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On multivariate scalar invariant maps and applications

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Abstract. Scalar invariance is a fundamental property in the theory of risk measures. In this paper, we provide a complete characterization of multivariate scalar-invariant mappings and examine their main structural properties. To support this analysis, we introduce a novel topology on the linear space of bounded random vectors, together with a new duality framework tailored to this setting. Building on these foundations, we establish a new representation result for multivariate scalar-invariant mappings.

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1. Introduction.

The axiomatic approach of risk measures originates from the introduction of the coherence axioms in Artzner et al. [1]. Since then, numerous extensions of this framework have been developed, notably encompassing convex, coherent and distortion risk measures as presented in [2], [3], [4], [5], [6] and [7]. These risk measures are inherently univariate, in the sense that the risk they quantify is aggregated into a single scalar at a future time.

To account for risks arising from multiple sources, univariate risk measures have been extended to the multivariate setting. The study of multivariate risk measures was first initiated by Burgert and Rüschendorf [8], where the authors introduced a class of multivariate convex risk measures of the form

$$\Psi_A(h) = \sup_{u\in A} \Psi_u(f_u(h)),$$

for $h \in L^\infty_d$, where L^∞_d denotes the Lebegue space of bounded random variables taking values in $\mathbb{R}^d$, $(\Psi_u)_{u\in A}$ is a family of univariate risk measures and $(f_u)_{u\in A}$ is a family of real functions on $\mathbb{R}^d$. Alternatively, they consider representations of the form

$$\Psi_M(h) = \sup_{\mu\in M} \int_A \Psi_u(f_u(h))\, d\mu(u),$$

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for a family M of weighted measures on A. They establish consistency property of these measures with respect to various convex and dependence orderings, and derive a Legendre-Fenchel-type representation for multivariate convex risk measures.

Further developments are provided in Rüschendorf [9], Ekeland and Schachermayer [10], and Ekeland et al. [11], where the class of law-invariant multivariate convex risk measures is characterized, and a Kusuoka-type dual representation is obtained. In particular, these measures are expressed as the supremum over a family of maximal correlation risk measures. More precisely, it is shown that a map $\phi : L_d^\infty \to \mathbb{R}$ is a law-invariant convex risk measure if and only if there exists a penalty α such that

$$\phi(h) = \sup_{\mu \in M} (\phi_\mu(h) - \alpha(\mu)),$$

for $h \in L_d^\infty$, where ϕ_μ is the maximal correlation measure with baseline μ and M is a set of vector distributions.

An axiomatic approach for such measures, referred to as systemic risk measures, is introduced in Chen et al. [12], Feinstein et al. [13], Armenti et al. [14], Kromer et al. [15] and Biagini et al. [16, 17]. Dual representations under suitable assumptions are further established in Arduca et al. [18] and in Ararat and Rudloff [19].

In general, multivariate financial positions are modeled as random vectors representing future capital positions of financial institutions. Thus, one considers a system of risky positions $h \in L_d^\infty$ and a mapping $\phi : L_d^\infty \to \mathbb{R}$ as a measure of the associated multivariate risk. The construction of such measures relies on a flexible methodological framework that incorporates multi-dimensional acceptance sets and aggregation functions. In particular, multivariate risk is typically reduced to a univariate setting through a combination of aggregation and risk measurement procedures.

Two widely studied approaches in the literature are the following. The first, known as first aggregate, then allocate, consists of aggregating the components of the risk vector and then applying a univariate risk measure. In this case, the multivariate risk measure is defined by

$$\phi(h) = \rho(\eta(h)),$$

for $h \in L_d^\infty$, where $\eta : L_d^\infty \to L^\infty$ is a random aggregate function and $\rho : L^\infty \to \mathbb{R}$ is a univariate risk measure. The second approach, called first allocate, then aggregate, proceeds by evaluating each component of the risk vector separately via univariate risk measures and then aggregating the results. In this setting, the multivariate risk measure takes the form

$$\phi(h) = \pi(\rho_1(h_1), \ldots, \rho_d(h_d)),$$

for $h \in L_d^\infty$, where $\pi : \mathbb{R}^d \to \mathbb{R}$ is a deterministic aggregate function and $\rho_1, \ldots, \rho_d : L^\infty \to \mathbb{R}$ are univariate risk measures.

In this paper we consider a deterministic linear aggregate function $\pi : \mathbb{R}^d \to \mathbb{R}$ and denote by $\underline{\Phi}_\pi$, the set of multivariate scalar invariant maps with respect to π. A map $\phi : L_d^\infty \to \mathbb{R}$ is said to be scalar invariant with respect to π if

$$\phi(h + y) = \phi(h) + \pi(y),$$

for all $h \in L_d^\infty$ and $y \in \mathbb{R}^d$. To each $\phi \in \underline{\Phi}_\pi$, we associate its acceptance set

$$\mathcal{B}_\phi := \{h \in L_d^\infty :\ \phi(h) \leq 0\}.$$

We show in particular that $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi$ for any subset $\mathcal{A}$ in L_d^∞, where

$$\psi_{\pi,\mathcal{A}}(h) = \inf\{\pi(c) :\ c \in \mathbb{R}^d,\ h - c \in \mathcal{A}\},$$

for $h \in L_d^\infty$ and that $\phi = \psi_{\pi,\mathcal{B}_\phi}$ if and only if $\phi \in \underline{\Phi}_\pi$. We also introduce two new properties called π-closeness and π-solidity and show that these conditions are necessary and sufficient for a set $\mathcal{A}$ to satisfy $\mathcal{A} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$.

For the purpose of establishing dual representations of multivariate scalar-invariant maps under minimal assumptions, we introduce a new topology τ on the vector space L_d^∞, defined in terms of the family of acceptable sets. We show that τ is the coarsest topology on L_d^∞, that renders all elements of $\underline{\Phi}_\pi$ lower semi-continuous. Furthermore we construct a duality between L_d^∞ and $\underline{\Phi}_\pi$ via the non-bilinear pairing b, defined by $b(\phi, h) = \phi(h)$ for $\phi \in \underline{\Phi}_\pi$ and $h \in L_d^\infty$, and establish a version of the classical bipolar theorem. Finally we investigate dual representation for elements in $\underline{\Phi}_\pi$.

We emphasize that the present work extends several results from [20] to the multivariate setting.

2. Preliminary results.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and introduce the main definitions and concepts used throughout this paper. We denote by Π_d, the set of real-valued linear functions on $\mathbb{R}^d$. For $\eta \in \Pi_d$, we define the set of units associated with η, by

$$\mathbf{e}(\eta) := \{e \in \mathbb{R}^d,\ \eta(e) = 1\}.$$

In the remainder of the paper, we fix a linear map $\pi \in \Pi_d$.

Definition 1. *(i) A subset $\mathcal{A}$ in L_d^∞, is called a π-pseudo-acceptance set (or pseudo-acceptance set with respect to π) if the map $\psi_{\pi,\mathcal{A}}$, defined by*

$$\psi_{\pi,\mathcal{A}}(h) := \inf\{\pi(c) :\ c \in \mathbb{R}^d,\ h - c \in \mathcal{A}\},$$

is finite for all $h \in L_d^\infty$. We denote by $\underline{\mathbb{A}}^{ps}(\pi)$, the set of all π-pseudo-acceptance sets.

(ii) A subset $\mathcal{A}$ in L_d^∞ is called a π-acceptance set if there exists a map $\phi \in \underline{\Phi}_\pi$ such that

$$\mathcal{A} = \mathcal{B}_\phi := \{h \in L_d^\infty :\ \phi(h) \leq 0\}.$$

We denote by $\underline{\mathbb{A}}(\pi)$, the set of all π-acceptance sets.

A simple example of a pseudo-acceptance set is given below, illustrating that the associated acceptance set may be significantly larger.

Example 1. *Consider $\pi(c) = c_1 + c_2$ for $c \in \mathbb{R}^2$ and define $\mathcal{A} := L^\infty_{2-}$, as the set of non-positive elements in L^∞_2. Then $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ since*

$$\psi_{\pi,\mathcal{A}}(h) = esssup_\omega(h_1 + h_2)(\omega),$$

which is finite for all $h \in L^\infty_2$. Consequently

$$\mathcal{B}_{\psi_{\pi,\mathcal{A}}} = \{h \in L^\infty_2 : \ h_1 + h_2 \leq 0\}.$$

We investigate necessary and sufficient conditions on $\pi \in \Pi_d$, $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ and $\phi \in \underline{\Phi}_\pi$, in order to ensure that $\mathcal{A} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$ and $\phi = \psi_{\pi,\mathcal{B}_\phi}$. To this end, we introduce two new notions, namely π-closeness and π-solidity.

