}_\varepsilon}\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\,\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\psi(\varsigma)\,\mathrm{d}\varsigma\right| \\
&\leq\sum_{\varepsilon=-\infty}^{\iota-2}\int_{\mathbf{R}_\varepsilon}\left|\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\right|\left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right||\psi(\varsigma)|\,\mathrm{d}\varsigma.
\end{aligned}$$

Since $\varpi\in\mathbf{R}_\iota$, $\varsigma\in\mathbf{R}_\varepsilon$, and $\varepsilon\leq\iota-2$, we have

$$|\varpi-\varsigma|\geq|\varpi|-|\varsigma|\gtrsim 2^{\iota}.$$

Thus, the integrand satisfies

$$\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\lesssim 2^{-\iota(\mathfrak{n}-\alpha+\gamma)}.$$

Hence,

$$\left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right|\lesssim 2^{-\iota(\mathfrak{n}-\alpha+\gamma)}\sum_{\varepsilon=-\infty}^{\iota-2}\int_{\mathbf{R}_\varepsilon}|\Omega(\cdot)|\cdot|\psi(\varsigma)|\,\mathrm{d}\varsigma.$$

From the previous estimate for $\left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right|$, and by Lemma 1, we multiply both sides by $2^{\iota\mathfrak{a}(0)}$:

$$2^{\iota\mathfrak{a}(0)}\left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right|\lesssim 2^{\iota\mathfrak{a}(0)}\sum_{\varepsilon=-\infty}^{\iota-2}\int_{\mathbf{R}_\varepsilon}\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\left|\Omega\Big(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\Big)\right||\psi(\varsigma)|\,\mathrm{d}\varsigma.$$

Since $\varpi\in\mathbf{R}_\iota$ and $\varsigma\in\mathbf{R}_\varepsilon\subset\mathbf{B}_{\iota-2}$, we have $|\varpi-\varsigma|\gtrsim 2^{\iota}$. This allows us to estimate the kernel terms:

$$\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}}\lesssim 2^{-\iota(\mathfrak{n}-\alpha+\gamma)}.$$

Due to the log-Hölder continuity of $\mathfrak{a}$, we obtain

$$2^{\iota\mathfrak{a}(0)}\leq \mathrm{C}'\,2^{(\iota-\varepsilon)\mathfrak{a}^{+}}\,2^{\varepsilon\mathfrak{a}(\varsigma)}.$$

Indeed, decompose

$$2^{\iota\mathfrak{a}(0)}=2^{(\iota-\varepsilon)\mathfrak{a}(0)}\,2^{\varepsilon\mathfrak{a}(\varsigma)}\,2^{\varepsilon(\mathfrak{a}(0)-\mathfrak{a}(\varsigma))}.$$

Since $\mathfrak{a}(0) \le \mathfrak{a}^+$, we have

$$2^{(\iota-\varepsilon)\mathfrak{a}(0)} \le 2^{(\iota-\varepsilon)\mathfrak{a}^+}.$$

Moreover, log-Hölder continuity at the origin implies

$$|\mathfrak{a}(0) - \mathfrak{a}(\varsigma)| \le \frac{c_{\log}(\mathfrak{a})}{\log\left(\mathrm{e} + \frac{1}{|\varsigma|}\right)} \le c_{\log}(\mathfrak{a}),$$

and therefore

$$2^{\varepsilon(\mathfrak{a}(0)-\mathfrak{a}(\varsigma))} \le 2^{\varepsilon c_{\log}(\mathfrak{a})} =: \mathrm{C}'.$$

Combining these bounds yields

$$2^{\iota\mathfrak{a}(0)} \le \mathrm{C}'\, 2^{(\iota-\varepsilon)\mathfrak{a}^+}\, 2^{\varepsilon\mathfrak{a}(\varsigma)},$$

which is the desired inequality.

$$\begin{aligned} 2^{\iota\mathfrak{a}(0)} \left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi \cdot \chi_{\mathbf{B}_{\iota-2}}\right)(\varpi)\right| &\lesssim \sum_{\varepsilon=-\infty}^{\iota-2} 2^{(\iota-\varepsilon)\mathfrak{a}^+} \cdot 2^{-\iota(\mathfrak{n}-\alpha+\gamma)} \\ &\quad \times \int_{\mathbf{R}_\varepsilon} \left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| 2^{\varepsilon\mathfrak{a}(\varsigma)} |\psi(\varsigma)|\, \mathrm{d}\varsigma. \end{aligned}$$

Next, we take the $\mathrm{L}^{\mathrm{p}^*(\cdot)}$-norm restricted to $\mathbf{R}_\iota$. Applying generalized Minkowski's inequality to move the outer $\mathrm{L}^{\mathrm{p}^*(\cdot)}$-norm inside the summation gives

$$\begin{aligned} &\left\|2^{\iota\mathfrak{a}(0)} \cdot \mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi \cdot \chi_{\mathbf{B}_{\iota-2}}\right) \cdot \chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}} \\ &\lesssim \sum_{\varepsilon=-\infty}^{\iota-2} 2^{(\iota-\varepsilon)\mathfrak{a}^+} \cdot 2^{-\iota(\mathfrak{n}-\alpha+\gamma)} \\ &\quad \times \left\|\left(\varpi \mapsto \int_{\mathbf{R}_\varepsilon} \left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| 2^{\varepsilon\mathfrak{a}(\varsigma)}|\psi(\varsigma)|\, \mathrm{d}\varsigma\right) \cdot \chi_\iota(\varpi)\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}. \end{aligned}$$

Now we apply the generalized Hölder inequality in variable exponent spaces. For each fixed $\varpi \in \mathbf{R}_\iota$, the integral over $\mathbf{R}_\varepsilon$ can be seen as the dual pairing of three functions:

$$\left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| \cdot \chi_\varepsilon(\varsigma), \quad 2^{\varepsilon\mathfrak{a}(\varsigma)}\psi(\varsigma) \cdot \chi_\varepsilon(\varsigma), \quad \text{and } \chi_\iota(\varpi).$$

By Hölder's inequality with exponents $\mathrm{p}(\cdot)$, $\mathrm{p}'(\cdot)$, and $\mathrm{p}^*(\cdot)$, we obtain

$$\begin{aligned} &\left\|\left(\varpi \mapsto \int_{\mathbf{R}_\varepsilon} \left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right| 2^{\varepsilon\mathfrak{a}(\varsigma)}|\psi(\varsigma)|\, \mathrm{d}\varsigma\right) \cdot \chi_\iota(\varpi)\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}} \\ &\lesssim \|\chi_\varepsilon\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}} \cdot \|2^{\varepsilon\mathfrak{a}(\cdot)}\psi \cdot \chi_\varepsilon\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \cdot \|\chi_\iota\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}. \end{aligned}$$

Here, the first factor arises from bounding the kernel part with χ_ε in the dual space $\mathrm{L}^{\mathrm{p}'(\cdot)}$, the second factor corresponds to the weighted function $2^{\varepsilon\mathfrak{a}(\cdot)}\psi$ localized to $\mathbf{R}_\varepsilon$, and the

third factor comes from the cutoff χ_ι ensuring restriction to $\mathbf{R}_\iota$. Thus, both applications of Hölder's inequality are consistent, and the exponent indices match the variable exponent framework.
Finally, invoking Lemma 4, we first estimate the auxiliary characteristic function. Since the condition $\mathrm{s} > \mathrm{p}'(\cdot)^-$ alone does not ensure that the decomposition

$$\frac{1}{\mathrm{p}'(\varpi)} = \frac{1}{\widetilde{\mathrm{p}}(\varpi)} + \frac{1}{\mathrm{s}}$$

is meaningful, we additionally assume that

$$\mathrm{s} > \mathrm{p}'^{+} := \operatorname*{ess\,sup}_{\varpi \in \mathbf{R}^{\mathfrak{n}}} \mathrm{p}'(\varpi),$$

which guarantees that

$$\frac{1}{\mathrm{p}'(\varpi)} - \frac{1}{\mathrm{s}} > 0 \quad \text{and hence} \quad \widetilde{\mathrm{p}}(\varpi) > 1.$$

