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A Novel Hybrid Approach to Convergence of Picard-S_n Iterations for Suzuki Generalized Nonexpansive Mappings

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Abstract. Recent investigations indicate that incorporating the Picard iteration into various iterative frameworks can significantly enhance convergence efficiency and approximation accuracy, as demonstrated in hybrid approaches like Picard–Noor, Picard–S, Picard–Ishikawa, and Picard–Mann. Building upon these insights, we introduce a novel iteration process that integrates the Picard method with the S_n scheme. The effectiveness of this Picard-S_n hybrid is demonstrated through a relevant numerical illustration. In addition, we establish strong as well as weak convergence theorems for Suzuki generalized nonexpansive mappings via this process. As a further application, the proposed approach is utilized to solve delay differential equations.

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Key Words and Phrases: Picard-S_n iteration, Suzuki generalized nonexpansive mappings, fixed point, convergence, delay differential equations

1. Introduction and Preliminary Results

Functional analysis provides a rigorous framework for studying and solving both linear and nonlinear problems arising in normed spaces that are not necessarily finite-dimensional. This scenario naturally occurs in many real-world problems. Fixed-point theory offers a valuable

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insight into the hidden stability of mathematical structures through points that remain unchanged during complex transformations. One of the most effective methods in contemporary mathematics is fixed point theory, particularly in topology, pure and applied analysis, and the theories of lattices, sets, and categories. Poincar'e first introduced the fixed point method in 1886 for the analysis of differential equations arising in celestial mechanics. The Banach and Brouwer theorems are two classical fixed point results that highlight the distinctions between the two primary branches of the theory, namely metric fixed point theory and topological fixed point theory, and remain among the most influential findings in the field. For over a century, fixed point theory has been a key component of nonlinear analysis and has been applied in a wide range of fields, including economics, control theory, optimization, and game theory.

The idea of Banach spaces, which offers a precise and organized framework for comprehending convergence, is the foundation of current research. Mathematicians can investigate the characteristics of linear operators and limits as well as the specifics of functional behavior using this paradigm. Banach spaces provide a framework for linear and nonlinear functional analysis, operator theory, abstract analysis, probability, optimization and other branches of mathematics. As research advances into nonlinear and fractional domains, Banach spaces continue to be a crucial point of reference that directs both theoretical comprehension and analytical advancement.

Let $\mathcal{D}$ be a normed linear space and let $K \subseteq \mathcal{D}$ be a nonempty, closed, and convex subset. A mapping $\mathcal{M} : K \to K$ is said to be a contraction on K if there exists a constant $\alpha \in [0, 1)$ such that

$$\|\mathcal{M}x - \mathcal{M}y\| \leq \alpha\|x - y\|, \quad \text{for all } x, y \in K. \tag{1.1}$$

The study of approximating solutions for nonlinear operator equations plays a fundamental role in numerical mathematics. Among the various approaches, fixed point iterative procedures are particularly prominent and have been extensively applied to such problems.

Fixed point theory traces its origins to the Banach Contraction Principle (BCP) [1] and the classical Picard iterative scheme [2]. Since then, the field has witnessed significant growth through the contributions of researchers including Browder [3], Gohde [4], Ishikawa [5], Karapinar [6], Khatoon et al. [7], and Schu [8], who extended these ideas into broader settings.

In 2008, Suzuki [9] introduced a new class of mappings, called *generalized nonexpansive mappings*, characterized by the so-called *condition (C)*. A mapping $\mathcal{M} : K \to K$ is said to satisfy condition (C) if, for all $x, y \in K$,

$$\frac{1}{2}\|x - \mathcal{M}x\| \leq \|x - y\| \quad \Longrightarrow \quad \|\mathcal{M}x - \mathcal{M}y\| \leq \|x - y\|.$$

Suzuki proved that this class lies strictly between the classes of nonexpansive and quasi-nonexpansive mappings.

Since then, extensive research has been carried out on fixed point theory for mappings satisfying condition (C); see, for example, [10] and the references therein.

We now recall several fundamental properties of Suzuki generalized nonexpansive mappings that will be used throughout this work.

Lemma 1 ([9]). *Let $\mathcal{D}$ be a uniformly convex Banach (UCB) space and let $K \subseteq \mathcal{D}$ be a nonempty, closed, and convex subset. Suppose that $\mathcal{M} : K \to K$. Then the following statements hold:*

(i) Every nonexpansive mapping is Suzuki generalized nonexpansive.

(ii) If $\mathcal{M}$ is Suzuki generalized nonexpansive and admits a fixed point, then $\mathcal{M}$ is quasi-nonexpansive.

(iii) If $\mathcal{M}$ is Suzuki generalized nonexpansive, then for all $x, y \in K$,

$$\|x - \mathcal{M}y\| \leq 3\|\mathcal{M}x - x\| + \|x - y\|.$$

Lemma 2 ([9]). *Let K be a subset of a Banach space $\mathcal{D}$ having the Opial property, and let $\mathcal{M} : K \to K$ satisfy condition (C). If a sequence $\{x_n\} \subseteq K$ converges weakly to z and*

$$\lim_{n\to\infty} \|\mathcal{M}x_n - x_n\| = 0,$$

then z is a fixed point of $\mathcal{M}$, that is, $\mathcal{M}z = z$.

Lemma 3 ([9]). *If $\mathcal{M}$ is a mapping defined on a subset K of a Banach space $\mathcal{D}$ and satisfies condition (C), then $\mathcal{M}$ has at least one fixed point.*

Lemma 4 ([9]). *Let $\mathcal{D}$ be a uniformly convex Banach space, and let $\{m_n\} \subset (0,1)$ satisfy $0 < a \leq m_n \leq b < 1$ for all $n \geq 1$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{D}$ satisfy*

$$\limsup_{n\to\infty} \|x_n\| \leq c, \qquad \limsup_{n\to\infty} \|y_n\| \leq c,$$

and

$$\lim_{n\to\infty} \|m_n x_n + (1-m_n)y_n\| = c$$

for some $c \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Definition 1. *Let $\mathcal{D}$ be a Banach space and let $\{x_n\} \subseteq \mathcal{D}$ be a bounded sequence. For a nonempty, closed, and convex subset $K \subseteq \mathcal{D}$, the* asymptotic radius *of $\{x_n\}$ with respect to K is defined by*

$$r(K, \{x_n\}) = \inf_{s_0 \in K} \left\{ \limsup_{n\to\infty} \|x_n - s_0\| \right\}.$$

The asymptotic center *of $\{x_n\}$ relative to K is given by*

$$\mathcal{A}(K, \{x_n\}) = \left\{ s_0 \in K : \limsup_{n\to\infty} \|x_n - s_0\| = r(K, \{x_n\}) \right\}.$$

Definition 2 ([8]). *A mapping $\mathcal{M} : K \to K$ is said to satisfy* condition (I) *if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$\|x - \mathcal{M}x\| \geq f(d(x, F_{\mathcal{M}})), \quad \forall\, x \in K,$$

where $d(x, F_{\mathcal{M}}) = \inf_{s_0 \in F_{\mathcal{M}}} \|x - s_0\|$.