Definition 2. *Let $\mathcal{A} \subseteq L^\infty_d$.*

(i) The set $\mathcal{A}$ is said to be π-closed if for every $h \in L^\infty_d$, $c \in \mathbb{R}^d$ and every sequence $(c_n)_{n\geq 1} \subseteq \mathbb{R}^d$ such that the sequence $(\pi(c_n))_{n\geq 1}$ is decreasing and converges to $\pi(c)$, with $h - c_n \in \mathcal{A}$ for all $n \geq 1$, it follows that $h - c \in \mathcal{A}$.

(ii) The set $\mathcal{A}$ is said to be π-solid if for every $h \in \mathcal{A}$ and every $c \in \mathbb{R}^d$ such that $\pi(c) \leq 0$, one has $h + c \in \mathcal{A}$.

The next result collects the fundamental properties of the map $\psi_{\pi,\mathcal{A}}$ associated with a pseudo-acceptance set $\mathcal{A}$. In particular, it shows that every pseudo-acceptance set naturally generates a multivariate scalar-invariant map.

Proposition 1. *Let $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ and define $\tilde{\mathcal{A}} := \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$. Then*

(1) for every $h \in L^\infty_d$, there exists a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}^d$, such that the sequence $(\pi(c_n))_{n\geq 1}$ is decreasing, converges to $\psi_{\pi,\mathcal{A}}(h)$ and satisfies $h - c_n \in \mathcal{A}$ for all $n \geq 1$.

(2) $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi$.

(3) If $\mathcal{C} \subseteq \mathcal{A}$ with $\mathcal{C} \in \underline{\mathbb{A}}^{ps}(\pi)$, then $\psi_{\pi,\mathcal{A}} \leq \psi_{\pi,\mathcal{C}}$.

(4) $\psi_{\pi,\mathcal{A}} = \psi_{\pi,\tilde{\mathcal{A}}}$.

Proof. (1) The result follows directly from the definition of $\psi_{\pi,\mathcal{A}}(h)$ as an infimum.

(2) let $h \in L^\infty_d$ and $y \in \mathbb{R}^d$. By assertion (1), there exists a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}^d$, such that the sequence $(\pi(c_n))_{n\geq 1}$ is decreasing, converges to $\psi_{\pi,\mathcal{A}}(h)$ and $h - c_n \in \mathcal{A}$ for all $n \geq 1$. Then

$$(h + y) - (c_n + y) = h - c_n \in \mathcal{A},$$

so

$$\psi_{\pi,\mathcal{A}}(h + y) \leq \pi(c_n + y) = \pi(c_n) + \pi(y).$$

Take the limit as $n \to \infty$, we obtain that

$$\psi_{\pi,\mathcal{A}}(h + y) \leq \psi_{\pi,\mathcal{A}}(h) + \pi(y).$$

Replace (h, y) by $(h + y, -y)$, we get

$$\psi_{\pi,\mathcal{A}}(h) \leq \psi_{\pi,\mathcal{A}}(h + y) - \pi(y).$$

Combining the two inequalities yields

$$\psi_{\pi,\mathcal{A}}(h + y) = \psi_{\pi,\mathcal{A}}(h) + \pi(y),$$

which shows that $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi$.

(3) Let $h \in L^\infty_d$. Since $\mathcal{C} \subseteq \mathcal{A}$, we have

$$\{\pi(c) :\ c \in \mathbb{R}^d,\ h - c \in \mathcal{C}\} \subseteq \{\pi(c) :\ c \in \mathbb{R}^d,\ h - c \in \mathcal{A}\}.$$

Taking infima yields $\psi_{\pi,\mathcal{A}}(h) \leq \psi_{\pi,\mathcal{C}}(h)$.

(4) Since $\mathcal{A} \subseteq \tilde{\mathcal{A}}$, assertion (3) implies $\psi_{\pi,\tilde{\mathcal{A}}} \leq \psi_{\pi,\mathcal{A}}$. Conversely let $h \in L^\infty_d$, and consider a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that the sequence $(\pi(c_n))_{n\geq 1}$ is decreasing, converges to $\psi_{\pi,\tilde{\mathcal{A}}}(h)$ and satisfies $h - c_n \in \tilde{\mathcal{A}}$ for all $n \geq 1$. By definition of $\tilde{\mathcal{A}}$, we have $\psi_{\pi,\mathcal{A}}(h - c_n) \leq 0$, which implies $\psi_{\pi,\mathcal{A}}(h) \leq \pi(c_n)$ for all n. Taking the limit gives $\psi_{\pi,\mathcal{A}}(h) \leq \psi_{\pi,\tilde{\mathcal{A}}}(h)$. Hence $\psi_{\pi,\mathcal{A}} = \psi_{\pi,\tilde{\mathcal{A}}}$.

Proposition 1 above shows that the correspondence

$$\mathcal{A} \to \psi_{\pi,\mathcal{A}},$$

preserves the scalar invariance property. The next result investigates the structural properties of the associated acceptance set $\mathcal{B}_{\psi_{\pi,\mathcal{A}}}$, and shows that it provides a canonical enlargement of the original set $\mathcal{A}$.

Proposition 2. *Let $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ and define $\tilde{\mathcal{A}} := \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$. Then*

(1) $\tilde{\mathcal{A}}$ is π-solid.

(2) $\tilde{\mathcal{A}}$ is π-closed.

(3) If $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$, then $\mathcal{A} = \tilde{\mathcal{A}}$.

(4) $\tilde{\mathcal{A}}$ is the smallest π-acceptance set containing $\mathcal{A}$.

Proof. (1) let $h \in \tilde{\mathcal{A}}$ and $y \in \mathbb{R}^d$ such that $\pi(y) \leq 0$. Since $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi$ from assertion (2) in Proposition 1, we have

$$\psi_{\pi,\mathcal{A}}(h + y) = \psi_{\pi,\mathcal{A}}(h) + \pi(y) \leq 0.$$

Hence $h + y \in \tilde{\mathcal{A}}$, which shows that $\tilde{\mathcal{A}}$ is π-solid.

(2) let $h \in L^\infty_d$, $c \in \mathbb{R}^d$ and $(c_n)_{n\geq 1} \subseteq \mathbb{R}^d$ such that $(\pi(c_n))_{n\geq 1}$ is decreasing and converges to $\pi(c)$, with $h - c_n \in \tilde{\mathcal{A}}$ for all $n \geq 1$. Then $\psi_{\pi,\mathcal{A}}(h - c_n) \leq 0$, which implies $\psi_{\pi,\mathcal{A}}(h) \leq \pi(c_n)$ for all $n \geq 1$. Taking the limit yields $\psi_{\pi,\mathcal{A}}(h) \leq \pi(c)$, hence $h - c \in \tilde{\mathcal{A}}$. Therefore $\tilde{\mathcal{A}}$ is π-closed.

(3) Suppose $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$. Then there exists $\mathcal{C} \in \underline{\mathbb{A}}^{ps}(\pi)$ such that $\mathcal{A} = \mathcal{B}_{\psi_{\pi,\mathcal{C}}}$. By assertion (4) in Proposition 1, we have $\psi_{\pi,\mathcal{A}} = \psi_{\pi,\mathcal{C}}$, and thus

$$\tilde{\mathcal{A}} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}} = \mathcal{B}_{\psi_{\pi,\mathcal{C}}} = \mathcal{A}.$$

(4) By definition, $\tilde{\mathcal{A}} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$ and $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi$, so $\tilde{\mathcal{A}} \in \underline{\mathbb{A}}(\pi)$. Now let $\mathcal{D} \in \underline{\mathbb{A}}(\pi)$ such that $\mathcal{A} \subseteq \mathcal{D}$. Then $\psi_{\pi,\mathcal{D}} \leq \psi_{\pi,\mathcal{A}}$, which implies

$$\tilde{\mathcal{A}} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}} \subseteq \mathcal{B}_{\psi_{\pi,\mathcal{D}}} = \mathcal{D}.$$

Thanks to Proposition 1, we can characterize the class of multivariate scalar invariant maps.