Under this strengthened condition, the relation

$$\frac{1}{\mathrm{p}'(\varpi)} = \frac{1}{\widetilde{\mathrm{p}}(\varpi)} + \frac{1}{\mathrm{s}}, \qquad \varpi \in \mathbf{R}^{\mathfrak{n}},$$

holds, and thus by the generalized Hölder inequality (Lemma 2.1.4 in [51]) we obtain

$$\begin{aligned}
\|\chi_\varepsilon\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}} &\leq \mathrm{C}\, \|\chi_\varepsilon\|_{\mathrm{L}^{\mathrm{s}}} \cdot \|\chi_\varepsilon\|_{\mathrm{L}^{\widetilde{\mathrm{p}}(\cdot)}} \\
&= \mathrm{C}\, |\mathbf{R}_\varepsilon|^{1/\mathrm{s}} \cdot \|\chi_\varepsilon\|_{\mathrm{L}^{\widetilde{\mathrm{p}}(\cdot)}} \\
&\lesssim 2^{\varepsilon \mathfrak{n}/\mathrm{s}} \cdot \|\chi_\varepsilon\|_{\mathrm{L}^{\widetilde{\mathrm{p}}(\cdot)}}.
\end{aligned}$$

Here the first step uses Hölder's inequality with mixed exponents $\mathrm{p}'(\cdot), \widetilde{\mathrm{p}}(\cdot)$, and the constant exponent s; the second step uses $\|\chi_\varepsilon\|_{\mathrm{L}^{\mathrm{s}}} = |\mathbf{R}_\varepsilon|^{1/\mathrm{s}}$; and the last step relies on the dyadic scaling relation $|\mathbf{R}_\varepsilon| \simeq 2^{\varepsilon \mathfrak{n}}$.

Substituting this bound into the previous estimate for the operator term, we obtain the consistent inequality

$$\begin{aligned}
&\left\| 2^{\iota \mathfrak{a}(0)}\, \mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\big(\psi \cdot \chi_\varepsilon\big) \cdot \chi_\iota \right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}} \\
&\lesssim 2^{(\iota-\varepsilon)(\mathfrak{a}^+ - (\mathfrak{n}-\alpha+\gamma))} \cdot \|2^{\varepsilon \mathfrak{a}(\cdot)} \psi \cdot \chi_\varepsilon\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \cdot \|\chi_\iota\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}.
\end{aligned}$$

We proceed to estimate the contribution from the local region, specifically, we consider

$$|\mathbf{R}_\varepsilon|^{-\frac{1}{\mathrm{p}(\varpi_\varepsilon)}}\, |\mathbf{R}_\iota|^{\frac{1}{\mathrm{p}(\varpi_\iota)}} \leq \mathrm{C} \cdot 2^{(\iota-\varepsilon)\frac{\mathfrak{n}}{\mathrm{p}^-}},$$

For any $\varpi_\varepsilon \in \mathbf{R}_\varepsilon$ and $\varpi_\iota \in \mathbf{R}_\iota$ with $\varepsilon \leq \iota - 2$, the Hölder continuity of $\mathrm{p}(\cdot) \in \mathbf{P}^{\log}_0(\mathbf{R}^{\mathfrak{n}}) \cap \mathbf{P}^{\log}_\infty(\mathbf{R}^{\mathfrak{n}})$, which further ensures that the conjugate and Sobolev exponents satisfy $\mathrm{p}'(\cdot), \mathrm{p}^*(\cdot) \in \mathbf{P}^{\log}_0 \cap \mathbf{P}^{\log}_\infty$.

Exploiting this property together with the structural form of the operator $\mathbf{T}_{\alpha,\gamma,\Omega}$, we deduce:

$$
\begin{aligned}
&\left\|2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\\
&\lesssim\sum_{\varepsilon=-\infty}^{\iota-2}2^{(\iota-\varepsilon)\mathfrak{a}^+}\cdot2^{-\iota(\mathfrak{n}-\gamma)}\cdot\left\|\left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|\cdot2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}}\\
&\quad\cdot|\mathbf{R}_\varepsilon|^{-\frac{1}{\mathrm{p}(\varpi_\varepsilon)}}\cdot|\mathbf{R}_\iota|^{\frac{1}{\mathrm{p}(\varpi_\iota)}}\\
&\lesssim\sum_{\varepsilon=-\infty}^{\iota-2}2^{(\iota-\varepsilon)\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\cdot\left\|\left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|\cdot2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}}.
\end{aligned}\tag{3.3}
$$

Noting that

$$\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}<0.$$

At this stage, invoking Lemma 3 for the Bessel–Riesz-type operator $\mathbf{T}_{\alpha,\gamma,\Omega}$ yields

$$
\begin{aligned}
&\left\{\sum_{\iota=-\infty}^{-1}\left\|2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}^{\mathrm{q}(0)}\right\}^{\frac{1}{\mathrm{q}(0)}}\\
&\lesssim\left\{\sum_{\iota=-\infty}^{-1}\left(\sum_{\varepsilon=-\infty}^{\iota-2}2^{(\iota-\varepsilon)\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\cdot\left\|\left|\Omega\left(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|\cdot2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\right)^{\mathrm{q}(0)}\right\}^{\frac{1}{\mathrm{q}(0)}}\\
&\lesssim\left\{\sum_{\iota=-\infty}^{-1}\left\|2^{\iota\mathfrak{a}(0)}\psi\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}^{\mathrm{q}(0)}\right\}^{\frac{1}{\mathrm{q}(0)}}\\
&\lesssim\|\psi\|_{\mathbf{K}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}.
\end{aligned}
$$

Consider the estimation of $2^{\iota\mathfrak{a}_\infty}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\big(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\big)$ in the $\ell^{\mathrm{q}_\infty}_{>}$-norm. We proceed analogously to the earlier case, but with $2^{\iota\mathfrak{a}_\infty}$ in place of $2^{\iota\mathfrak{a}(0)}$. Here, the validity of the estimate is ensured by the log-Hölder continuity at infinity, which guarantees the existence of constants $c_{\log}(\mathfrak{a})>0$ and $\mathfrak{a}_\infty\in\mathbf{R}$ such that

$$|\mathfrak{a}(\varpi)-\mathfrak{a}_\infty|\le\frac{c_{\log}(\mathfrak{a})}{\log(\mathrm{e}+|\varpi|)},\qquad\varpi\in\mathbf{R}^{\mathfrak{n}}.$$

Hence, for large $|\varpi|$ (in particular when $\varpi\in\mathbf{B}_\iota$ with large ι), we obtain

$$\big|\mathfrak{a}_\infty-\mathfrak{a}(\varpi)\big|\le c_{\log}(\mathfrak{a}),\qquad2^{\iota(\mathfrak{a}_\infty-\mathfrak{a}(\varpi))}\le2^{\iota c_{\log}(\mathfrak{a})}=:\mathrm{C}'.$$

Therefore,

$$2^{\iota\mathfrak{a}_\infty}=2^{\iota\mathfrak{a}(\varpi)}\,2^{\iota(\mathfrak{a}_\infty-\mathfrak{a}(\varpi))}\le\mathrm{C}'\,2^{\iota\mathfrak{a}(\varpi)}.$$

This shows that $2^{\iota\mathfrak{a}_\infty}$ can be controlled by $2^{\iota\mathfrak{a}(\varpi)}$ up to a harmless multiplicative constant depending only on the log–Hölder constant. With this observation the subsequent estimate

$$\begin{aligned}&\left\|2^{\iota\mathfrak{a}_\infty}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^\mathfrak{n})}\\ &\lesssim\sum_{\varepsilon=-\infty}^{\iota-2}2^{(\iota-\varepsilon)\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\cdot\left\|\left|\Omega\left(\frac{\varpi-\varsigma}{|\varpi-\varsigma|}\right)\right|\cdot2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^\mathfrak{n})}\end{aligned}$$

follows exactly as in the case of the origin.