Remark 1. *If $\mathcal{D}$ is a uniformly convex Banach space, then the asymptotic center $\mathcal{A}(K, \{x_n\})$ consists of a unique point (see [11]). Moreover, if K is convex and weakly compact, then $\mathcal{A}(K, \{x_n\})$ is also convex (see [12, 13]).*

Definition 3 ([14]). *A Banach space $\mathcal{D}$ is said to satisfy* Opial's condition *if, for every sequence $\{x_n\} \subseteq \mathcal{D}$ that converges weakly to $s_0 \in \mathcal{D}$, one has*

$$\limsup_{n\to\infty} \|x_n - s_0\| < \limsup_{n\to\infty} \|x_n - e_0\|, \quad \forall\, e_0 \in \mathcal{D} \setminus \{s_0\}.$$

The numerical treatment of nonlinear operators remains an active and important area of mathematical analysis. Obtaining fixed point approximations typically requires effective iteration schemes. Several such algorithms have been developed by Ullah [15, 16], Ahmad [17], Noor [18], Mann [19], Abbas [20], and Kanayo Stella and Hudson [21], all aimed at approximating fixed points of generalized nonexpansive mappings.

Several classical and modern iterative processes are outlined below.

1.1. Established Iterative Schemes

Agarwal's two-step procedure [22]:

$$\begin{cases} x_0 \in K, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{M}x_n, \\ x_{n+1} = (1-\alpha_n)\mathcal{M}x_n + \alpha_n \mathcal{M}v_n. \end{cases} \tag{1.2}$$

Noor's three-step iteration [18]:

$$\begin{cases} x_0 \in K, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{M}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{M}w_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{M}v_n. \end{cases} \tag{1.3}$$

The Picard–Noor iteration [21]:

$$\begin{cases} x_0 \in K, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{M}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n \mathcal{M}w_n, \\ u_n = (1-\alpha_n)x_n + \alpha_n \mathcal{M}v_n, \\ x_{n+1} = \mathcal{M}u_n. \end{cases} \tag{1.4}$$

The S_n process [23]:

$$\begin{cases} x_0 \in K, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{M}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n w_n, \\ x_{n+1} = (1-\alpha_n)\mathcal{M}v_n + \alpha_n \mathcal{M}w_n. \end{cases} \tag{1.5}$$

1.2. The Proposed Hybrid Scheme

In the present study, we introduce the **Picard-S_n hybrid iteration**, which combines the multi-step structure of the S_n process with a Picard-type acceleration step:

$$\begin{cases} x_0 \in K, \\ w_n = (1-\gamma_n)x_n + \gamma_n \mathcal{M}x_n, \\ v_n = (1-\beta_n)x_n + \beta_n w_n, \\ u_n = (1-\alpha_n)\mathcal{M}v_n + \alpha_n \mathcal{M}w_n, \\ x_{n+1} = \mathcal{M}u_n. \end{cases} \tag{1.6}$$

Highlights of our contributions:

- **Novel Iterative Framework:** We introduce the Picard-S_n iteration process, a novel hybrid scheme that integrates Picard iteration into the S_n framework to enhance convergence efficiency.

- **Theoretical Foundation:** We establish rigorous strong and weak convergence theorems for the class of Suzuki generalized nonexpansive mappings.

- **Computational Efficiency:** Through comparative numerical experiments, we demonstrate that the proposed hybrid process converges faster than several well-known existing iterative methods.

- **Practical Application:** We showcase the real-world utility of the proposed scheme by applying it to solve a delay differential equation.

The main goal of this work is to present a new iterative procedure that successfully combines the Picard method and the S_n scheme to improve convergence in abstract spaces. We will establish both strong and weak convergence results using the hybrid approach to Suzuki generalized nonexpansive mappings. This work also covers the practical implementation of the proposed scheme for delay differential equations, supported by numerical examples. We aim to provide theoretical analysis and computational examples to offer nonlinear analysis researchers and practitioners a dependable and flexible approach.

The organization of the paper is as follows. Section 1 provides a comprehensive introduction and outlines the motivation behind the study; we also revisit key concepts and definitions that lay the groundwork for our analysis. In Section 2, we present the main contributions. In Section 3, we present applications to delay differential equations. Finally, Section 4 gives concluding remarks and suggests some possible future research directions.

2. The Main Results

In this section, we establish convergence results for Suzuki-type generalized nonexpansive mappings in uniformly convex Banach spaces using the proposed Picard–S_n iteration scheme. It is worth emphasizing that the Picard–S_n process exhibits a faster rate of convergence than several well-known iterative methods, including the Abbas, S, Picard–Noor, S_n, Noor, Ishikawa, Agarwal, and Mann iterations.

Lemma 5. *Let $\mathcal{D}$ be a uniformly convex Banach (UCB) space, and let $K \subseteq \mathcal{D}$ be nonempty, closed, and convex. Suppose that $\mathcal{M} : K \to K$ satisfies condition (C), and that $F_{\mathcal{M}} \neq \emptyset$. Let $\{x_n\}$ be generated via the Picard-S_n process given in* (1.6)*. Then, for every $s_0 \in F_{\mathcal{M}}$, the limit $\lim_{n\to\infty} \|x_n - s_0\|$ exists.*

Proof. Select any $s_0 \in F_{\mathcal{M}}$ and fix an arbitrary $z \in K$. Since $\mathcal{M}$ fulfills condition (C), we have

$$\frac{1}{2}\|s_0 - \mathcal{M}s_0\| = 0 \leq \|s_0 - z\| \quad \Rightarrow \quad \|\mathcal{M}s_0 - \mathcal{M}z\| \leq \|s_0 - z\|.$$