Proposition 3. *Let $\phi : L_d^\infty \to \mathbb{R}$. Then $\phi \in \underline{\Phi}_\pi$ if and only if there exists a set $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ such that $\phi = \psi_{\pi,\mathcal{A}}$.*

Proof. For the direct implication, let $h \in L_d^\infty$. Then

$$\psi_{\pi,\mathcal{B}_\phi}(h) = \inf\{\pi(c) :\ c \in \mathbb{R}^d,\ h - c \in \mathcal{B}_\phi\} = \inf\{\pi(c) :\ c \in \mathbb{R}^d,\ \phi(h - c) \leq 0\}.$$

Since $\phi \in \underline{\Phi}_\pi$, we have $\phi(h - c) = \phi(h) - \pi(c)$, and thus the condition $\phi(h - c) \leq 0$ is equivalent to $\phi(h) \leq \pi(c)$. Hence,

$$\psi_{\pi,\mathcal{B}_\phi}(h) = \inf\{\pi(c) :\ c \in \mathbb{R}^d,\ \phi(h) \leq \pi(c)\} = \phi(h),$$

which shows that $\mathcal{B}_\phi \in \underline{\mathbb{A}}^{ps}(\pi)$. The converse implication follows directly from Proposition 1.

Thanks to Proposition 2, we now show next that the properties of π-solidity and π-closedness for a set $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ are necessary and sufficient for $\mathcal{A}$ to belong to $\underline{\mathbb{A}}(\pi)$. Note that in Example 1, the set L_{2-}^∞ is not an acceptance set. Indeed for $h = (0, 0)$ and $c = (1, -2)$, we have $\pi(c) = -1 < 0$, but $h + c = c \notin L_{2-}^\infty$, hence L_{2-}^∞ is not π-solid.

Theorem 1. *Let $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$. Then the following assertions are equivalent:*

(1) $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$.

(2) $\mathcal{A} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$.

(3) $\mathcal{A}$ is π-closed and π-solid.

(4) $\mathcal{A}$ is π-solid and $h - \psi_{\pi,\mathcal{A}}(h)e \in \mathcal{A}$ for all $e \in \mathbf{e}(\pi)$ and $h \in L_d^\infty$.

(5) $\mathcal{A}$ is π-solid and there exists $e \in \mathbf{e}(\pi)$ such that $h - \psi_{\pi,\mathcal{A}}(h)e \in \mathcal{A}$ for all $h \in L_d^\infty$.

Proof. (1) $\Rightarrow$ (2) This follows from assertion (3) in Proposition 2.

(2) $\Rightarrow$ (3) This follows from assertions (1) and (2) in Proposition 2.

(3) $\Rightarrow$ (4) Let $h \in L_d^\infty$ and $e \in \mathbf{e}(\pi)$. By assertion (1) in Proposition 1, there exists a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that $(\pi(c_n))_{n\geq 1}$ decreases to $\psi_{\pi,\mathcal{A}}(h)$ and $h - c_n \in \mathcal{A}$ for all $n \geq 1$. Since $\pi(c_n) \to \pi(c)$ with $c := \psi_{\pi,\mathcal{A}}(h)e$ and $\mathcal{A}$ is π-closed, then $h - c \in \mathcal{A}$.

(4) $\Rightarrow$ (5) This implication is immediate.

(5) $\Rightarrow$ (2) The inclusion $\mathcal{A} \subseteq \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$ is straightforward. For the reverse inclusion, let $h \in \tilde{\mathcal{A}} := \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$. For $e \in \mathbf{e}(\pi)$, define $c := \psi_{\pi,\mathcal{A}}(h)e$. Then $h = (h-c) + c \in \mathcal{A}$, where $h - c \in \mathcal{A}$, $\pi(c) = \psi_{\pi,\mathcal{A}}(h) \leq 0$ and $\mathcal{A}$ is π-solid. Hence $h \in \mathcal{A}$ and therefore $\mathcal{A} = \mathcal{B}_{\psi_{\pi,\mathcal{A}}}$.

(2) $\Rightarrow$ (1) This follows directly from Definition 1.

Corollary 1. *Let $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$. Then for all $h \in L_d^\infty$,*

$$\psi_{\pi,\mathcal{A}}(h) = \inf_{e\in\mathbf{e}(\pi)} \inf\{a \in \mathbb{R} :\ h - ae \in \mathcal{A}\}.$$

Proof. For $e \in \mathbf{e}(\pi)$, define

$$\varphi_e(h) = \inf\{a \in \mathbb{R} :\ h - ae \in \mathcal{A}\},$$

and set $\varphi(h) = \inf_{e\in\mathbf{e}(\pi)} \varphi_e(h)$. For each $h \in L_d^\infty$ and $e \in \mathbf{e}(\pi)$, there exists a sequence $(a_n)_{n\geq 1}$, converging to $\varphi_e(h)$ such that $h - a_n e \in \mathcal{A}$. This implies $\psi_{\pi,\mathcal{A}}(h \leq a_n$, and hence by passing to the limit, $\psi_{\pi,\mathcal{A}}(h) \leq \varphi_e(h)$. Taking the infimum over e yields $\psi_{\pi,\mathcal{A}}(h) \leq \varphi(h)$.

Conversely, let $h \in L_d^\infty$, and consider a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}^d$, such that $(\pi(c_n))_{n\geq 1}$ converges to $\psi_{\pi,\mathcal{A}}(h)$ and $h - c_n \in \mathcal{A}$ for all $n \geq 1$. Define

$$e_n := \frac{c_n}{\pi(c_n)} \in \mathbf{e}(\pi).$$

Then $h - \pi(c_n)e_n \in \mathcal{A}$, which implies $\varphi(h) \leq \varphi_{e_n}(h) \leq \pi(c_n)$. Passing to the limit yields $\varphi(h) \leq \psi_{\pi,\mathcal{A}}(h)$.

We now present some examples of multivariate maps. We denote by Φ, the set of univariate scalar invariant maps and by $\mathbb{A}$, the set of univariate acceptance sets. We say that $\pi \in \Pi_d$ is monotone if $\pi(x) \leq \pi(y)$ for all $x, y \in \mathbb{R}^d$ such that $x \leq y$. We also set $\underline{\Phi} := \cup_{\eta\in\Pi_d}\underline{\Phi}_\eta$.

Example 2. *A map $\phi \in \underline{\Phi}$ is said to be of type (A) if there exist $\eta \in \Pi_d$ and $\rho = (\rho_1, \ldots, \rho_d) \in \times_{i=1}^d \Phi$ such that*

$$\phi = \phi^{\eta,\rho}, \quad \textit{where } \phi^{\eta,\rho}(h) := \eta(\rho_1(h_1), \ldots, \rho_d(h_d)),$$

for all $h \in L_d^\infty$. Then

(i) $\phi^{\eta,\rho} \in \underline{\Phi}_\eta$.

(ii) If η is monotone, then

$$\mathcal{B}_{\phi^{\eta,\rho}} = \mathcal{B}_{\rho_1} \times \ldots \times \mathcal{B}_{\rho_d}.$$

(iii) if η is monotone, then a map $\phi \in \underline{\Phi}_\eta$ is of type (A) if and only if there exist $\mathcal{A}_1, \dots, \mathcal{A}_d \in \mathbb{A}$ such that

$$\mathcal{B}_\phi = \mathcal{A}_1 \times \dots \times \mathcal{A}_d.$$

Proof. (1) Let $h \in L_d^\infty$ and $c \in \mathbb{R}^d$. Then

$$\phi^{\eta,\rho}(h+c) = \eta(\rho_1(h_1+c_1), \dots, \rho_d(h_d+c_d)) = \eta(\rho_1(h_1)+c_1, \dots, \rho_d(h_d)+c_d),$$

since each $\rho_i \in \Phi$. Hence

$$\phi^{\eta,\rho}(h+c) = \eta((\rho_1(h_1), \dots, \rho_d(h_d)) + c) = \phi^{\pi,\rho}(h) + \eta(c),$$

which shows that $\phi^{\eta,\rho} \in \underline{\Phi}_\eta$.