Fixing $\iota\geq0$ and adopting the convention that $\sum_{\varepsilon=1}^{\iota-2}\cdots=0$ if $\iota\in\{0,1,2\}$, we can apply Lemma 3 once more, leading to

$$\begin{aligned}&\left\{\sum_{\iota=0}^{\infty}\left\|2^{\iota\mathfrak{a}_\infty}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^\mathfrak{n})}^{\mathrm{q}_\infty}\right\}^{\frac{1}{\mathrm{q}_\infty}}\\ &\leq \mathrm{C}\left\{\sum_{\iota=0}^{\infty}\left(2^{\iota\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\cdot\sum_{\varepsilon=-\infty}^{0}2^{-\varepsilon\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\sup_{\varepsilon\leq0}2^{\varepsilon\mathfrak{a}(0)}\left\|\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^\mathfrak{n})}\right)^{\mathrm{q}_\infty}\right\}^{\frac{1}{\mathrm{q}_\infty}}\\ &\quad+\ \mathrm{C}\left\{\sum_{\iota=0}^{\infty}\left(\sum_{\varepsilon=1}^{\iota-2}2^{(\iota-\varepsilon)\left(\mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}\right)}\cdot\left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^\mathfrak{n})}\right)^{\mathrm{q}_\infty}\right\}^{\frac{1}{\mathrm{q}_\infty}}.\end{aligned}$$

$$\text{Since } \mathfrak{a}^+-\mathfrak{n}+\frac{\mathfrak{n}}{\mathrm{p}^-}<0,\ \text{ we conclude:}$$

$$\left\{\sum_{\iota=0}^{\infty}\left\|2^{\iota\mathfrak{a}_\infty}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{B}_{\iota-2}}\right)\cdot\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^\mathfrak{n})}^{\mathrm{q}_\infty}\right\}^{\frac{1}{\mathrm{q}_\infty}}\leq\mathrm{C}\left\|\psi\right\|_{\mathbf{K}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^\mathfrak{n})}.$$

In estimating the middle component,

$$\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right),$$

by using the boundedness of $\mathbf{T}_{\alpha,\gamma,\Omega}$ from $\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^\mathfrak{n})$ to $\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^\mathfrak{n})$, we obtain:

$$\begin{aligned}&\left\|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right)\right\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^\mathfrak{n})}\\ &\lesssim\left\|\left\{2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right)\cdot\chi_\iota\right\}\right\|_{\ell^{\mathrm{q}(0)}_{<}\left(\mathrm{L}^{\mathrm{p}^*(\cdot)}\right)}+\left\|\left\{2^{\iota\mathfrak{a}_\infty}\cdot\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right)\cdot\chi_\iota\right\}\right\|_{\ell^{\mathrm{q}_\infty}_{>}\left(\mathrm{L}^{\mathrm{p}^*(\cdot)}\right)}\\ &\lesssim\left\|\left\{2^{\iota\mathfrak{a}(0)}\cdot\left\|\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}}\right\}\right\|_{\ell^{\mathrm{q}(0)}_{<}}+\mathrm{C}\cdot\left\|\left\{2^{\iota\mathfrak{a}_\infty}\cdot\left\|\psi\cdot\chi_{\widetilde{\mathbf{R}}_\iota}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}}\right\}\right\|_{\ell^{\mathrm{q}_\infty}_{>}}\\ &\leq\mathrm{C}\left\|\psi\right\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^\mathfrak{n})}.\end{aligned}$$

Next, we turn to the estimate of the far-field component:

$$\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}}\setminus \mathbf{B}_{\iota+2}}\right), \qquad \varpi \in \mathbf{R}_\iota,\ \iota < 0.$$

Recalling the definition of $\mathbf{T}_{\alpha,\gamma,\Omega}$, we can express

$$\begin{aligned}
&2^{\iota\mathfrak{a}(0)}\left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}}\setminus \mathbf{B}_{\iota+2}}\right)(\varpi)\right| \\
&= 2^{\iota\mathfrak{a}(0)}\left|\int_{\mathbf{R}^{\mathfrak{n}}\setminus \mathbf{B}_{\iota+2}} \frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}} \left|\Omega\Big(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\Big)\right| |\psi(\varsigma)|\, \mathrm{d}\varsigma\right| \\
&= 2^{\iota\mathfrak{a}(0)} \sum_{\varepsilon=\iota+3}^{\infty} \int_{\mathbf{R}_\varepsilon} \frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}} \left|\Omega\Big(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\Big)\right| |\psi(\varsigma)|\, \mathrm{d}\varsigma.
\end{aligned}$$

For $\varpi \in \mathbf{R}_\iota$ and $\varsigma \in \mathbf{R}_\varepsilon$ with $\varepsilon \geq \iota+3$ we have

$$|\varpi-\varsigma| \geq |\varsigma| - |\varpi| \geq 2^{\varepsilon-1} - 2^{\iota} \gtrsim 2^{\varepsilon},$$

hence the kernel satisfies the pointwise bound

$$\frac{|\varpi-\varsigma|^{\alpha-\mathfrak{n}}}{(1+|\varpi-\varsigma|)^{\gamma}} \left|\Omega\Big(\tfrac{\varpi-\varsigma}{|\varpi-\varsigma|}\Big)\right| \lesssim \|\Omega\|_{\mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathfrak{n}-1})}\, 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)}.$$

Using the log–Hölder control of $\mathfrak{a}$ (cf. Lemma 1), we may compare weights and obtain

$$2^{\iota\mathfrak{a}(0)} \lesssim 2^{(\iota-\varepsilon)\mathfrak{a}_-}\, 2^{\varepsilon\mathfrak{a}(\varsigma)},$$

so that

$$\begin{aligned}
&2^{\iota\mathfrak{a}(0)}\left|\mathbf{T}_{\alpha,\gamma,\Omega}\left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}}\setminus \mathbf{B}_{\iota+2}}\right)(\varpi)\right| \\
&\lesssim \|\Omega\|_{\mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathfrak{n}-1})} \sum_{\varepsilon=\iota+3}^{\infty} 2^{(\iota-\varepsilon)\mathfrak{a}_-}\, 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)} \int_{\mathbf{R}_\varepsilon} 2^{\varepsilon\mathfrak{a}(\varsigma)}\, |\psi(\varsigma)|\, \mathrm{d}\varsigma.
\end{aligned}$$

Next take the $\mathrm{L}^{\mathrm{p}^*(\cdot)}$-norm over $\varpi \in \mathbf{R}_\iota$. By Hölder's inequality in variable Lebesgue spaces we have

$$\int_{\mathbf{R}_\varepsilon} 2^{\varepsilon\mathfrak{a}(\varsigma)}|\psi(\varsigma)|\, \mathrm{d}\varsigma \lesssim \left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\, \chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \cdot \left\|\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}}.$$

Therefore

$$\begin{aligned}
&\left\|2^{\iota\mathfrak{a}(0)} \cdot \mathbf{T}_{\alpha,\gamma,\Omega}\big(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}}\setminus \mathbf{B}_{\iota+2}}\big) \cdot \chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}} \\
&\lesssim \|\Omega\|_{\mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathfrak{n}-1})} \sum_{\varepsilon=\iota+3}^{\infty} 2^{(\iota-\varepsilon)\mathfrak{a}_-}\, 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)} \left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\, \chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}} \left\|\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}} \left\|\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}.
\end{aligned}$$

Finally, using Lemma 4 to control the characteristic norms $\left\|\chi_\varepsilon\right\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}}$ and $\left\|\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}$, and summing in ι (applying Lemma 3 where necessary), one obtains the expected Herz–space bound

$$\left\{ \sum_{\iota=-\infty}^{-1} \left\|2^{\iota\mathfrak{a}(0)}\mathbf{T}_{\alpha,\gamma,\Omega}(\psi\chi_{\mathbf{R}\setminus \mathbf{B}_{\iota+2}})\chi_\iota\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}}^{\mathrm{q}(0)} \right\}^{1/\mathrm{q}(0)} \lesssim \|\Omega\|_{\mathrm{L}^{\mathrm{s}}(\mathbf{S}^{\mathfrak{n}-1})}\, \|\psi\|_{\mathbf{K}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}}.$$