By Lemma 1(ii) and the iteration (1.6), we obtain

$$\begin{aligned}
\|w_n - s_0\| &= \|(1-\gamma_n)x_n + \gamma_n \mathcal{M}x_n - s_0\| \\
&= \|(1-\gamma_n)(x_n - s_0) + \gamma_n(\mathcal{M}x_n - s_0)\| \\
&\leq (1-\gamma_n)\|x_n - s_0\| + \gamma_n\|x_n - s_0\| \\
&= \|x_n - s_0\|. \qquad (2.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|v_n - s_0\| &= \|(1-\beta_n)x_n + \beta_n w_n - s_0\| \\
&= \|(1-\beta_n)(x_n - s_0) + \beta_n(w_n - s_0)\| \\
&\leq (1-\beta_n)\|x_n - s_0\| + \beta_n\|w_n - s_0\| \\
&\leq \|x_n - s_0\|. \qquad (2.2)
\end{aligned}$$

For the third step,

$$\begin{aligned}
\|u_n - s_0\| &= \|(1-\alpha_n)\mathcal{M}v_n + \alpha_n \mathcal{M}w_n - s_0\| \\
&= \|(1-\alpha_n)(\mathcal{M}v_n - s_0) + \alpha_n(\mathcal{M}w_n - s_0)\| \\
&\leq (1-\alpha_n)\|\mathcal{M}v_n - s_0\| + \alpha_n\|\mathcal{M}w_n - s_0\|
\end{aligned}$$

$$\begin{aligned} &\le (1-\alpha_n)\|v_n - s_0\| + \alpha_n\|w_n - s_0\| \\ &\le \|x_n - s_0\|. \end{aligned} \tag{2.3}$$

Consequently,

$$\begin{aligned} \|x_{n+1} - s_0\| &= \|\mathcal{M}u_n - s_0\| \\ &\le \|u_n - s_0\| \le \|x_n - s_0\|. \end{aligned} \tag{2.4}$$

From (2.1)–(2.4), it follows that $\{\|x_n - s_0\|\}$ is nonincreasing and bounded below, which implies convergence. Thus, the limit $\lim_{n\to\infty}\|x_n - s_0\|$ exists for every $s_0 \in F_{\mathcal{M}}$.

Theorem 1. *Let $\mathcal{D}$ be a uniformly convex Banach space, and let $K \subseteq \mathcal{D}$ be nonempty, closed, and convex. Suppose that $\mathcal{M} : K \to K$ satisfies condition (C), and let $\{x_n\}$ be generated by the Picard-S_n iteration* (1.6) *with an arbitrary initial point $x_0 \in K$. Then, $F_{\mathcal{M}} \neq \emptyset$ if and only if the sequence $\{x_n\}$ is bounded and*

$$\lim_{n\to\infty}\|x_n - \mathcal{M}x_n\| = 0.$$

Proof. Assume first that $F_{\mathcal{M}} \neq \emptyset$ and choose $s_0 \in F_{\mathcal{M}}$. By Lemma 5, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - s_0\|$ exists. Denote

$$\lim_{n\to\infty}\|x_n - s_0\| = e. \tag{2.5}$$

We aim to prove that $\lim_{n\to\infty}\|x_n - \mathcal{M}x_n\| = 0$.

From (2.1),

$$\begin{aligned} &\|w_n - s_0\| \le \|x_n - s_0\| \\ &\Rightarrow \limsup_{n\to\infty}\|w_n - s_0\| \le \limsup_{n\to\infty}\|x_n - s_0\| = e. \end{aligned} \tag{2.6}$$

Moreover, since $s_0 \in F_{\mathcal{M}}$, Lemma 1(ii) gives

$$\begin{aligned} &\|\mathcal{M}x_n - s_0\| \le \|x_n - s_0\| \\ &\Rightarrow \limsup_{n\to\infty}\|\mathcal{M}x_n - s_0\| \le \limsup_{n\to\infty}\|x_n - s_0\|. \end{aligned} \tag{2.7}$$

From (2.4) we also have $\|x_{n+1} - s_0\| \le \|w_n - s_0\|$. Combining this with (2.5) gives

$$e \le \liminf_{n\to\infty}\|w_n - s_0\|. \tag{2.8}$$

By (2.6) and (2.8), we obtain

$$\lim_{n\to\infty}\|w_n - s_0\| = e. \tag{2.9}$$

Observe that

$$\|w_n - s_0\| = \|(1-\gamma_n)(x_n - s_0) + \gamma_n(\mathcal{M}x_n - s_0)\|.$$

Substituting into (2.9) leads to

$$e = \lim_{n\to\infty}\|(1-\gamma_n)(x_n - s_0) + \gamma_n(\mathcal{M}x_n - s_0)\|. \tag{2.10}$$

Now, using (2.5), (2.7), and (2.10) in conjunction with Lemma 4, we deduce

$$\lim_{n\to\infty}\|x_n - \mathcal{M}x_n\| = 0.$$

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - \mathcal{M}x_n\| = 0$. We claim that $F_{\mathcal{M}} \neq \emptyset$. Let $s_0 \in \mathcal{A}(K, \{x_n\})$. By Lemma 1(iii),

$$r(\mathcal{M}s_0, \{x_n\}) = \limsup_{n\to\infty}\|x_n - \mathcal{M}s_0\|$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty} \left(3\|\mathcal{M}x_n - x_n\| + \|x_n - s_0\|\right) \\
&= \limsup_{n\to\infty} \|x_n - s_0\| = r(s_0, \{x_n\}).
\end{aligned}
$$

Therefore, $\mathcal{M}s_0 \in \mathcal{A}(K, \{x_n\})$. Since the asymptotic center is unique, $\mathcal{M}s_0 = s_0$, so $s_0 \in F_{\mathcal{M}}$. This proves $F_{\mathcal{M}} \neq \emptyset$.

Theorem 2. *Let $\mathcal{D}$ be a uniformly convex Banach (UCB) space, and let $K \subseteq \mathcal{D}$ be nonempty, closed, and convex. Suppose $\mathcal{M} : K \to K$ satisfies condition (C), and let $\{x_n\}$ be generated by the Picard-S_n iteration* (1.6) *from some $x_0 \in K$. If $F_{\mathcal{M}} \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of $\mathcal{M}$.*

Proof. Since $\mathcal{D}$ is uniformly convex, Theorem 1 ensures that $\{x_n\}$ is bounded. By the Banach–Alaoglu theorem, there exists a subsequence $\{x_{n_m}\}$ converging weakly to some $x_0 \in K$. Furthermore, Theorem 1 gives

$$\lim_{m\to\infty} \|x_{n_m} - \mathcal{M}x_{n_m}\| = 0.$$

By Lemma 2, x_0 is a fixed point of $\mathcal{M}$, i.e., $x_0 \in F_{\mathcal{M}}$.