(2) Let $h \in L_d^\infty$ and set $\mathcal{B} := \times_{i=1}^d \mathcal{B}_{\rho_i}$. Then

$$\psi_{\eta,\mathcal{B}}(h) = \inf\{\pi(c) : \ c \in \mathbb{R}^d, \ h - c \in \times_{i=1}^d \mathcal{B}_{\rho_i}\} = \inf\{\pi(c) : \ c \in \mathbb{R}^d, \ \rho_i(h_i) \le c_i \text{ for } i = 1 \dots d\}.$$

By monotonicity of η, the infimum is attained at $c_i = \rho_i(h_i)$, hence

$$\psi_{\eta,\mathcal{B}}(h) = \eta(\rho_1(h_1), \dots, \rho_d(h_d)) = \phi^{\eta,\rho}(h).$$

Therefore $\mathcal{B}_{\phi^{\eta,\rho}} = \mathcal{B}_{\rho_1} \times \dots \times \mathcal{B}_{\rho_d}$.

(3) This follows directly from (2).

We recall that a map $\phi : L_d^\infty \to \mathbb{R}$ is subadditive if

$$\phi(h+h') \le \phi(h) + \phi(h'), \ \text{ for all } \ h, h' \in L_d^\infty.$$

Example 3. *A map $\phi \in \underline{\Phi}$ is said to be of type (B) if there exist a subadditive map $\rho \in \Phi$ and a random vector $f \in L_d^\infty$ such that the map $c \in \mathbb{R}^d \to \rho(f.c)$ is linear on $\mathbb{R}^d$ and*

$$\phi = \phi^{\rho,f}, \ \text{ where } \ \phi^{\rho,f}(h) := \rho(f.h),$$

for all $h \in L_d^\infty$. Then

- *(i) $\phi^{\rho,f} \in \underline{\Phi}_\eta$, where $\eta(c) = \rho(f.c)$ for $c \in \mathbb{R}^d$.*
- *(ii) $\phi^{\rho,f}$ is subadditive.*
- *(iii) If $\phi \in \underline{\Phi}_\eta$ is subadditive, then ϕ is of type (B) if and only if there exits a one-dimensional random vector space C in $\mathbb{R}^d$ such that*

$$\phi = \phi \circ p_C,$$

where p_C denotes the pointwise projection onto C.

Proof. (1) Let $g \in L^\infty$ and $c \in \mathbb{R}^d$. Since ρ is subadditive, we have

$$\rho(g + c.f) \leq \rho(g) + \rho(c.f).$$

Moreover

$$\rho(g) = \rho(g + c.f - c.f) \leq \rho(g + c.f) + \rho(-c.f),$$

and since $\rho(-c.f) = -\rho(c.f)$, it follows that

$$\rho(g + c.f) = \rho(g) + \rho(c.f).$$

Hence

$$\phi^{\rho,f}(h + c) = \rho((h + c).f) = \rho(h.f + c.f) = \phi^{\rho,f}(h) + \eta(c),$$

for all $h \in L_d^\infty$ and $c \in \mathbb{R}^d$.

(2) Let $h, h' \in L_d^\infty$. Then

$$\phi^{\rho,f}(h + h') = \rho((h + h').f) = \rho(h.f + h'.f) \leq \rho(h.f) + \rho(h'.f) = \phi^{\rho,f}(h) + \phi^{\rho,f}(h').$$

(3) Assume that ϕ is of type (B). Define the one-dimensional random vector space C spanned by f, normalized as kf with $k := 1/(f.f)$. Then for $h \in L_d^\infty$,

$$\phi \circ p_C(h) = \phi((h.f)kf) = \rho((h.f)kf.f) = \rho(h.f) = \phi(h).$$

Conversely, let f be a generator of C such that $f.f = 1$, and assume

$$\phi(h) = \phi \circ p_C(h) = \phi((h.f)f),$$

for $h \in L_d^\infty$. We define $\rho(g) = \phi(gf)$ for $g \in L^\infty$. Then

$$\phi(h) = \rho(h.f),$$

for all $h \in L_d^\infty$. Since ϕ is subadditive, so is ρ. Moreover

$$\rho(c.f) = \phi(c) = \phi(0) + \eta(c),$$

for all $c \in \mathbb{R}^d$, which shows that ϕ is of type (B).

In the following lemma, we show that under a suitable roundedness condition, the intersection of a family of acceptance sets is again an acceptance set.

Lemma 1. *Let $(\mathcal{A}_\alpha)_{\alpha\in\lambda} \subseteq \underline{\mathbb{A}}^{ps}(\pi)$ and assume that $\sup_{\alpha\in\lambda} \psi_{\pi,\mathcal{A}_\alpha}$ is finitely valued. Then, for*

$$\mathcal{A} := \cap_{\alpha\in\lambda}\mathcal{A}_\alpha,$$

we have $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$ and

$$\psi_{\pi,\mathcal{A}} = \sup_{\alpha\in\lambda} \psi_{\pi,\mathcal{A}_\alpha}.$$

Moreover if $(\mathcal{A}_\alpha)_{\alpha\in\lambda} \subseteq \underline{\mathbb{A}}(\pi)$, then $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$.

Proof. Let $h \in L_d^\infty$ and $c \in \mathbb{R}^d$. Then

$$\begin{aligned} h - c \in \mathcal{A} &\Leftrightarrow h - c \in \mathcal{A}_\alpha \text{ for all } \alpha \in \lambda \\ &\Leftrightarrow \psi_{\pi,\mathcal{A}_\alpha}(h) \leq \pi(c) \text{ for all } \alpha \in \lambda \\ &\Leftrightarrow \sup_{\alpha\in\lambda} \psi_{\pi,\mathcal{A}_\alpha}(h) \leq \pi(c). \end{aligned}$$

Hence

$$\psi_{\pi,\mathcal{A}}(h) := \inf\{\pi(c) : \ c \in \mathbb{R}^d, \, h - c \in \mathcal{A}\} = \sup_{\alpha\in\lambda} \psi_{\pi,\mathcal{A}_\alpha}(h),$$

which shows that $\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi)$.

Now assume that $(\mathcal{A}_\alpha)_{\alpha\in\lambda} \subseteq \underline{\mathbb{A}}(\pi)$. We show that $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$ by verifying condition (3) of Theorem 1.

First, let $h \in \mathcal{A}$ and $c \in \mathbb{R}^d$ such that $\pi(c) \leq 0$. Since each $\mathcal{A}_\alpha$ is π-solid, we have $h \in \mathcal{A}_\alpha$ implying $h + c \in \mathcal{A}_\alpha$ for all $\alpha \in \lambda$. Therefore $h + c \in \mathcal{A}$.

Second, let $h \in L_d^\infty$, $c \in \mathbb{R}^d$ and $(c_n)_{n\geq1}$ in $\mathbb{R}^d$ such that the sequence $(\pi(c_n))_{n\geq1}$ is decreasing, converges to $\pi(c)$ with $h - c_n \in \mathcal{A}$ for all $n \geq 1$. Then $h - c_n \in \mathcal{A}_\alpha$ for all $\alpha \in \lambda$, which implies $h - c \in \mathcal{A}_\alpha$. Consequently $h - c \in \mathcal{A}$. This completes the proof.

3. On coherent multivariate scalar invariant maps.

In this section, we establish a dual representation for elements of $\underline{\Phi}_\pi$ under coherency axioms. We refer to [19], [18] and [17] for more details on dual representation of convex systemic risk measures.

Definition 3. *A map $\phi \in \underline{\Phi}_\pi$ is said to satisfy:*

(i) the subadditivity property (SA) if

$$\phi(h + g) \leq \phi(h) + \phi(g) \ \textit{ for all } \ h, g \in L_d^\infty.$$

(ii) the positive homogeneity (PH) if

$$\phi(\lambda h) = \lambda\phi(h) \ \textit{ for all } \ h \in L_d^\infty \textit{ and } \lambda \in \mathbb{R}_+.$$

(iii) the monotonicity property (M) if

$$\phi(h) \leq \phi(g) \ \textit{ for all } \ h, g \in L_d^\infty \textit{ such that } h \leq g.$$

(iv) the Fatou property (F) if for every bounded sequence $(h_n)_{n\geq1} \subseteq L_d^\infty$,

$$\phi(\liminf_{n\to\infty} h_n) \leq \liminf_{n\to\infty} \phi(h_n).$$

Definition 4. *A map $\phi \in \underline{\Phi}_\pi$ is said to be coherent if it satisfies the properties (SA), (M), (PH) and (F).*

We recall that there exists a vector $a \in \mathbb{R}^d$ such that $\pi(c) = \pi^a(c) := a.c$ for $c \in \mathbb{R}^d$. In particular π is monotone if and only if $a \geq 0$.