In order to control the tail component of the operator, $\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)$, for fixed $\varpi\in\mathbf{R}_{\iota}$, $\iota<0$, we write:

$$\begin{aligned}&2^{\iota\mathfrak{a}(0)}\left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)(\varpi)\right|\\ &\lesssim\|\Omega\|_{\mathrm{L}^{s}(\mathbb{S}^{\mathfrak{n}-1})}\cdot 2^{\iota\mathfrak{a}(0)}\sum_{\varepsilon=\iota+3}^{\infty}\int_{\mathbf{R}_{\varepsilon}}\frac{|\psi(\varsigma)|}{|\varpi-\varsigma|^{\mathfrak{n}-\alpha}(1+|\varpi-\varsigma|)^{\gamma}}\,\mathrm{d}\varsigma.\end{aligned}$$

Since $|\varpi-\varsigma|\geq 2^{\varepsilon-3}$ for $\varsigma\in\mathbf{R}_{\varepsilon}$, we estimate

$$\frac{1}{|\varpi-\varsigma|^{\mathfrak{n}-\alpha}(1+|\varpi-\varsigma|)^{\gamma}}\lesssim 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)},$$

and hence,

$$\begin{aligned}2^{\iota\mathfrak{a}(0)}\left|\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)(\varpi)\right|\lesssim\|\Omega\|_{\mathrm{L}^{s}(\mathbb{S}^{\mathfrak{n}-1})}&\sum_{\varepsilon=\iota+3}^{\infty}2^{(\iota-\varepsilon)\mathfrak{a}_-}\cdot 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)}\\ &\times\int_{\mathbf{R}_{\varepsilon}}2^{\varepsilon\mathfrak{a}(\varsigma)}\cdot|\psi(\varsigma)|\,\mathrm{d}\varsigma.\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}&\left\|2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)\cdot\chi_{\iota}\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\\ &\lesssim\|\Omega\|_{\mathrm{L}^{s}(\mathbb{S}^{\mathfrak{n}-1})}\sum_{\varepsilon=\iota+3}^{\infty}2^{(\iota-\varepsilon)\mathfrak{a}_-}\cdot 2^{-\varepsilon(\mathfrak{n}-\alpha+\gamma)}\left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_{\varepsilon}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\|\chi_{\varepsilon}\|_{\mathrm{L}^{\mathrm{p}'(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\|\chi_{\iota}\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}.\end{aligned}$$

Using the volume scaling relation

$$|\mathbf{R}_{\varepsilon}|^{-1/\mathrm{p}(\varsigma)}\cdot|\mathbf{R}_{\iota}|^{1/\mathrm{p}(\varpi)}\leq\mathrm{C}\cdot 2^{(\iota-\varepsilon)\frac{\mathfrak{n}}{\mathrm{p}_+}},$$

we get

$$\begin{aligned}&\left\|2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)\cdot\chi_{\iota}\right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\\ &\lesssim\|\Omega\|_{\mathrm{L}^{s}(\mathbb{S}^{\mathfrak{n}-1})}\sum_{\varepsilon=\iota+3}^{\infty}2^{(\iota-\varepsilon)\left(\mathfrak{a}_-+\frac{\mathfrak{n}}{\mathrm{p}_-}-\gamma\right)}\left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_{\varepsilon}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}.\end{aligned}$$

Observing that

$$\mathfrak{a}_-+\frac{\mathfrak{n}}{\mathrm{p}_-}-\gamma>0\quad\text{and}\quad 2^{\varepsilon\mathfrak{a}(\varsigma)}\approx 2^{\varepsilon\mathfrak{a}_\infty},\quad\text{for all }\varsigma\in\mathbf{R}_{\varepsilon},\ \varepsilon\geq 0,$$

and applying Lemma 3, we obtain

$$\begin{aligned}&\left\{\sum_{\iota=-\infty}^{-1}\left\|2^{\iota\mathfrak{a}(0)}\cdot\mathbf{T}^{\upsilon}_{\alpha,\gamma,\Omega}\left(\psi\cdot\chi_{\mathbf{R}^{\mathfrak{n}}\setminus\mathbf{B}_{\iota+2}}\right)\cdot\chi_{\iota}\right\|^{\mathrm{q}(0)}_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\right\}^{1/\mathrm{q}(0)}\\ &\lesssim\|\Omega\|_{\mathrm{L}^{s}(\mathbb{S}^{\mathfrak{n}-1})}\left\{\sum_{\iota=-\infty}^{-1}\left(\sum_{\varepsilon=\iota+3}^{\infty}2^{(\iota-\varepsilon)\left(\mathfrak{a}_-+\frac{\mathfrak{n}}{\mathrm{p}_+}-\gamma\right)}\cdot\left\|2^{\varepsilon\mathfrak{a}(\cdot)}\psi\cdot\chi_{\varepsilon}\right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}\right)^{\mathrm{q}(0)}\right\}^{1/\mathrm{q}(0)}.\end{aligned}$$

We proceed by estimating each sum separately. The first is directly absorbed through Lemma 3, from which we obtain

$$\left\{ \sum_{\iota=-\infty}^{-1} \left\| 2^{\iota \mathfrak{a}(0)} \psi \cdot \chi_\iota \right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})}^{\mathrm{q}(0)} \right\}^{1/\mathrm{q}(0)} .$$

Regarding the second term, employing the supremum and isolating the summable geometric decay yields

$$\left\{ \sum_{\iota=-\infty}^{-1} \left(2^{\iota\left(\mathfrak{a}_- + \frac{\mathfrak{n}}{\mathrm{p}_+} - \gamma\right)} \sum_{\varepsilon=0}^{\infty} 2^{-\varepsilon\left(\mathfrak{a}_- + \frac{\mathfrak{n}}{\mathrm{p}_+} - \gamma\right)} \cdot \sup_{\varepsilon \geq 0} 2^{\varepsilon \mathfrak{a}_\infty} \left\| \psi \cdot \chi_\varepsilon \right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} \right)^{\mathrm{q}(0)} \right\}^{1/\mathrm{q}(0)} .$$

By combining the two preceding estimates, we obtain

$$\left\| \mathbf{T}_{\alpha,\gamma,\Omega} \left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}} \setminus \mathbf{B}_{\iota+2}} \right) \right\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} \leq \mathrm{C} \left\| \psi \right\|_{\widetilde{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} .$$

To control the estimate

$$2^{\iota \mathfrak{a}_\infty} \cdot \mathbf{T}_{\alpha,\gamma,\Omega} \big(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}} \setminus \mathbf{B}_{\iota+2}} \big)$$

in the $\ell_{>}^{\mathrm{q}_\infty}$ norm, we proceed analogously to the argument in the local part, replacing the factor $2^{\iota \mathfrak{a}(0)}$ with $2^{\iota \mathfrak{a}_\infty}$. This leads to the estimate

$$\begin{aligned}
&\left\| 2^{\iota \mathfrak{a}_\infty} \cdot \mathbf{T}_{\alpha,\gamma,\Omega} \left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}} \setminus \mathbf{B}_{\iota+2}} \right) \cdot \chi_\iota \right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})} \\
&\lesssim \left\| \Omega \right\|_{\mathrm{L}^s(\mathbb{S}^{\mathfrak{n}-1})} \sum_{\varepsilon=\iota+3}^{\infty} 2^{(\iota-\varepsilon)\left(\mathfrak{a}_- + \frac{\mathfrak{n}}{\mathrm{p}_-} - \gamma\right)} \cdot \left\| 2^{\varepsilon \mathfrak{a}(\cdot)} \psi \cdot \chi_\varepsilon \right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} .
\end{aligned}$$