To show uniqueness of the weak limit, suppose for contradiction that another subsequence $\{x_{n_s}\}$ converges weakly to some $x_0' \neq x_0$. Again, Theorem 1 implies $\lim_{s\to\infty} \|x_{n_s} - \mathcal{M}x_{n_s}\| = 0$, and Lemma 2 yields $x_0' \in F_{\mathcal{M}}$.

Applying Opial's condition and Lemma 5, we have

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - x_0\| &= \lim_{m\to\infty} \|x_{n_m} - x_0\| \\
&< \lim_{m\to\infty} \|x_{n_m} - x_0'\| \\
&= \lim_{n\to\infty} \|x_n - x_0'\| \\
&= \lim_{s\to\infty} \|x_{n_s} - x_0'\| \\
&< \lim_{s\to\infty} \|x_{n_s} - x_0\| \\
&= \lim_{n\to\infty} \|x_n - x_0\|,
\end{aligned}
$$

which gives the contradiction $\lim_{n\to\infty} \|x_n - x_0\| < \lim_{n\to\infty} \|x_n - x_0\|$.

Hence $x_0 = x_0'$, and the entire sequence $\{x_n\}$ converges weakly to a fixed point of $\mathcal{M}$.

Theorem 3. *Let $\mathcal{D}$ be a uniformly convex Banach (UCB) space, and let $K \subseteq \mathcal{D}$ be nonempty, closed, and convex. Suppose that $\mathcal{M} : K \to K$ satisfies condition (C), and let $\{x_n\}$ be produced by the Picard-S_n iteration* (1.6) *from any starting point $x_0 \in K$. If $F_{\mathcal{M}} \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{M}$.*

Proof. Since $F_{\mathcal{M}} \neq \emptyset$, Lemma 3 applies. From Theorem 1 we have $\lim_{n\to\infty} \|x_n - \mathcal{M}x_n\| = 0$. Because K is convex and compact, the sequence $\{x_n\} \subseteq K$ possesses a subsequence $\{x_{n_m}\}$ that converges strongly to some $s_0 \in \mathcal{D}$, i.e., $\lim_{m\to\infty} \|x_{n_m} - s_0\| = 0$. Applying Lemma 1(iii) yields

$$\|x_{n_m} - \mathcal{M}s_0\| \leq 3\|\mathcal{M}x_{n_m} - x_{n_m}\| + \|x_{n_m} - s_0\|. \tag{2.11}$$

From Theorem 1, $\lim_{m\to\infty} \|x_{n_m} - \mathcal{M}x_{n_m}\| = 0$. Taking the limit in (2.11) gives $\lim_{m\to\infty} \|x_{n_m} - \mathcal{M}s_0\| = 0$, hence $\mathcal{M}s_0 = s_0$, so $s_0 \in F_{\mathcal{M}}$.

Furthermore, Lemma 5 guarantees that $\{\|x_n - s_0\|\}$ converges. This implies $x_n \to s_0$ in norm, establishing strong convergence to a fixed point of $\mathcal{M}$.

Theorem 4. *Let $\mathcal{D}$ be a uniformly convex Banach (UCB) space, and let $K \subseteq \mathcal{D}$ be nonempty, closed, and convex. Suppose that $\mathcal{M} : K \to K$ satisfies condition (C) and condition (I), and let $\{x_n\}$ be generated by the Picard-S_n iteration* (1.6) *from any $x_0 \in K$. If $F_{\mathcal{M}} \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{M}$.*

Proof. Since $F_{\mathcal{M}} \neq \emptyset$, Lemma 5 guarantees that $\lim_{n\to\infty} \|x_n - s_0\|$ exists for every $s_0 \in F_{\mathcal{M}}$, and hence $\lim_{n\to\infty} d(x_n, F_{\mathcal{M}})$ also exists. By Theorem 1,

$$\lim_{n\to\infty} \|x_n - \mathcal{M}x_n\| = 0.$$

Since $\mathcal{M}$ satisfies condition (I), there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$\|x_n - \mathcal{M}x_n\| \geq f(d(x_n, F_{\mathcal{M}})), \quad \forall\, n \geq 1.$$

Taking the limit as $n \to \infty$ on both sides and using the nondecreasingness of f,

$$0 = \lim_{n\to\infty} \|x_n - \mathcal{M}x_n\| \geq \lim_{n\to\infty} f(d(x_n, F_{\mathcal{M}})) = f\Big(\lim_{n\to\infty} d(x_n, F_{\mathcal{M}})\Big) \geq 0.$$

Hence $f(\lim_{n\to\infty} d(x_n, F_{\mathcal{M}})) = 0$, which by the properties of f forces

$$\lim_{n\to\infty} d(x_n, F_{\mathcal{M}}) = 0.$$

It follows that there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ and a sequence $\{s_k\} \subseteq F_{\mathcal{M}}$ satisfying

$$\|x_{n_k} - s_k\| < \frac{1}{2^k}, \quad \forall\, k \geq 1.$$

We show $\{s_k\}$ is Cauchy. For $k, j \geq 1$, Lemma 5 gives

$$\begin{aligned} \|s_k - s_j\| &\leq \|s_k - x_{n_k}\| + \|x_{n_k} - s_j\| \\ &< \frac{1}{2^k} + \|x_{n_k} - s_j\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - s_j\|$ exists and $\|x_{n_k} - s_j\| \leq \|x_{n_k} - s_k\| + \|s_k - s_j\| < 2^{-k} + \|s_k - s_j\|$, the sequence $\{s_k\}$ is Cauchy in the closed set $F_{\mathcal{M}} \subseteq K$, and therefore converges to some $s_0 \in F_{\mathcal{M}}$.

Now, using the triangle inequality,

$$\|x_n - s_0\| \leq \|x_n - s_k\| + \|s_k - s_0\|.$$

By Lemma 5, $\lim_{n\to\infty} \|x_n - s_k\|$ exists and is bounded by $\|x_{n_k} - s_k\| < 2^{-k}$. Letting $k \to \infty$,

$$\lim_{n\to\infty} \|x_n - s_0\| \leq 0.$$

Hence $\{x_n\}$ converges strongly to $s_0 \in F_{\mathcal{M}}$, completing the proof.