For a vector probability measure $\underline{\mathbb{Q}} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d)$ and $h \in L^\infty_d$, we define

$$\mathbb{E}^{\underline{\mathbb{Q}}}(h) = (\mathbb{E}^{\mathbb{Q}_1}(h_1), \ldots, \mathbb{E}^{\mathbb{Q}_d}(h_d)).$$

Theorem 2. *Suppose that $\pi = \pi^a$ with $a > 0$ and let $\phi \in \underline{\Phi}_\pi$. Then the following assertions are equivalent:*

(i) ϕ is coherent.

(ii) $\mathcal{B}_\phi$ is a weak-closed convex cone in L^∞_d with $L^\infty_{d-} \subseteq \mathcal{B}_\phi$.*

(iii) There exists a closed convex set $\underline{\mathcal{Q}}^\phi$ of $\mathbb{P}$-equivalent vector probability measures such that

$$\phi(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}^\phi} \pi(\mathbb{E}^{\underline{\mathbb{Q}}}(h)), \ \textit{for all}\, h \in L^\infty_d.$$

Before proving this theorem, we establish the following auxiliary result.

Theorem 3. *Let $\phi \in \underline{\Phi}_\pi$. Then*

(i) ϕ is subadditive if and only if $\mathcal{B}_\phi + \mathcal{B}_\phi \subseteq \mathcal{B}_\phi$.

(ii) ϕ is convex if and only if $\mathcal{B}_\phi$ is convex, i.e,

$$\lambda \mathcal{B}_\phi + (1-\lambda)\mathcal{B}_\phi \subseteq \mathcal{B}_\phi \ \textit{for all}\, \lambda \in [0, 1].$$

(iii) ϕ is positively homogeneous if and only if $\mathcal{B}_\phi$ is a cone, i.em

$$\lambda \mathcal{B}_\phi \subseteq \mathcal{B}_\phi \ \textit{for all}\, \lambda \in \mathbb{R}_+.$$

(iv) ϕ is monotone if and only if π is monotone and

$$\mathcal{B}_\phi + L^\infty_{d-} \subseteq \mathcal{B}_\phi.$$

Proof. (1) Assume that ϕ is subadditive. For $h, g \in \mathcal{B}_\phi$, we have

$$\phi(h+g) \leq \phi(h) + \phi(g) \leq 0,$$

so $h + g \in \mathcal{B}_\phi$, and thus $\mathcal{B}_\phi + \mathcal{B}_\phi \subseteq \mathcal{B}_\phi$.

Conversely, let $h, g \in L^\infty_d$ and let $e \in \mathbf{e}(\pi)$. Define $h' := h - \phi(h)e \in \mathcal{B}_\phi$ and $g' := g - \phi(g)e \in \mathcal{B}_\phi$. Then $h' + g' \in \mathcal{B}_\phi$, so

$$(h+g) - (\phi(h) + \phi(g))e \in \mathcal{B}_\phi,$$

which implies $\phi(h+g) \leq \phi(h) + \phi(g)$.

(2) Assume that ϕ is convex. For $h, g \in \mathcal{B}_\phi$ and $\lambda \in (0,1)$, we have

$$\phi(\lambda h + (1-\lambda)g) \leq \lambda\phi(h) + (1-\lambda)\phi(g) \leq 0,$$

so $\lambda h + (1-\lambda)g \in \mathcal{B}_\phi$.

Conversely, let $h, g \in L_d^\infty$ and $\lambda \in (0,1)$. For $e \in \mathbf{e}(\pi)$, define $h' := h - \phi(h)e \in \mathcal{B}_\phi$ and $g' := g - \phi(g)e \in \mathcal{B}_\phi$. Then

$$\lambda h' + (1-\lambda)g' \in \mathcal{B}_\phi,$$

which implies

$$(\lambda h + (1-\lambda)g) - (\lambda\phi(h) + (1-\lambda)\phi(g))e \in \mathcal{B}_\phi.$$

Hence $\phi(\lambda h + (1-\lambda)g) \leq \lambda\phi(h) + (1-\lambda)\phi(g)$.

(3) Assume that ϕ is positively homogeneous. For $h \in \mathcal{B}_\phi$ and $\lambda \in \mathbb{R}_+$, we have

$$\phi(\lambda h) = \lambda\phi(h) \leq 0,$$

so $\lambda h \in \mathcal{B}_\phi$.

Conversely, let $h \in L_d^\infty$ and $\lambda \in \mathbb{R}_+$. For $e \in \mathbf{e}(\pi)$, we have $h - \phi(h)e \in \mathcal{B}_\phi$, hence

$$\lambda h - \lambda\phi(h)e = \lambda(h - \phi(h)e) \in \mathcal{B}_\phi,$$

which implies $\phi(\lambda h) \leq \lambda\phi(h)$. Applying this inequality again, we obtain

$$\phi(h) = \phi(\lambda^{-1}(\lambda h)) \leq \lambda^{-1}\phi(\lambda h),$$

and therefore $\phi(\lambda h) = \lambda\phi(h)$.

(4) Assume that ϕ is monotone. For $h \in \mathcal{B}_\phi$ and $g \in L_{d-}^\infty$, we have $h + g \leq h$, hence

$$\phi(h+g) \leq \phi(h) \leq 0,$$

so $h + g \in \mathcal{B}_\phi$.

Conversely let $h, g \in L_d^\infty$ such that $h \leq g$. For $e \in \mathbf{e}(\pi)$, we write

$$h - \phi(g)e = (g - \phi(g)e) + (h - g).$$

Since $g - \phi(g)e \in \mathcal{B}_\phi$ and $h - g \in L_{d-}^\infty$, we deduce that

$$h - \phi(g)e \in \mathcal{B}_\phi + L_{d-}^\infty \subseteq \mathcal{B}_\phi,$$

and therefore $\phi(h) \leq \phi(g)$.

Proof. (of Theorem 2)

(1) $\Rightarrow$ (2)

The fact that $\mathcal{B}_\phi$ is a convex cone follows directly from the subadditivity and positive homogeneity of ϕ, together with assertions (1) and (3) of Theorem 3.

Let $h \in L^\infty_{d-}$. By the monotonicity of ϕ, we have

$$\phi(h) \leq \phi(0) = 0,$$

hence $h \in \mathcal{B}_\phi$ and therefore $L^\infty_{d-} \subseteq \mathcal{B}_\phi$.

We now prove that $\mathcal{B}_\phi$ is weak*-closed. Let $\tilde{\mathcal{B}} := \mathcal{B}_\phi \cap B$, where B denotes the unit ball of L^∞_d. It suffices to show that $\tilde{\mathcal{B}}$ is closed in L^0_d. Let $(h_n)_{n\geq 1} \subseteq \tilde{\mathcal{B}}$ be a sequence converging to h in L^0_d. Then $h \in B$ and by the Fatou property,

$$\phi(h) \leq \liminf_{n\to\infty} \phi(h_n) \leq 0.$$

so $h \in \tilde{\mathcal{B}}$. Hence $\mathcal{B}_\phi$ is weak*-closed.

(2) $\Rightarrow$ (3)

let $\mathcal{Z}$ be the polar cone of $\mathcal{B}_\phi$ in L^1_d. Since $L^\infty_{d-} \subseteq \mathcal{B}_\phi$, it follows that $\mathcal{Z} \subseteq L^1_{d+}$.