By once more invoking Lemma 3 together with the summability of the geometric series, we deduce:

$$\begin{aligned}
&\left\{ \sum_{\iota=0}^{\infty} \left\| 2^{\iota \mathfrak{a}_\infty} \cdot \mathbf{T}_{\alpha,\gamma,\Omega} \left(\psi \cdot \chi_{\mathbf{R}^{\mathfrak{n}} \setminus \mathbf{B}_{\iota+2}} \right) \cdot \chi_\iota \right\|_{\mathrm{L}^{\mathrm{p}^*(\cdot)}(\mathbf{R}^{\mathfrak{n}})}^{\mathrm{q}_\infty} \right\}^{1/\mathrm{q}_\infty} \\
&\lesssim \left\| \Omega \right\|_{\mathrm{L}^s(\mathbb{S}^{\mathfrak{n}-1})} \left\{ \sum_{\iota=0}^{\infty} \left(\sum_{\varepsilon=\iota+3}^{\infty} 2^{(\iota-\varepsilon)\left(\mathfrak{a}_- + \frac{\mathfrak{n}}{\mathrm{p}_+} - \gamma\right)} \cdot \left\| 2^{\varepsilon \mathfrak{a}(\cdot)} \psi \cdot \chi_\varepsilon \right\|_{\mathrm{L}^{\mathrm{p}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} \right)^{\mathrm{q}_\infty} \right\}^{1/\mathrm{q}_\infty} \\
&\leq \mathrm{C} \left\| \psi \right\|_{\dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathfrak{n}})} .
\end{aligned}$$

This concludes the argument and thereby establishes the validity of Theorem 2.

Remark 5. *Theorem 2 encompasses several known results as special cases and provides refined extensions. For instance, when* $\mathfrak{a}(\cdot) = \mathfrak{a}$, $\gamma = 0$, *and the variable exponents reduce to constants* $\mathrm{p}(\cdot) = \mathrm{p}$, $\mathrm{q}(\cdot) = \mathrm{q}$, *the statement sharpens Theorem 2.2 of [54]. Furthermore, if one additionally sets* $\Omega \equiv 1$, *the result specializes to a refinement of Theorem 2.3 in [48]. Thus, the present framework not only unifies but also strengthens earlier boundedness criteria in the literature.*

4. Regularity Framework for Delayed Two-Dimensional Parabolic Equations

In this section, we examine the regularity behavior of a nonlinear family of two-dimensional parabolic equations containing a time-delay component. The analysis is carried out by employing key boundedness results for fractional-type potential operators, which play a central role in both elliptic and parabolic regularity theory. Such operators are instrumental in characterizing qualitative properties of solutions and in establishing precise control over their local and global behavior.

Our methodology is motivated in part by the contributions of [50], where these techniques are shown to be particularly effective in addressing fine regularity phenomena. In addition, the developments in [29] on elliptic equations posed in Herz spaces with variable exponents, together with the insights obtained in [55] for problems formulated in Morrey-type spaces, provide a robust analytical foundation that supports the framework adopted in the present study.

To provide analytical context, we now discuss the related parabolic framework that forms the basis for studying regularity and well-posedness. The Stokes system, derived as a linearized form of the Navier–Stokes equations, captures the behavior of incompressible viscous fluids in regimes where inertial forces are negligible.

Let $\emptyset \neq \Omega \subset \mathbf{R}^{\mathrm{n}}$, where $\mathrm{n} \geq 3$, be a bounded domain, and define the temporal interval $\mathbf{T} := (0, \mathrm{T})$. Within this framework, we investigate the following initial-boundary value problem [56]:

We consider the following initial–boundary value problem of parabolic type:

$$\begin{cases} \dfrac{\partial \varpi}{\partial \mathrm{t}} - \mathbf{L}\varpi = \Psi, & \text{in } \Omega_{\mathbf{T}}, \\ \varpi = 0, & \text{on } \partial_{\mathrm{P}}\Omega_{\mathbf{T}}, \\ \varpi(0, \cdot) = \varpi_0, & \text{in } \Omega, \end{cases}$$

with data specified as

$$\Psi \in \mathrm{L}^{\mathrm{p}(\cdot)}(\Omega_{\mathbf{T}}), \qquad \varpi_0 \in \mathrm{L}^2(\Omega).$$

Here, the space–time cylinder is defined by $\Omega_{\mathbf{T}} := \Omega \times \mathbf{T}$, while $\partial_{\mathrm{P}}\Omega_{\mathbf{T}}$ denotes its parabolic boundary, incorporating both the spatial boundary of Ω and the initial time layer.

The evolution is driven by a second-order differential operator of parabolic type, denoted by $\mathbf{L}$, which acts on the spatial variables and governs the diffusion mechanism of the system.

$$\mathbf{L}\varpi := \sum_{\mathrm{i,j}=1}^{\mathrm{n}} \partial_{\varpi_{\mathrm{i}}} \left(\mathrm{a}^{\mathrm{ij}}(\varpi, \mathrm{t})\, \partial_{\varpi_{\mathrm{j}}} \varpi\right),$$

with the coefficient matrix $\left(\mathrm{a}^{\mathrm{ij}}\right)$ satisfying:

$$\mathrm{a}^{\mathrm{ij}} \in \mathrm{L}^{\infty}(\Omega_{\mathbf{T}}) \cap \mathrm{VMO}(\Omega_{\mathbf{T}}), \tag{1}$$

$$\mathrm{a}^{\mathrm{ij}} = \mathrm{a}^{\mathrm{ji}} \quad \text{(symmetry)}, \tag{2}$$

$$\nu^{-1}|\xi|^2 \leq \sum_{\mathrm{i,j}=1}^{\mathrm{n}} \mathrm{a}^{\mathrm{ij}}\xi_{\mathrm{i}}\xi_{\mathrm{j}} \leq \nu|\xi|^2, \quad \forall \xi \in \mathbf{R}^{\mathrm{n}}. \tag{3}$$

These ensure uniform ellipticity and regularity of the operator $\mathbf{L}$.

For a fixed point $(\varpi_0, \theta_0) \in \Omega_{\mathbf{T}}$, the associated frozen-coefficient operator

$$\mathbf{L}_0 \mathrm{v} := \partial_{\mathrm{t}} \mathrm{v} - \sum_{\mathrm{i,j}=1}^{\mathrm{n}} \mathrm{a}^{\mathrm{ij}}(\varpi_0, \theta_0)\, \partial_{\varpi_{\mathrm{i}}\varpi_{\mathrm{j}}} \mathrm{v}$$

admits the fundamental solution:

$$\Gamma((\varpi_0, \theta_0), (\tau, \theta)) := \frac{1}{(4\pi(\theta_0 - \theta))^{\mathrm{n}/2}\sqrt{\det \mathrm{A}(\varpi_0, \theta_0)}} \exp\left(-\frac{1}{4(\theta_0 - \theta)} \sum_{\mathrm{i,j}=1}^{\mathrm{n}} \mathrm{A}^{\mathrm{ij}}(\varpi_0, \theta_0)(\varpi_{\mathrm{i}} - \tau_{\mathrm{i}})(\varpi_{\mathrm{j}} - \tau_{\mathrm{j}})\right),$$

where $\left(\mathrm{A}^{\mathrm{ij}}\right)$ is the inverse of $\left(\mathrm{a}^{\mathrm{ij}}(\varpi_0, \theta_0)\right)$.

We define the spatial derivatives:

$$\Gamma_{\mathrm{i}} := \frac{\partial}{\partial \tau_{\mathrm{i}}}\Gamma, \quad \Gamma_{\mathrm{ij}} := \frac{\partial^2}{\partial \tau_{\mathrm{i}} \partial \tau_{\mathrm{j}}}\Gamma,$$

and introduce the bound

$$\mathbf{M} := \max_{\mathrm{i,j}} \max_{|\alpha| \leq 2\mathrm{n}} \left\| \frac{\partial^{|\alpha|}\Gamma_{\mathrm{ij}}}{\partial \tau^{\alpha}} \right\|_{\mathrm{L}^{\infty}(\Omega_{\mathbf{T}} \times \Sigma)},$$

where α stands for a index, and $\Sigma \subset \mathbf{R}^{\mathrm{n}} \times (0, \mathrm{T})$ is a compact subset chosen so as to exclude the singular point $(\varpi, \mathrm{t}) = (\tau, \theta)$.