Theorem 5. *Let K be a nonempty, closed, and convex subset of a uniformly convex Banach space D. Suppose $M : K \to K$ satisfies condition (C) with $F_M \neq \emptyset$. If $\{x_n\}$ is generated by the Picard-S_n iteration* (1.6)*, then*

$$d(x_{n+1}, F_M) \leq d(x_n, F_M), \quad \forall\, n \geq 0.$$

Proof. Let $s_0 \in F_M$ be arbitrary. By Lemma 5, we have

$$\|x_{n+1} - s_0\| \leq \|x_n - s_0\|.$$

Taking the infimum over all $s_0 \in F_M$ on both sides gives

$$d(x_{n+1}, F_M) = \inf_{s_0 \in F_M} \|x_{n+1} - s_0\| \leq \inf_{s_0 \in F_M} \|x_n - s_0\| = d(x_n, F_M).$$

Hence the distance sequence $\{d(x_n, F_M)\}$ is nonincreasing.

Example 1. *Let $K = [0,5] \subset \mathbb{R}$, and define the mapping $\widetilde{\mathcal{M}} : K \to K$ by*

$$\widetilde{\mathcal{M}}(x) = \begin{cases} \dfrac{x+12}{5}, & \text{if } x \in [0,3], \\ \dfrac{x+10}{4}, & \text{if } x \in (3,5]. \end{cases}$$

We claim that $\widetilde{\mathcal{M}}$ is not nonexpansive, but satisfies Suzuki's condition: if $\frac{1}{2}\|x - \widetilde{\mathcal{M}}x\| \le \|x-y\|$, then $\|\widetilde{\mathcal{M}}x - \widetilde{\mathcal{M}}y\| \le \|x-y\|$.

Proof. We verify the two properties separately.

(1) Not nonexpansive. *Take $x = 3$, $y = 3.2$:*

$$\widetilde{\mathcal{M}}(3) = \frac{3+12}{5} = 3, \qquad \widetilde{\mathcal{M}}(3.2) = \frac{3.2+10}{4} = 3.3.$$

Since

$$\begin{aligned} \|\widetilde{\mathcal{M}}(3) - \widetilde{\mathcal{M}}(3.2)\| &= 0.3 > 0.2 \\ &= \|3 - 3.2\|, \end{aligned}$$

the mapping is not nonexpansive.

(2) Suzuki condition holds. *We check representative cases.*
Case I: *$x, y \in [0,3]$. Take $x = 2$, $y = 2.5$:*

$$\begin{aligned} \widetilde{\mathcal{M}}(2) = 2.8, &\quad \widetilde{\mathcal{M}}(2.5) = 2.9, \\ \tfrac{1}{2}\|2 - 2.8\| &= 0.4 \\ &\le 0.5 = \|2 - 2.5\|, \\ \|\widetilde{\mathcal{M}}(2) - \widetilde{\mathcal{M}}(2.5)\| &= 0.1 \le 0.5. \end{aligned}$$

Case II: *$x, y \in (3,5]$. Let $x = 4$, $y = 5$:*

$$\begin{aligned} \widetilde{\mathcal{M}}(4) = 3.5, &\quad \widetilde{\mathcal{M}}(5) = 3.75, \\ \tfrac{1}{2}\|4 - 3.5\| &= 0.25 \le 1 \\ &= \|4 - 5\|, \\ \|\widetilde{\mathcal{M}}(4) - \widetilde{\mathcal{M}}(5)\| &= 0.25 \le 1. \end{aligned}$$

Case III: *$x \in [0,3]$, $y \in (3,5]$. Let $x = 2.5$, $y = 4.5$:*

$$\begin{aligned} \widetilde{\mathcal{M}}(2.5) = 2.9, &\widetilde{\mathcal{M}}(4.5) = 3.625, \\ \tfrac{1}{2}\|2.5 - 2.9\| &= 0.2 \\ &\le 2 = \|2.5 - 4.5\|, \\ \|\widetilde{\mathcal{M}}(2.5) - \widetilde{\mathcal{M}}(4.5)\| &= 0.725 \le 2. \end{aligned}$$

In all cases, the Suzuki condition holds. Hence $\widetilde{\mathcal{M}}$ is a Suzuki-type generalized nonexpansive mapping on K, but is not nonexpansive.

Table 1: Numerical outcomes using the Picard-S_n, Agarwal, Noor, Abbas and S_n iterative schemes for the mapping in Example 1.

n	**Picard-S_n**	**Agarwal**	**Noor**	**Abbas**	$\boldsymbol{S_n}$
1	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
2	2.91966000	2.75080000	2.66334000	2.61566000	2.59830000
3	2.99741819	2.97515974	2.95466400	2.94091310	2.93545484
4	2.99991703	2.99752392	2.99389480	2.99091621	2.98962888
5	2.99999733	2.99975318	2.99917780	2.99860349	2.99833356
6	2.99999991	2.99997539	2.99988920	2.99978530	2.99973223
7	3.00000000	2.99999754	2.99998500	2.99996699	2.99995697
8	3.00000000	2.99999975	2.99999790	2.99999492	2.99999308
9	3.00000000	3.00000000	2.99999970	2.99999921	2.99999888
10	3.00000000	3.00000000	3.00000000	2.99999988	2.99999982
11	3.00000000	3.00000000	3.00000000	3.00000000	3.00000000

Table 2: Comparison of the number of iterations required to obtain the fixed point under different initial points and parameter choices.

	Initial Points					
Scheme	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**
Case 1: $\alpha_r = 0.22,\ \beta_r = 0.66,\ \gamma_r = 0.25$						
Agarwal	14	13	16	11	12	13
Abbas	13	11	12	10	11	12
Noor	13	11	12	10	11	12
Picard–S_n	**9**	**8**	**7**	**7**	**8**	**8**
Case 2: $\alpha_r = \frac{r}{(r+3)^{11/10}},\ \beta_r = \frac{1}{(r+5)^{3/2}},\ \gamma_r = \frac{r}{4r+2}$						
Agarwal	15	15	16	17	17	17
Abbas	11	12	13	13	13	14
Noor	11	12	13	13	13	14
Picard–S_n	**9**	**8**	**7**	**8**	**8**	**8**
Case 3: $\alpha_r = 1 - \sqrt{\frac{1}{6r+4}},\ \beta_r = r^{-3},\ \gamma_r = 1 - \frac{1}{r}$						
Agarwal	16	15	16	16	17	17
Abbas	12	12	12	13	13	13
Noor	12	12	12	13	13	13
Picard–S_n	**8**	**8**	**7**	**7**	**8**	**8**
Case 4: $\alpha_r = \sqrt[25]{\frac{2r}{10r+1}},\ \beta_r = 1 - \sqrt[5]{\frac{1}{r+1}},\ \gamma_r = 1 - \frac{6r}{(7r+5)^4}$						
Agarwal	14	14	15	15	16	16
Abbas	11	11	12	12	13	13
Noor	11	11	12	12	13	13
Picard–S_n	**8**	**8**	**7**	**7**	**8**	**8**

Table 3: Number of iterations to reach the fixed point of $\mathcal{M}$ in Example 1 under $\|x_r - x^*\| < 10^{-15}$.