Let $e, e' \in \mathbf{e}(\pi)$. Then $\pm(e - e') \in \mathcal{B}_\phi$, which implies

$$\mathbb{E}(Z).(e - e') = 0 \ \text{ for all } Z \in \mathcal{Z}.$$

Hence, there exists $\lambda^Z \geq 0$ such that

$$\mathbb{E}(Z) = \lambda^Z a.$$

For $h \in L^\infty_d$ and $e \in \mathbf{e}(\pi)$, we have $h - \phi(h)e \in \mathcal{B}_\phi$, so

$$\mathbb{E}(Z.h) \leq \lambda^Z \phi(h), \ \text{ for all } Z \in \mathcal{Z}.$$

On the other hand, for every $\varepsilon > 0$, we have $h - (\phi(h) - \varepsilon)e \notin \mathcal{B}_\phi$. Therefore there exists $Z^\varepsilon \in \mathcal{Z}$ such that

$$\mathbb{E}(Z^\varepsilon.h) > \lambda^{Z^\varepsilon}(\phi(h) - \varepsilon),$$

which implies

$$\lambda^{Z^\varepsilon}\phi(h) \geq \mathbb{E}(Z^\varepsilon.h) > \lambda^{Z^\varepsilon}(\phi(h) - \varepsilon),$$

showing that

$$\lambda^{Z^\varepsilon}\phi(h) > \lambda^{Z^\varepsilon}(\phi(h) - \varepsilon),$$

for all $h \in L^\infty_d$ and $\varepsilon > 0$. We choose h such that $\phi(h) > 0$ and $\varepsilon := \phi(h)$, therefore $\lambda^{Z^\varepsilon} > 0$. Define

$$\tilde{\mathcal{Z}} := \{Z/\lambda^Z : \lambda^Z > 0, \ Z \in \mathcal{Z}\}.$$

Then for every $\varepsilon > 0$,

$$\varepsilon + \sup_{Z\in\tilde{\mathcal{Z}}} \mathbb{E}(Z.h) \geq \varepsilon + \mathbb{E}((Z^\varepsilon/\lambda^{Z^\varepsilon}).h) \geq \phi(h) \geq \sup_{Z\in\tilde{\mathcal{Z}}} \mathbb{E}(Z.h).$$

Letting $\varepsilon \to 0$, we obtain

$$\phi(h) = \sup_{Z\in\tilde{\mathcal{Z}}} \mathbb{E}(Z.h).$$

Writing this componentwise,

$$\phi(h) = \sup_{Z \in \tilde{\mathcal{Z}}} \sum_{i=1}^{d} \mathbb{E}(Z_i)\mathbb{E}\left(\frac{Z_i}{\mathbb{E}(Z_i)} h_i\right).$$

Define probability measure $\mathbb{Q}_i^Z$ by

$$\frac{d\mathbb{Q}_i^Z}{d\mathbb{P}} = \frac{Z_i}{\mathbb{E}(Z_i)}, \quad i = 1 \dots d.$$

Since $\mathbb{E}(Z) = a$ for all $Z \in \tilde{\mathcal{Z}}$, it follows that

$$\phi(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}} a.\mathbb{E}^{\underline{\mathbb{Q}}}(h),$$

where $\underline{\mathcal{Q}} := \{\underline{\mathbb{Q}}^Z : \ Z \in \tilde{\mathcal{Z}}\}$.

(3) $\Rightarrow$ (1)

This implication follows directly, since the supremum of linear functionals preserves subadditivity, positive homogeneity, monotonicity, and the Fatou property.

The examples 2 and 3 are reexamined under coherency property.

Example 4. *Recall from example 2 that a map $\phi \in \underline{\Phi}$ is said to be of type (A) if $\phi = \phi^{\pi,\rho}$ for some $\rho = (\rho_1, \dots, \rho_d)$ with $\rho_1, \dots, \rho_d \in \Phi$. Then a coherent map $\phi \in \underline{\Phi}_\pi$ is of type (A) if and only if*

$$\underline{\mathcal{Q}}^\phi = \mathcal{Q}^1 \times \dots \times \mathcal{Q}^d,$$

for some $\mathcal{Q}^1, \dots, \mathcal{Q}^d \subseteq L^1$.

Indeed, let $\phi \in \underline{\Phi}_\pi$ be a coherent map of type (A), then $\mathcal{B}_\phi = \mathcal{B}_{\rho_1} \times \dots \times \mathcal{B}_{\rho_d}$. Let $(e_1, \dots, e_d)$ denote the canonical basis of $\mathbb{R}^d$. For each $i = 1 \dots d$ and $g \in L^\infty$, we have

$$\rho_i(g) = \phi(ge_i)/a_i.$$

It follows that each ρ_i is coherent. Therefore, by the univariate dual representation, we obtain

$$\underline{\mathcal{Q}}^\phi = \mathcal{Q}^{\rho_1} \times \dots \times \mathcal{Q}^{\rho_d}.$$

Conversely, let $\mathcal{Q}^i$ be associated with ρ_i for $i = 1 \dots d$. Then, for $h \in L^\infty_d$,

$$\phi(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}^\phi} a.\mathbb{E}^{\underline{\mathbb{Q}}}(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}^\phi} \sum_{i=1}^{d} a_i \mathbb{E}^{\mathbb{Q}_i}(h_i) = \sum_{i=1}^{d} \sup_{\mathbb{Q}_i \in \mathcal{Q}_i} a_i \mathbb{E}^{\mathbb{Q}_i}(h_i) = \sum_{i=1}^{d} a_i \rho_i(h_i).$$

We say that a map $\phi \in \underline{\Phi}$ is relevant if for every $h \in L^\infty_{d+}$ such that $\mathbb{P}(h > 0) > 0$, one has $\phi(h) > 0$.

Example 5. *Recall from example 3 that a map $\phi \in \underline{\Phi}$ is of type (B) if $\phi = \phi^{\rho,f}$ for some subadditive map $\rho \in \Phi$ and $f \in L^\infty_d$ such that the mapping $c \in \mathbb{R}^d \to \rho(f.c)$ is linear. Then a relevant coherent map $\phi \in \underline{\Phi}$ is of type (B) if and only if there exists $k \in L^\infty_{d+}$ and a closed convex set $\mathcal{Q} \subseteq L^1$ such that*

$$\underline{\mathcal{Q}}^\phi = \{\underline{\mathbb{Q}} : \ d\mathbb{Q}_i = k_i \, d\mathbb{Q} \text{ for some } \mathbb{Q} \in \mathcal{Q}, \ i = 1 \dots d\}.$$

Indeed, let $\phi \in \underline{\Phi}$ be a relevant coherent map of type (B) and write $\phi(h) = \rho(h.f)$. We first show that $f \geq 0$. Suppose by contradiction that there exists $i \in \{1, \dots, d\}$ such that the set $F := \{f_i < 0\}$ has positive probability. Let $(e_1, \dots, e_d)$ denote the canonical basis of $\mathbb{R}^d$. Then $-\mathbf{1}_F f_i e_i \geq 0$, and by monotonicity of ϕ,

$$0 \leq \phi(-\mathbf{1}_F f_i e_i) = \rho(-\mathbf{1}_F |f_i|^2) \leq 0,$$

which implies $\phi(-\mathbf{1}_F f_i e_i) = 0$, contradicting the relevance of ϕ. Hence $f \geq 0$.

Moreover, using the representation $\rho(g) = \phi(gf/|f|^2))$, it follows that ρ is coherent. Therefore

$$\phi(h) = \rho(h.f) = \sup_{\mathbb{Q} \in \mathcal{Q}^\rho} \mathbb{E}^{\mathbb{Q}}(h.f) = \sup_{\mathbb{Q} \in \mathcal{Q}^\rho} \sum_i \mathbb{E}^{\mathbb{Q}}(f_i h_i).$$

Define $k_i := f_i/\mathbb{E}(f_i)$ for $i = 1 \dots d$, and set

$$\underline{\mathcal{Q}} := \{\underline{\mathbb{Q}} : \ d\underline{\mathbb{Q}} = k \, d\mathbb{Q} \text{ for some } \mathbb{Q} \in \mathcal{Q}^\rho\}.$$

Then

$$\phi(h) = \rho(h.f) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}} \sum_i \mathbb{E}(f_i) \mathbb{E}^{\mathbb{Q}_i}(h_i) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}} \mathbb{E}(f).\mathbb{E}^{\underline{\mathbb{Q}}}(h).$$

Since $\pi(c) = a.c = \phi(c) = \mathbb{E}(f).c$ for all $c \in \mathbb{R}^d$, it follows that $\mathbb{E}(f) = a$, and thus

$$\phi(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}} a.\mathbb{E}^{\underline{\mathbb{Q}}}(h).$$

Conversely, suppose that

$$\phi(h) = \sup_{\underline{\mathbb{Q}} \in \underline{\mathcal{Q}}^\phi} \sum_i a_i \mathbb{E}^{\mathbb{Q}_i}(h_i).$$

Then, writing $f = (a_1 k_1, \dots, a_d k_d)$ and letting ρ be the univariate coherent risk measure associated with $\mathcal{Q}$, we obtain

$$\phi(h) = \rho(f.h),$$

which shows that ϕ is of type (B).