Theorem 3. *Suppose that assumptions* (1)–(3) *are fulfilled, and the variable exponent mappings* $\mathrm{p}(\cdot), \mathrm{q}(\cdot)$, *and* $\alpha(\cdot)$ *comply with the structural requirements stated in Theorem 2. Under these hypotheses, one can find positive constants*

$$\mathrm{C} = \mathrm{C}(\mathrm{n}, \mathrm{p}(\cdot), \mathrm{q}(\cdot), \alpha(\cdot), \mathbf{M}), \quad \rho_0 = \rho_0(\mathrm{C}, \mathrm{n}),$$

such that for any ball $\mathbf{B}_{\mathrm{r}} \Subset \Omega$ *with* $\mathrm{r} < \rho_0$*, and any weak solution* $\mathrm{u} \in \mathbf{W}_0^{2,\mathrm{p}(\cdot)}(\mathbf{B}_{\mathrm{r}})$ *of the following incompressible Navier–Stokes system with delay:*

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial \mathrm{s}} - \mathrm{v}\Delta \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla \mathrm{p} = \psi(\mathrm{s}) + \Phi(\mathrm{s}, \mathrm{u}(\mathrm{s}+\theta)), & \textit{in } (\sigma, \mathrm{s}) \in \Omega \times (\iota, \mathrm{T}), \\ \operatorname{div} \mathrm{u} = 0, & \textit{in } (\sigma, \mathrm{s}) \in \Omega \times (\iota, \mathrm{T}), \\ \mathrm{u} = 0, & \textit{on } (\sigma, \mathrm{s}) \in \partial\Omega \times (\iota, \mathrm{T}), \\ \mathrm{u}(\sigma, \iota+\theta) = \phi(\sigma, \theta), & \sigma \in \Omega, \quad \theta \in (-\infty, 0], \end{cases}$$

where u *denotes the velocity field,* p *the pressure,* ψ *the external force,* $\mathrm{v} > 0$ *the viscosity coefficient, and* Φ *a prescribed delay-dependent function. Then the following a priori estimate is satisfied:*

$$\left\| \partial_{\varpi_{\mathrm{i}}\varpi_{\mathrm{j}}} \mathrm{u}^{\ell} \right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}(\mathbf{B}_{\mathrm{r}})} \leq \mathrm{C} \left\| \psi^{\ell} + \Phi^{\ell}(\cdot, \mathrm{u}(\cdot+\theta)) \right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{B}_{\mathrm{r}})}.$$

Here, the Sobolev conjugate exponent $\mathrm{p}^*(\cdot)$ *is defined by*

$$\frac{1}{\mathrm{p}^*(\varpi)} := \frac{1}{\mathrm{p}(\varpi)} - \frac{\alpha}{\mathrm{n}}.$$

Proof. Fix a ball $\mathbf{B}_{\mathrm{r}} \subset \Omega$ with $\mathrm{r} < \rho_0$, and let $\mathrm{u} \in \mathbf{W}_0^{2,\mathrm{p}(\cdot)}(\mathbf{B}_{\mathrm{r}})$ be a weak solution of the above system. The nonlinear term $(\mathrm{u}\cdot\nabla)\mathrm{u}$, together with the delay contribution $\Phi(\cdot, \mathrm{u}(\cdot+\theta))$, is incorporated into the effective forcing term $\psi + \Phi$.

Using the standard parametrix representation for the linearized Stokes system, the second-order derivatives of the velocity field u satisfy

$$\begin{aligned} \partial_{\varpi_{\mathrm{i}}\varpi_{\mathrm{j}}} \mathrm{u}^{\ell}(\varpi, \mathrm{s}) = \mathrm{P.V.} &\int_{\mathbf{B}_{\mathrm{r}}} \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta) \left(\psi^{\mathrm{n}}(\vartheta) + \Phi^{\mathrm{n}}(\vartheta, \mathrm{u}(\vartheta+\theta)) \right) \mathrm{d}\tau \\ &+ \mathrm{P.V.} \int_{\mathbf{B}_{\mathrm{r}}} \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta) \left[(\mathrm{u}\cdot\nabla)\mathrm{u}^{\mathrm{n}}(\varpi, \mathrm{s}) - (\mathrm{u}\cdot\nabla)\mathrm{u}^{\mathrm{n}}(\tau, \vartheta) \right] \mathrm{d}\tau \\ &+ \mathbb{S}^{\ell}(\varpi, \mathrm{s}), \end{aligned}$$

valid $\forall\ \mathrm{i}, \mathrm{j} \in \{1, \ldots, \mathrm{n}\}$.

The kernel $\mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta)$ arises as the second-order spatial derivative, with respect to ϖ, of the velocity component of the fundamental solution to the time-dependent Stokes system. It satisfies the parabolic Calderón–Zygmund estimate

$$\left| \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta) \right| \leq \frac{\mathrm{C}}{\rho^{\,\mathrm{n}+2}} = \frac{\mathrm{C}}{\left(|\varpi - \tau|^2 + \nu\, |\mathrm{s} - \vartheta| \right)^{(\mathrm{n}+2)/2}},$$

where

$$\rho^2 := |\varpi - \tau|^2 + \nu\, |\mathrm{s} - \vartheta|,$$

and $\nu > 0$ denotes the kinematic viscosity. In the steady-state case $\mathrm{s} = \vartheta$, this bound reduces to the classical Calderón–Zygmund singular kernel estimate

$$\left| \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau) \right| \leq \frac{\mathrm{C}}{|\varpi - \tau|^{\,\mathrm{n}}},$$

which has the same singular structure as the kernel of a standard singular integral operator (see Remark 2). If $\alpha = 0$, $\Omega \equiv 1$, and $\gamma = 0$, the operator $\mathbf{T}_{\alpha,\gamma,\Omega}$ reduces to

$$\mathbf{T}_{0,0,1} = \int_{\mathbf{R}^{\mathrm{n}}} \frac{\psi(\varsigma)}{|\varpi - \varsigma|^{\mathrm{n}}}\, \mathrm{d}\varsigma,$$

which has the same singularity profile as $\mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}$. Thus, $\mathbf{T}_{0,0,1}$ corresponds to the classical singular integral operator generated by a homogeneous kernel of order $-\mathrm{n}$, for which boundedness theory applies. In general, we introduce the fractional-type operator

$$\mathbf{T}_{\alpha,\gamma,\Omega}\psi(\varpi) := \int_{\mathbb{R}^{\mathrm{n}}} \frac{|\varpi - \varsigma|^{\alpha-\mathrm{n}}}{(1 + |\varpi - \varsigma|)^{\gamma}}\, \Omega\left(\frac{\varpi - \varsigma}{|\varpi - \varsigma|}\right) \psi(\varsigma)\, \mathrm{d}\varsigma,$$

which, by Theorem 2, yields the bounded mapping

$$\mathbf{T}_{\alpha,\gamma,\Omega} : \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathrm{n}}) \longrightarrow \dot{\mathbf{K}}^{\mathfrak{a}(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}(\mathbf{R}^{\mathrm{n}}).$$

Applied to the effective source term $\psi + \Phi$, this gives

$$\left\| \mathrm{P.V.} \int \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau)\big(\psi^{\mathrm{n}} + \Phi^{\mathrm{n}}\big)(\tau)\, \mathrm{d}\tau \right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}} \leq \mathrm{C}\, \big\|\psi + \Phi(\cdot, \mathrm{u}(\cdot + \theta))\big\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}}.$$

For the nonlinear contribution, assuming VMO-type regularity,

$$\sup_{|\varpi-\tau|<\delta} \left|\partial_{\xi_k}\big[(\mathrm{u}\cdot\nabla)\mathrm{u}\big]^{\mathrm{n}}(\varpi) - \partial_{\xi_k}\big[(\mathrm{u}\cdot\nabla)\mathrm{u}\big]^{\mathrm{n}}(\tau)\right| < \varepsilon,$$

we deduce

$$\left\| \mathrm{P.V.} \int \mathbb{K}_{\mathrm{ij}}^{\ell\mathrm{n}}(\varpi, \tau)\big((\mathrm{u}\cdot\nabla)\mathrm{u}^{\mathrm{n}}(\varpi) - (\mathrm{u}\cdot\nabla)\mathrm{u}^{\mathrm{n}}(\tau)\big)\, \mathrm{d}\tau \right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}} \leq \varepsilon\, \|\nabla \mathrm{u}\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}}.$$

This small term may be absorbed into the left-hand side for sufficiently small ε.