	Number of Iterations				
Scheme	**Picard–S_n**	**Agarwal**	**Noor**	**Abbas**	$\boldsymbol{S_n}$
Initial point: $x_0 = 0.5$					
Iterations	**7**	10	10	11	11

Note: Bold values indicate the most efficient iterative scheme.

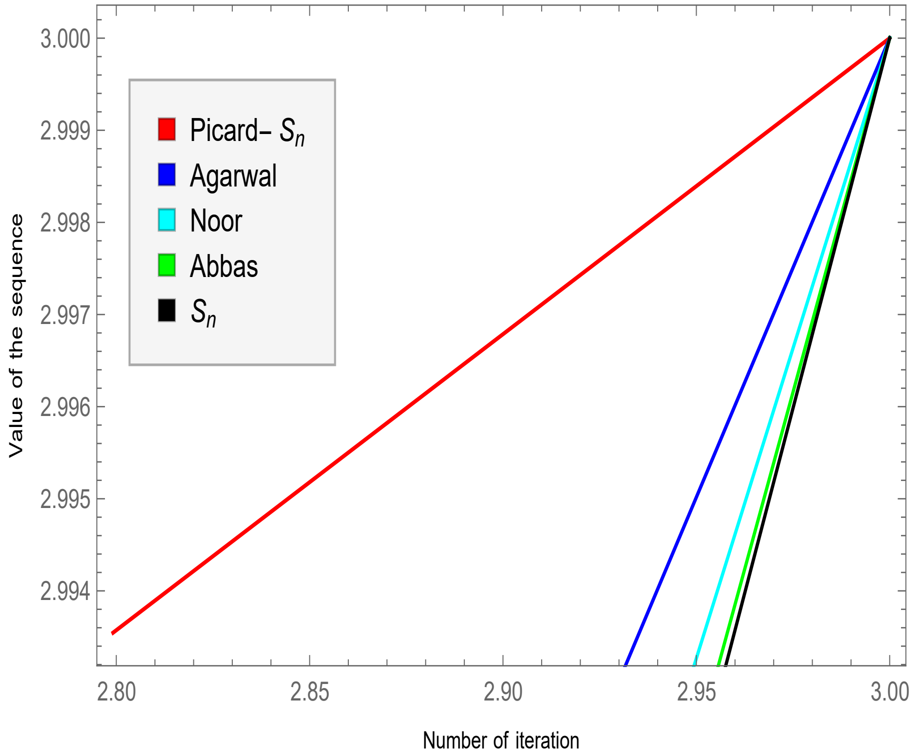

Figure 1: Graphical analysis of iteration schemes towards the fixed point of $\mathcal{M}$ in Example 1.

Example 2. *Let $K = [-2, 4] \subset \mathbb{R}$ and define the piecewise operator*

$$\widetilde{M}(x) = \begin{cases} \dfrac{x}{3} + 1, & x \in [-2, 1], \\ \dfrac{x}{5} + \dfrac{6}{5}, & x \in (1, 4]. \end{cases}$$

We show that $\widetilde{M}$ is not nonexpansive on K, but satisfies Suzuki's condition:

$$\frac{1}{2}|x - \widetilde{M}(x)| \leq |x - y| \quad \Longrightarrow \quad |\widetilde{M}(x) - \widetilde{M}(y)| \leq |x - y|.$$

(1) Failure of nonexpansiveness. *Take $x = 1$, $y = 1.3$:*

$$\widetilde{M}(1) = \tfrac{4}{3}, \qquad \widetilde{M}(1.3) = 1.26.$$

Now choose $x = -1$, $y = 1.3$: $\widetilde{M}(-1) = \frac{2}{3}$, so

$$|\widetilde{M}(-1) - \widetilde{M}(1.3)| = \left|\tfrac{2}{3} - 1.26\right| \approx 0.593 > 0.3 = |-1 - 1.3|.$$

Thus $\widetilde{M}$ is not nonexpansive.

(2) Verification of Suzuki's condition.

Case I: $x, y \in [-2, 1]$***.*** *Here*

$$\begin{aligned}|\widetilde{M}(x) - \widetilde{M}(y)| &= \frac{1}{3}|x-y| \\ &\leq |x-y|.\end{aligned}$$

Condition (C) holds trivially.
Case II: $x, y \in (1, 4]$***.*** *Then*

$$\begin{aligned}|\widetilde{M}(x) - \widetilde{M}(y)| &= \frac{1}{5}|x-y| \\ &\leq |x-y|.\end{aligned}$$

Case III: $x \in [-2, 1],\ y \in (1, 4]$***.*** *Let* $x = 0.6$*,* $y = 3.2$*:*

$$\begin{aligned}\widetilde{M}(0.6) &= 1.2, \quad \widetilde{M}(3.2) = 2.24, \\ \tfrac{1}{2}|0.6 - 1.2| &= 0.3 \leq 2.6 \\ &= |0.6 - 3.2|, \\ |\widetilde{M}(0.6) - \widetilde{M}(3.2)| &= 1.04 \leq 2.6.\end{aligned}$$

Since all cases satisfy condition (C), $\widetilde{M}$ *is a Suzuki-type generalized nonexpansive mapping on* $K = [-2, 4]$*.*

Table 4: Numerical outcomes using the Picard-S_n, Agarwal, Noor, Abbas and S_n iterative schemes for the mapping in Example 2.

n	**Picard-S_n**	**Abbas**	**Agarwal**	$\boldsymbol{S_n}$	**Noor**
1	3.000000000	3.000000000	3.000000000	3.000000000	3.000000000
2	1.548204000	1.730604000	1.649520000	1.741020000	1.701996000
3	1.501549083	1.535452136	1.514904150	1.538727094	1.527201589
4	1.500049781	1.505450270	1.501485640	1.506222669	1.503663075
5	1.500001600	1.500837903	1.500148089	1.500999859	1.500493284
6	1.500000051	1.500128816	1.500014762	1.500160657	1.500066428
7	1.500000002	1.500019804	1.500001471	1.500025814	1.500008945
8	1.500000000	1.500003045	1.500000147	1.500004148	1.500001205
9	1.500000000	1.500000468	1.500000015	1.500000666	1.500000162
10	1.500000000	1.500000072	1.500000001	1.500000107	1.500000022
11	1.500000000	1.500000011	1.500000000	1.500000017	1.500000003
12	1.500000000	1.500000002	1.500000000	1.500000003	1.500000000
13	1.500000000	1.500000000	1.500000000	1.500000000	1.500000000
14	1.500000000	1.500000000	1.500000000	1.500000000	1.500000000
15	1.500000000	1.500000000	1.500000000	1.500000000	1.500000000

Table 5: Number of iterations to reach the fixed point of $\widetilde{M}$ in Example 2 under $\|x_r - x^*\| < 10^{-15}$.