4. Topology on L_d^∞, topological dual and dual representation.

The dual representation of (multivariate) coherent risk measures rely on the duality between Lebesgue spaces, and in particular on the bipolar theorem, which states that a set $\mathcal{A} \subseteq L_d^p$ coincides with its bipolar if and only if $\mathcal{A}$ is closed and convex. When weaker assumptions are imposed on the risk measure, these properties of closeness and convexity may no longer hold. This motivates the introduction of a new topology on L_d^∞, together with a corresponding dual space and a suitable version of the bipolar theorem in this framework.

Assume throughout this section that the map π is monotone.

4.1. Topology on L_d^∞.

In this subsection, we introduce a new topology on L_d^∞. Define

$$\underline{\Phi}_\pi^+ := \{\phi \in \underline{\Phi}_\pi : \ \phi \text{ is monotone}\},$$

and

$$\underline{\mathbb{A}}_+(\pi) := \{\mathcal{A} \in \underline{\mathbb{A}}(\pi) : \ \psi_{\pi,\mathcal{A}} \in \underline{\Phi}_\pi^+\}.$$

By assertion (4) of Theorem 3, a set $\mathcal{A} \in \underline{\mathbb{A}}(\pi)$ belongs to $\underline{\mathbb{A}}_+(\pi)$ if and only if

$$L_{d-}^\infty + \mathcal{A} \subseteq \mathcal{A}.$$

Theorem 4. *The family τ, defined by:*

$$\tau := \{\mathcal{A}' : \ \mathcal{A} \in \underline{\mathbb{A}}_+(\pi)\} \cup \{\emptyset, L_d^\infty\},$$

is a topology on L_d^∞, where $\mathcal{A}'$ denotes the complement of $\mathcal{A}$ in L_d^∞.

Proof. We verify the axioms of a topology.

(1) By definition $\emptyset$ and L_d^∞ belong to τ.

(2) Let $(\mathcal{A}_\alpha)_{\alpha\in\Delta} \subseteq \underline{\mathbb{A}}_+(\pi)$ and consider $\mathcal{B} := \cap_\alpha \mathcal{A}_\alpha$. If $\mathcal{B} = \emptyset$, then its complement belongs to τ. Otherwise, fix $f \in \mathcal{B}$. For any $h \in L_d^\infty$, we can write

$$h \le f + \|h - f\|_{L_d^\infty} \mathbf{1},$$

where $\mathbf{1}$ is the vector of ones in $\mathbb{R}^d$. Since each $\psi_{\pi,\mathcal{A}_\alpha}$ is monotone, we obtain

$$\psi_{\pi,\mathcal{A}_\alpha}(h) \le \psi_{\pi,\mathcal{A}_\alpha}(f + \|h-f\|_{L_d^\infty}\mathbf{1}) = \psi_{\pi,\mathcal{A}_\alpha}(f) + \pi(\|h-f\|_{L_d^\infty}\mathbf{1}) \le \pi(\|h-f\|_{L_d^\infty}\mathbf{1}).$$

Taking the supremum over $\alpha \in \Delta$, we get

$$\sup_{\alpha\in\Delta} \psi_{\pi,\mathcal{A}_\alpha}(h) \le \pi(\|h-f\|_{L_d^\infty}\mathbf{1}) < \infty.$$

Hence, by Lemma 1, $\psi_{\pi,\mathcal{B}} = \sup_{\alpha\in\Delta} \psi_{\pi,\mathcal{A}_\alpha}$ and $\mathcal{B} \in \underline{\mathbb{A}}(\pi)$. Moreover $\psi_{\pi,\mathcal{B}}$ is monotone, so $\mathcal{B} \in \underline{\mathbb{A}}_+(\pi)$. Therefore $\cup_\alpha \mathcal{A}'_\alpha = \mathcal{B}' \in \tau$.

(3) Let $(\mathcal{A}_k)_{k=1}^m \subseteq \underline{\mathbb{A}}_+(\pi)$ and set $\mathcal{C} := \cup_{k=1}^m \mathcal{A}_k$. Then

$$\psi_{\pi,\mathcal{C}} = \inf_k \psi_{\pi,\mathcal{A}_k},$$

which is finitely valued, hence $\mathcal{C} \in \underline{\mathbb{A}}^{ps}(\pi)$. One verifies that $\mathcal{C}$ is both π-solid and π-closed and therefore $\mathcal{C} \in \underline{\mathbb{A}}(\pi)$. Since $\psi_{\pi,\mathcal{C}}$ is monotone, we conclude that $\mathcal{C} \in \underline{\mathbb{A}}_+(\pi)$, and thus $\cap_{k=1}^m \mathcal{A}'_k = \mathcal{C}' \in \tau$.

This proves that τ is a topology on L_d^∞.

We now derive some properties of the topological space (L_d^∞, τ). To this end, we first characterise the τ-convergence of nets.

Proposition 4. *A net $(h_\alpha)_{\alpha\in\Delta}$ in L_d^∞ converges to $h \in L_d^\infty$ with respect to τ if and only if*

$$\phi(h) \le \liminf_\alpha \phi(h_\alpha), \quad \textit{for all}\ \phi \in \underline{\Phi}_\pi^+.$$

Equivalently, for every $\varepsilon > 0$ and every $\phi \in \underline{\Phi}_\pi^+$, there exists $\alpha_0 \in \Delta$ such that for all $\alpha \succeq \alpha_0$,

$$\phi(h) \le \phi(h_\alpha) + \varepsilon.$$

Proof. For the direct implication, assume that (h_α) converges to $h \in L_d^\infty$ in the topology τ. Then for every τ-neighbourhood V of h, there exists $\alpha_0 \in \Delta$ such that $h_\alpha \in V$ for all $\alpha \succeq \alpha_0$. In particular, for any $\varphi \in \underline{\Phi}_\pi^+$ such that $\varphi(h) > 0$, there exists $\alpha_0 \in \Delta$ such that $\varphi(h_\alpha) > 0$ for all $\alpha \succeq \alpha_0$.

Fix $\varepsilon > 0$ and $\phi \in \underline{\Phi}_\pi^+$, and define $\varphi = \phi + \varepsilon - \phi(h)$. Then $\varphi \in \underline{\Phi}_\pi^+$ and $\varphi(h) = \varepsilon > 0$. Hence there exists α_0 such that $\varphi(h_\alpha) > 0$ for all $\alpha \succeq \alpha_0$, which yields

$$\phi(h_\alpha) + \varepsilon - \phi(h) > 0,$$

and therefore $\phi(h) < \phi(h_\alpha) + \varepsilon$.

Conversely, assume that the inequality holds for all $\phi \in \underline{\Phi}_\pi^+$. Let $\phi \in \underline{\Phi}_\pi^+$ be such that $\phi(h) > 0$. For $\delta \in (0, 1)$, set $\varepsilon := \delta\phi(h) > 0$. Then there exists α_0 such that for all $\alpha \succeq \alpha_0$

$$\phi(h) \le \phi(h_\alpha) + \varepsilon,$$

which implies

$$\phi(h_\alpha) \ge \phi(h) - \varepsilon = (1 - \delta)\phi(h) > 0.$$

This proves the τ-convergence of (h_α) to h.

We recall that a map $\varphi : (L_d^\infty, \tau) \to \mathbb{R}$ is said to be lower semicontinuous if, for every $h_0 \in L_d^\infty$ and $\varepsilon > 0$, there exists a neighbourhood V of h_0 such that

$$\varphi(h_0) \le \varphi(h) + \varepsilon \ \text{ for all } h \in V,$$

or equivalently, if $\varphi^{-1}((-\infty, y])$ is τ-closed for all $y \in \mathbb{R}$.

Proposition 5. *Suppose π is strictly monotone.*

(i) The topology τ is the smallest topology on L_d^∞ for which each $\phi \in \underline{\Phi}_\pi^+$ is lower semicontinuous.

(ii) The space (L_d^∞, τ) is a T_0-space.

(iii) The space (L_d^∞, τ) is not a T_1-space.