Finally, the term $\mathbb{S}^{\ell}(\varpi, \mathrm{s})$ is the remainder (or error) term. Crucially, the function $\mathbb{S}^{\ell}$ is expected to be smoother than the singular integral terms. The small radius $\mathrm{r} < \rho_0$ ensures that $\mathbb{S}^{\ell}$ behaves as a lower-order term. Specifically, since $\mathbb{S}^{\ell}$ is controlled by a non-singular integral operator acting on the force term, we have

$$\mathbb{S}^{\ell}(\varpi, \mathrm{s}) = \int_{\mathbf{B}_{\mathrm{r}}} \mathbb{H}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta)\Big(\psi^{\mathrm{n}}(\tau) + \Phi^{\mathrm{n}}(\tau, \mathrm{u}(\tau + \theta))\Big)\, \mathrm{d}\tau,$$

where the remainder kernel satisfies the uniform smooth estimate

$$\left|\mathbb{H}^{\ell\mathrm{n}}(\varpi, \tau; \mathrm{s}, \vartheta)\right| \leq \frac{\mathrm{C}}{\mathrm{r}^{\mathrm{n}}}.$$

Since $\mathbb{H}^{\ell\mathrm{n}}$ is bounded, it follows that

$$\left|\mathbb{S}^{\ell}(\varpi)\right| \leq \mathrm{C} \int_{\mathbf{B}_{\mathrm{r}}} \left|\psi(\tau) + \Phi(\tau, \mathrm{u}(\tau + \theta))\right| \mathrm{d}\tau.$$

Hence, $\mathbb{S}^{\ell}$ is controlled by a Hardy-type averaging operator:

$$\mathbb{S}^{\ell}(\varpi) \leq \mathrm{C}\,\mathbf{M}\Big(\psi + \Phi(\cdot, \mathrm{u}(\cdot + \theta))\Big)(\varpi),$$

where $\mathbf{M}$ denotes the Hardy–Littlewood maximal operator.

By Theorem 2, the maximal operator is bounded on the variable exponent Herz space:

$$\mathbf{M} : \dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)} \longrightarrow \dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}.$$

Using the embedding

$$\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)} \hookrightarrow \dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)} \qquad (\text{since } \mathrm{p}^*(\varpi) \leq \mathrm{p}(\varpi)),$$

we conclude that

$$\left\|\mathbb{S}^{\ell}\right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}^*(\cdot),\mathrm{q}(\cdot)}} \leq \mathrm{C}\left\|\psi + \Phi(\cdot, \mathrm{u}(\cdot + \theta))\right\|_{\dot{\mathbf{K}}^{\alpha(\cdot)}_{\mathrm{p}(\cdot),\mathrm{q}(\cdot)}}.$$

This uniform bound for the remainder term, combined with the singular integral estimates for $\mathbb{K}^{\ell\mathrm{n}}_{\mathrm{ij}}$, completes the a priori estimate.

5. Conclusions

In this work, we introduced and analyzed a new class of Bessel–Riesz type fractional potential operators with spatial roughness. Using a refined decomposition method, we established their boundedness in Herz-type spaces with variable growth, thereby unifying and extending classical results for Riesz and Bessel potentials.

We further demonstrated the applicability of this framework to the two-dimensional Navier–Stokes system with delay terms, where the finiteness of these operators plays a key role in establishing existence and regularity of mild solutions.

Several directions for future research remain open, including extensions to coupled nonlinear systems (e.g., MHD, Oldroyd-type fluids), development of numerical discretization schemes, investigation of endpoint estimates in Herz and Lorentz-Herz spaces, and applications to stochastic Navier–Stokes equations with fractional noise.

Acknowledgements

The authors would like to thank the anonymous reviewers and editors for their valuable suggestions and comments, which helped to improve the quality of this manuscript. The authors would like to acknowledge the financial support from the University Center for Exact Sciences and Engineering, University of Guadalajara.

References

[1] R. Temam. *Navier–Stokes Equations and Nonlinear Functional Analysis*. SIAM, 1995.

[2] H. Beirão da Veiga. Existence and asymptotic behavior for strong solutions of the navier–stokes equations in the whole space. *Indiana University Mathematics Journal*, 36(1):149–166, 1987.

[3] H. Triebel. *Hybrid Function Spaces, Heat and Navier–Stokes Equations*. European Mathematical Society, 2015.

[4] W. Afzal. Regularity of parabolic ornstein–uhlenbeck equations via boundedness of fractional muckenhoupt-type weighted singular operators in variable herz spaces. *Journal of Pseudo-Differential Operators and Applications*, 17(1):2, 2026.

[5] R. Niu, H. Zheng, and B. Zhang. Navier–stokes equations with variable viscosity in variable exponent spaces of clifford-valued functions. *Boundary Value Problems*, 2015:30, 2015.

[6] B. Zhang, Y. Fu, and V. D. Rădulescu. The stationary navier–stokes equations in variable exponent spaces of clifford-valued functions. *Advances in Applied Clifford Algebras*, 24(1):231–252, 2014.

[7] M. Z. Abidin and J. Chen. Global well-posedness for fractional navier–stokes equations in variable exponent fourier–besov–morrey spaces. *Acta Mathematica Scientia*, 41(1):164–176, 2021.

[8] R. Abdulkadirov and P. Lyakhov. Estimates of mild solutions of navier–stokes equations in weak herz-type besov–morrey spaces. *Mathematics*, 10(5):680, 2022.

[9] Y. Tsutsui. Navier–stokes equations and weak herz spaces. *Mathematische Annalen*. to appear.

[10] M. Cannone and G. Wu. Global well-posedness for navier–stokes equations in critical fourier–herz spaces. *Nonlinear Analysis*, 75(9):3754–3760, 2012.

[11] Y. Nie and X. Zheng. Ill-posedness of the 3d incompressible hyperdissipative navier–stokes system in critical fourier–herz spaces. *Nonlinearity*, 31(7):3115, 2018.

[12] L. G. Longen. *Boa-colocação e soluções assintoticamente autossimilares para as equações de Navier-Stokes em espaços de Herz fracos*. PhD thesis, Doctoral Dissertation, 2016.

[13] D. Iftimie. The 3d navier-stokes equations seen as a perturbation of the 2d navier-stokes equations. *Bulletin de la Société Mathématique de France*, 127(4):473–517, 1999.

[14] D. F. Machado. *Uma análise das equações de Navier-Stokes e Euler não-homogêneas em espaços do tipo Besov*. PhD thesis, Doctoral Dissertation, 2023.

[15] R. Danchin and P. B. Mucha. A lagrangian approach for the incompressible navier–stokes equations with variable density. *Communications on Pure and Applied Mathematics*, 65(10):1458–1480, 2012.

[16] U. Iotti. Uniform boundedness of homogeneous incompressible flows in $\mathbf{R}^3$. *arXiv preprint*, arXiv:2503.03991, 2025.

[17] H. Ge and F. Du. Global well-posedness and determining nodes of non-autonomous

navier–stokes equations with infinite delay on bounded domains. *Mathematics*, 13:222, 2025.

[18] Z. A. Khan, W. Afzal, M. Abbas, J. Ro, and N. M. Aloraini. A novel fractional approach to finding the upper bounds of simpson and hermite-hadamard-type inequalities in tensorial hilbert spaces by using differentiable convex mappings. *AIMS Mathematics*, 9(12):35151–35180, 2024.

[19] M. Tariq, A. Waqar, M. Nadeem, A. E. Muñoz Zavala, and J. E. Mac. Novel fractional hermite–hadamard and product-type inequalities via raina function and preinvex mappings with entropy applications. *European Journal of Pure and Applied Mathematics*, 18(3):6556–6556, 2025.