	Number of Iterations				
Scheme	**Picard–S_n**	**Agarwal**	**Noor**	**Abbas**	$\boldsymbol{S_n}$
Initial point: $x_0 = 3$					
Iterations	**8**	11	12	13	13
Note: Bold value indicates the most efficient (fastest) iterative scheme.					

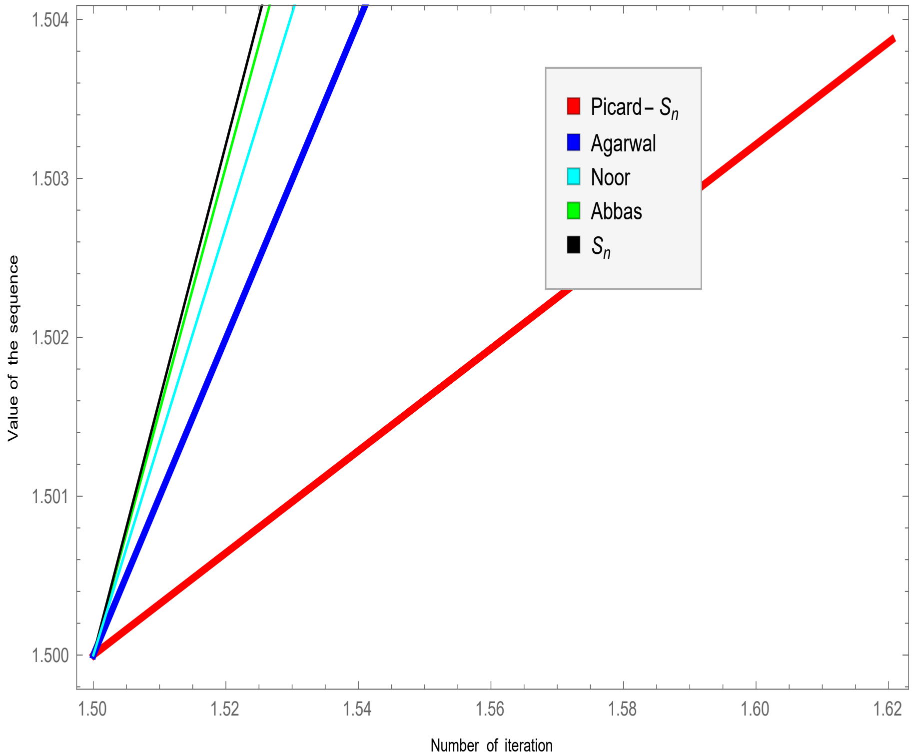

Figure 2: Graphical analysis of iteration schemes towards the fixed point of $\mathcal{M}$ in Example 2.

3. Solution of Functional Equations

Consider the function space $D([a,b])$, consisting of all real-valued functions on the interval $[a,b]$, equipped with the Chebyshev norm

$$\|x-y\|_\infty = \max_{t\in[a,b]} |x(t)-y(t)|.$$

It is known that $(D([a,b]), \|\cdot\|_\infty)$ forms a Banach space. We now focus on the following delay differential equation:

$$x'(t) = \mathcal{M}(t, x(t), x(t-\tau)), \quad t \in [t_0, b], \tag{3.1}$$

with the initial condition

$$x(t) = \varrho(t), \quad t \in [t_0-\tau, t_0]. \tag{3.2}$$

We impose the following assumptions:

(i) $t_0, b \in \mathbb{R}$, $\tau > 0$;

(ii) $\mathcal{M} \in D\big([t_0,b]\times\mathbb{R}^2, \mathbb{R}\big)$;

(iii) $\varrho \in D([t_0-\tau, b], \mathbb{R})$;

(iv) there exists a constant $L_{\mathcal{M}} > 0$ such that

$$|\mathcal{M}(t,u_1,u_2) - \mathcal{M}(t,v_1,v_2)| \le L_{\mathcal{M}} \sum_{i=1}^{2} |u_i - v_i|, \quad \forall\, u_i, v_i \in \mathbb{R},\ i=1,2,\ t\in[t_0,b]; \tag{3.3}$$

(v) $2L_{\mathcal{M}}(b-t_0) < 1$.

A function $p_0 \in D([t_0-\tau, b], \mathbb{R}) \cap D^1([t_0,b],\mathbb{R})$ is called a solution of problem (3.1)–(3.2).

The problem (3.1)–(3.2) can be equivalently expressed as the integral equation:

$$x(t) = \begin{cases} \varrho(t), & t \in [t_0 - \tau, t_0], \\ \varrho(t_0) + \displaystyle\int_{t_0}^{t} \mathcal{M}(s, x(s), x(s-\tau))\, ds, & t \in [t_0, b]. \end{cases} \tag{3.4}$$

Theorem 6. *Under assumptions* (i)–(v)*, the problem* (3.1)–(3.2) *admits a unique solution* $s_0 \in D([t_0-\tau, b], \mathbb{R}) \cap D^1([t_0, b], \mathbb{R})$*. Moreover, the Picard-*S_n *iteration* (1.6) *with real sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\} \subset [0,1]$ *satisfying* $\sum_{n=0}^{\infty} \alpha_n\beta_n\gamma_n = \infty$ *converges to* s_0*.*

Proof. Let $\{x_n\}$ be the sequence generated by the Picard-S_n scheme (1.6) for the operator $\mathcal{M} : D([t_0-\tau, b], \mathbb{R}) \to D([t_0-\tau, b], \mathbb{R})$ defined by

$$\mathcal{M}(x(t)) = \begin{cases} \varrho(t), & t \in [t_0 - \tau, t_0], \\ \varrho(t_0) + \displaystyle\int_{t_0}^{t} \mathcal{M}(s, x(s), x(s-\tau))\, ds, & t \in [t_0, b]. \end{cases}$$