[20] Clemente Cesarano et al. Generalized special functions in the description of fractional diffusive equations. *Commun. Appl. Ind. Math*, 10:31–40, 2019.

[21] Mohammed Zakarya, Mohamed Altanji, Ghada AlNemer, Hoda A Abd El-Hamid, Clemente Cesarano, and Haytham M. Rezk. Fractional reverse coposn's inequalities via conformable calculus on time scales. *Symmetry*, 13(4):542, 2021.

[22] W. Afzal, M. Abbas, and O. M. Alsalami. Bounds of different integral operators in tensorial hilbert and variable exponent function spaces. *Mathematics*, 12(16):1–33, 2024.

[23] L. Ji and A. Weber. Dynamics of the heat semigroup on symmetric spaces. *Ergodic Theory and Dynamical Systems*, 30(2):457–468, 2010.

[24] W. Afzal, M. Abbas, D. Breaz, and L. I. Cotîrlă. Fractional hermite–hadamard, newton–milne, and convexity involving arithmetic–geometric mean-type inequalities in hilbert and mixed-norm morrey spaces $\ell^{q(\cdot)}(m_{p(\cdot),v(\cdot)})$ with variable exponents. *Fractal and Fractional*, 8(9), 2024.

[25] J. García-Luengo, P. Marín-Rubio, and J. Real. Pullback attractors for 2d navier-stokes equations with delays and their regularity. *Advanced Nonlinear Studies*, 13(2):331–357, 2013.

[26] H. O. Bae and B. J. Jin. Upper and lower bounds of temporal and spatial decays for the navier–stokes equations. *Journal of Differential Equations*, 209(2):365–391, 2005.

[27] D. Drihem. Composition operators on herz-type triebel–lizorkin spaces with application to semilinear parabolic equations. *Banach Journal of Mathematical Analysis*, 16(2):29, 2022.

[28] A. Fiorenza, M. R. Formica, and A. Gogatishvili. On grand and small lebesgue and sobolev spaces and some applications to pde's. *Differential Equations and Applications*, 10(1):21–46, 2018.

[29] A. Scapellato. Regularity of solutions to elliptic equations on herz spaces with variable exponents. *Boundary Value Problems*, 2019(1):2, 2019.

[30] H. Rafeiro and S. Samko. On embeddings of morrey type spaces between weighted lebesgue or stummel spaces with application to herz spaces. *Banach Journal of Mathematical Analysis*, 15(3):48, 2021.

[31] D. Drihem and F. Seghiri. Notes on the herz-type hardy spaces of variable smoothness and integrability. *Mathematical Inequalities and Applications*, 19(1):145–165, 2016.

[32] M. A. Ragusa. Parabolic herz spaces and their applications. *Applied Mathematics*

Letters, 25(10):1270–1273, 2012.

[33] Y. Zhao and J. Zhou. Hardy spaces associated with some anisotropic mixed-norm herz spaces and their applications. *Open Mathematics*, 21(1):20220599, 2023.

[34] Y. Li, D. Yang, and L. Huang. *Real-variable theory of Hardy spaces associated with generalized Herz spaces of Rafeiro and Samko*, volume 2320. Springer Nature, 2023.

[35] S. Shi, Z. Fu, and F. Zhao. Estimates for operators on weighted morrey spaces and their applications to nondivergence elliptic equations. *Journal of Inequalities and Applications*, 2013(1):390, 2013.

[36] O. A. Omer, K. Saibi, M. Z. Abidin, and M. Osman. Parametric marcinkiewicz integral and its higher-order commutators on variable exponents morrey-herz spaces. *Journal of Function Spaces*, 2022(1):7209977, 2022.

[37] Y. Almalki, W. Afzal, K. Shabbir, D. Breaz, L. I. Cotîrlă, and A. Albaity. New structural properties and hermite–hadamard inequalities for godunova–levin mappings via a novel analytical approach. *Research in Mathematics*, 12(1):2574096, 2025.

[38] T. Saeed, W. Afzal, K. Shabbir, S. Treanţă, and M. De la Sen. Some novel estimates of hermite–hadamard and jensen type inequalities for (h_1, h_2)-convex functions pertaining to total order relation. *Mathematics*, 10(24):4777, 2022.

[39] X. Zhang, K. Shabbir, W. Afzal, H. Xiao, and D. Lin. Hermite–hadamard and jensen-type inequalities via riemann integral operator for a generalized class of godunova–levin functions. *Journal of Mathematics*, 2022(1):3830324, 2022.

[40] W. Afzal, N. M. Aloraini, M. Abbas, J. S. Ro, and A. A. Zaagan. Hermite-hadamard, fejér and trapezoid type inequalities using godunova-levin preinvex functions via bhunia's order and with applications to quadrature formula and random variable. *Mathematical Biosciences and Engineering*, 21(2):3422–3447, 2024.

[41] W. Afzal, K. Shabbir, T. Botmart, S. Treantă, W. Afzal, K. Shabbir, and S. Treantă. Some new estimates of well known inequalities for (h_1, h_2)-godunova–levin functions by means of center-radius order relation. *AIMS Mathematics*, 8:3101–3119, 2023.

[42] W. Afzal, M. Abbas, J. E. Macías-Díaz, A. Gallegos, and Y. Almalki. Boundedness and sobolev-type estimates for the exponentially damped riesz potential with applications to the regularity theory of elliptic pdes. *Fractal and Fractional*, 9(7):458, 2025.

[43] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky, J. Wirth, and S. Tikhonov. *Variable Lebesgue spaces and hyperbolic systems*. Birkhäuser, 2014.

[44] D. Drihem and F. Seghiri. Notes on the herz-type hardy spaces of variable smoothness and integrability. *Mathematical Inequalities and Applications*, 19:145–165, 2016.

[45] S. Lu and D. Yang. The littlewood-paley function and transform characterizations of a new hardy space hk_2 associated with the herz space. *Studia Mathematica*, 101:285–298, 1992.

[46] M. Izuki. Herz and amalgam spaces with variable exponent, the haar wavelets and greediness of the wavelet system. *East Journal on Approximations*, 15:87–109, 2009.

[47] C. Herz. Lipschitz spaces and bernstein's theorem on absolutely convergent fourier transforms. *Journal of Mathematics and Mechanics*, 18:283–324, 1968.

[48] A. Almeida and D. Drihem. Maximal, potential and singular type operators on herz

spaces with variable exponents. *Journal of Mathematical Analysis and Applications*, 394(2):781–795, 2012.

[49] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko. *Integral Operators in Non-Standard Function Spaces: Volume 2: Variable Exponent Hölder, Morrey–Campanato and Grand Spaces*, volume 249. Birkhäuser, 2016.

[50] M. A. Ragusa. Homogeneous herz spaces and regularity results. *Nonlinear Analysis: Theory, Methods & Applications*, 71:e1909–e1914, 2009.

[51] B. M. Acyl, S. Tao, and O. Khalill. Boundedness of rough operators on grand variable herz spaces. *Applied Mathematics*, 12(7):614–626, 2021.

[52] M. Izuki. Boundedness of sublinear operators on herz spaces with variable exponent and application to wavelet characterization. *Analysis Mathematica*, 36:33–50, 2010.

[53] X. Li and D. Yang. Boundedness of some sublinear operators on herz spaces. *Illinois Journal of Mathematics*, 40(3):484–501, 1996.

[54] G. Hu, S. Lu, and D. Yang. Boundedness of rough singular integral operators on homogeneous herz spaces. *Journal of the Australian Mathematical Society*, 66(2):201–223, 1999.

[55] D. Makharadze, A. Meskhi, and M. A. Ragusa. Regularity results in grand variable exponent morrey spaces and applications. *Electronic Research Archive*, 33:5, 2025.

[56] J. Lewis, P. Lindqvist, J. J. Manfredi, and S. Salsa. *Regularity estimates for nonlinear elliptic and parabolic problems: Cetraro, Italy 2009*, volume 2045. Springer, 2012.