Let s_0 be the fixed point of $\mathcal{M}$. For $t \in [t_0-\tau, t_0]$, we have $x_n \to s_0$ as $n \to \infty$. We now show convergence for $t \in [t_0, b]$. Following the iteration steps and using the Lipschitz property (iv), one obtains:

$$\begin{aligned} \|w_n - s_0\|_\infty &\le (1-\gamma_n)(1 - 2L_{\mathcal{M}}(b-t_0))\|x_n - s_0\|_\infty, \\ \|v_n - s_0\|_\infty &\le \big(2L_{\mathcal{M}}(b-t_0)\big)\big(1 - \beta_n\gamma_n(1 - 2L_{\mathcal{M}}(b-t_0))\big)\|x_n - s_0\|_\infty, \\ \|u_n - s_0\|_\infty &\le 2L_{\mathcal{M}}(b-t_0)^3\big(1 - \alpha_n\beta_n\gamma_n(1 - 2L_{\mathcal{M}}(b-t_0))\big)\|x_n - s_0\|_\infty. \end{aligned}$$

From (v), we deduce the recursive bound

$$\|x_{n+1} - s_0\|_\infty \le \big(1 - \alpha_n\beta_n\gamma_n(1 - 2L_{\mathcal{M}}(b-t_0))\big)\|x_n - s_0\|_\infty.$$

Iterating and using $1 - x \le e^{-x}$ for $x \in [0,1]$:

$$\|x_{n+1} - s_0\|_\infty \le \frac{\|x_0 - s_0\|_\infty}{\exp\big((1 - 2L_{\mathcal{M}}(b-t_0))\sum_{k=0}^{n} \alpha_k\beta_k\gamma_k\big)}.$$

Since $\sum_{k=0}^{\infty} \alpha_k\beta_k\gamma_k = \infty$, the right-hand side tends to zero, so $x_n \to s_0$ in the Chebyshev norm.

4. Conclusions

We proposed a new Picard-S_n hybrid iterative process to study Suzuki generalized nonexpansive mappings in this paper. This study has focused on analyzing the convergence properties of the Picard-S_n iterative method when applied to Suzuki-type generalized nonexpansive mappings. The Picard-S_n process has proven to be a practical and efficient tool for approximating fixed points of such mappings. To highlight its effectiveness, we employed the criterion $\|x_n - s^*\| < 10^{-10}$ in comparative performance evaluations with other iterative techniques. Numerical examples confirmed that the proposed scheme is both efficient and effective, and the numerical code is capable of facilitating practical computations. Also, the application to delay differential equations worked well to show how it can be used to solve real-world problems. This establishes the fact that mixing Picard iteration with S_n schemes is a clear gateway toward ensuring convergence, creating an interesting avenue for future research on iterative methods, nonlinear analysis, and applied mathematics.

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References

[1] S. Banach. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fundamenta Mathematicae*, 3:133–181, 1922.

[2] E. Picard. Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives. *Journal de Mathématiques Pures et Appliquées*, 6:145–210, 1880.

[3] F. E. Browder. Nonexpansive nonlinear operators in a banach space. *Proceedings of the National Academy of Sciences*, 54:1041–1044, 1965.

[4] D. Gohde. Zum prinzip der kontraktiven abbildung. *Mathematische Nachrichten*, 30:251–258, 1965.

[5] S. Ishikawa. Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, 44:147–150, 1974.

[6] E. Karapinar and K. Tas. Generalized (c)-conditions and related fixed point theorems. *Computers and Mathematics with Applications*, 61:3370–3380, 2011.

[7] S. Khatoon and I. B. Uddin. Approximation of fixed point and its application to fractional differential equation. *Journal of Applied Mathematics and Computing*, 2020:1–20, 2020.

[8] J. Schu. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bulletin of the Australian Mathematical Society*, 43:153–159, 1991.

[9] T. Suzuki. Fixed point theorems and convergence theorems for some generalized nonexpansive mapping. *Journal of Mathematical Analysis and Applications*, 340:1088–1095, 2008.

[10] B. S. Thakur, D. Thakur, and M. Postolache. A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. *Appl. Math. Comput.*, 275:147–155, 2016.

[11] J. A. Clarkson. Uniformly convex spaces. *Transactions of the American Mathematical Society*, 40:396–414, 1936.

[12] W. Takahashi. *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama, 2000.

[13] R. P. Agarwal, D. O'Regan, and D. R. Sahu. *Fixed Point Theory for Lipschitzian-Type Mappings with Applications*. Springer, New York, 2009.

[14] Z. Opial. Weak and strong convergence of the sequence of successive approximations for nonexpansive mappings. *Bulletin of the American Mathematical Society*, 73:591–597, 1967.

[15] K. Ullah and M. Arshad. Numerical reckoning fixed points for suzuki's generalized nonexpansive mappings via new iteration process. *Filomat*, 32:187–196, 2018.

[16] K. Ullah, J. Ahmed, and M. D. L. Sen. On generalized non-expansive maps in banach spaces. *Computation*, 8:61, 2020.

[17] J. Ahmad, K. Ullah, M. Arshad, and Z. Ma. A new iterative method for suzuki mappings in banach space. *Journal of Mathematics*, 2021:1–7, 2021.

[18] M. A. Noor. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*, 251:217–229, 2000.

[19] W. R. Mann. Mean value methods in iteration. *Proceedings of the American Mathematical Society*, 4:506–510, 1953.

[20] M. Abbas and T. Nazir. A new faster iteration process applied to constrained minimization and feasibility problems. *Mathematica Vesnik*, 66:223–234, 2014.

[21] K. Eke and H. Akewe. Equivalence of picard-type hybrid iterative algorithms for contractive mappings. *Asian Journal of Scientific Research*, 12(3):298–307, 2019.

[22] R. P. Agarwal, D. O'Regan, and D. R. Sahu. Iterative construction of fixed points of nearly asymptotically non-expansive mappings. *Journal of Nonlinear and Convex Analysis*, 8:61–79, 2007.

[23] W. Sintunavarat and A. Pitea. On a new iteration scheme for numerical reckoning fixed points of berinde mappings with convergence analysis. *Journal of Nonlinear Sciences and Applications*, 9:2553–2562, 2016.