Proof. (1) Let $\phi \in \underline{\Phi}_\pi^+$ and $y \in \mathbb{R}$. Then

$$\phi^{-1}((-\infty, y]) = \phi_y^{-1}((-\infty, 0]),$$

where $\phi_y := \phi - y \in \underline{\Phi}_\pi^+$. Hence

$$\phi^{-1}((-\infty, y]) = \mathcal{B}_{\phi_y} \in \underline{\mathbb{A}}_+(\pi),$$

which shows that ϕ is lower semicontinuous with respect to τ.

Now suppose that τ' is another topology on L_d^∞ for which every $\phi \in \underline{\Phi}_\pi^+$ is lower semicontinuous. For any $\mathcal{A} \in \underline{\mathbb{A}}_+(\pi)$, let $\phi := \psi_{\pi,\mathcal{A}}$. Then

$$\mathcal{A} = \mathcal{B}_\phi = \phi^{-1}((-\infty, 0]),$$

which is τ'-closed. Hence $\tau \subseteq \tau'$, proving minimality.

(2) Assume there exists $a \in \mathbb{R}^d$ such that $a > 0$ and $\pi(c) = a.c$ for all $c \in \mathbb{R}^d$. Let $f, g \in L_d^\infty$ with $f \neq g$. Define $f_a := (a_1 f_1, \ldots, a_d f_d)$ and $g_a := (a_1 g_1, \ldots, a_d g_d)$, then $f_a \neq g_a$. Therefore there exists a vector probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ such that

$$\sum_i \mathbb{E}^{\mathbb{Q}^i}(f_a^i) < \sum_i \mathbb{E}^{\mathbb{Q}^i}(g_a^i).$$

We define the map

$$\phi(h) := \sum_i a^i \mathbb{E}^{\mathbb{Q}^i}(h^i), \;\; h \in L_d^\infty.$$

Thus $\phi \in \underline{\Phi}_\pi^+$ and $\phi(f) < \phi(g)$. Define $\varphi := \phi - \phi(f)$. Then $\varphi \in \underline{\Phi}_\pi^+$, $\varphi(f) = 0$ and $\varphi(g) > 0$. Setting $\mathcal{A} := \mathcal{B}'_\varphi \in \tau$, we have $g \in \mathcal{A}$ and $f \notin \mathcal{A}$, which shows that (L_d^∞, τ) is T_0.

(3) A space is T_1 if and only if all singletons are closed. This is not the case here, since singletons are not π-solid. Therefore (L_d^∞, τ) is not a T_1-space.

4.2. Topological dual.

We define the topological dual of L_d^∞ with respect to the topology τ, as $\underline{\Phi}_\pi^+$. In order to introduce dual sets both in L_d^∞ and in $\underline{\Phi}_\pi^+$, and to state the bipolar theorem in this framework, we first introduce the following notation.

Define

$$\underline{\mathbb{H}}^{ps}(\pi) := \{\mathcal{H} \subseteq \underline{\Phi}_\pi^+ : \; \varphi_\mathcal{H} := \sup_{\varphi \in \mathcal{H}} \varphi \text{ is finitely valued}\},$$

and

$$\underline{\mathbb{H}}(\pi) := \{\mathcal{H} \in \underline{\mathbb{H}}^{ps}(\pi) : \ \varphi_{\mathcal{H}} \in \mathcal{H} \text{ and } \mathcal{H} \text{ is solid}\},$$

where a set $\mathcal{H} \subseteq \underline{\Phi}_{\pi}^{+}$ is said to be solid if for all $\phi \in \underline{\Phi}_{\pi}^{+}$ such that $\phi \leq \varphi$ and $\varphi \in \mathcal{H}$, it follows that $\phi \in \mathcal{H}$.

Definition 5. *Let $\mathcal{A} \subseteq L_d^{\infty}$ and $\mathcal{H} \subseteq \underline{\Phi}_{\pi}^{+}$. We define the generalised dual sets of $\mathcal{A}$ and $\mathcal{H}$, respectively by:*

$$\mathcal{A}^o := \{\phi \in \underline{\Phi}_{\pi}^{+} : \ \phi(h) \leq 0 \textit{ for all } h \in \mathcal{A}\},$$

and

$$\mathcal{H}^o := \{h \in L_d^{\infty} : \ \phi(h) \leq 0 \textit{ for all } \phi \in \mathcal{H}\}.$$

We now state conditions ensuring that these dual sets are nonempty and describe their properties. Define

$$\underline{\mathbb{A}}_{+}^{ps}(\pi) := \{\mathcal{A} \in \underline{\mathbb{A}}^{ps}(\pi) : \ \mathcal{A} + L_{d-}^{\infty} \subseteq \mathcal{A}\}.$$

Proposition 6. *Let $\mathcal{A} \in \underline{\mathbb{A}}_{+}^{ps}(\pi)$ and $\mathcal{H} \in \underline{\mathbb{H}}^{ps}(\pi)$. Then*

(i) $\mathcal{A}^o \neq \emptyset$ and $\mathcal{H}^o \neq \emptyset$.

(ii) $\mathcal{H}^o \in \underline{\mathbb{A}}_{+}^{ps}(\pi)$ and $\psi_{\pi,\mathcal{H}^o} = \varphi_{\mathcal{H}}$.

(iii) $\mathcal{A}^o \in \underline{\mathbb{H}}^{ps}(\pi)$ and $\varphi_{\mathcal{A}^o} = \psi_{\pi,\mathcal{A}}$.

Proof. (1) Since $\psi_{\pi,\mathcal{A}} \in \underline{\Phi}_{\pi}^{+}$ and $\psi_{\pi,\mathcal{A}}(h) \leq 0$ for all $h \in \mathcal{A}$, we have $\psi_{\pi,\mathcal{A}} \in \mathcal{A}^o$, hence $\mathcal{A}^o \neq \emptyset$.

Now let $h \in L_d^{\infty}$ and $e \in \mathbf{e}(\pi)$, and define

$$h' := h - \varphi_{\mathcal{H}}(h)e.$$

Then for every $\phi \in \mathcal{H}$,

$$\phi(h') = \phi(h) - \varphi_{\mathcal{H}}(h) \leq 0,$$

which shows that $h' \in \mathcal{H}^o$. Hence $\mathcal{H}^o \neq \emptyset$.

(2) Let $h \in L_d^{\infty}$ and $c \in \mathbb{R}^d$. Then

$$h - c \in \mathcal{H}^o \Leftrightarrow \phi(h) \leq \pi(c) \text{ for all } \phi \in \mathcal{H} \Leftrightarrow \varphi_{\mathcal{H}}(h) \leq \pi(c).$$

Therefore,

$$\psi_{\pi,\mathcal{H}^o} = \varphi_{\mathcal{H}},$$

and thus $\mathcal{H}^o \in \underline{\mathbb{A}}_{+}^{ps}(\pi)$.

(3) Let $h \in L_d^{\infty}$, then there exists a sequence $(c_n)_{n\geq 1} \subseteq \mathbb{R}^d$ such that $\pi(c_n) \searrow \psi_{\pi,\mathcal{A}}(h)$ and $h - c_n \in \mathcal{A}$ for all $n \geq 1$. Hence, for all $\phi \in \mathcal{A}^o$,

$$\phi(h) \leq \pi(c_n), \quad \text{for all } n \geq 1,$$

and passing to the limit gives $\phi(h) \leq \psi_{\pi,\mathcal{A}}(h)$. Therefore

$$\varphi_{\mathcal{A}^o}(h) \leq \psi_{\pi,\mathcal{A}}(h).$$

Since $\psi_{\pi,\mathcal{A}} \in \mathcal{A}^o$, equality holds, and thus $\varphi_{\mathcal{A}^o} = \psi_{\pi,\mathcal{A}}$. In particular $\mathcal{A}^o \in \underline{\mathbb{H}}^{ps}(\pi)$.

We now establish a one-to-one correspondence between $\underline{\mathbb{A}}_{+}(\pi)$ and $\underline{\mathbb{H}}(\pi)$.

Proposition 7. *Let* $\mathcal{A} \in \underline{\mathbb{A}}_+^{ps}(\pi)$ *and* $\mathcal{H} \in \underline{\mathbb{H}}^{ps}(\pi)$*. Then*

(i) $\mathcal{H}^o \in \underline{\mathbb{A}}_+(\pi)$.

(ii) $\mathcal{A}^o \in \underline{\mathbb{H}}(\pi)$.

Proof. (i) We have

$$\mathcal{H}^o = \cap_{\phi\in\mathcal{H}}\mathcal{B}_\phi \in \underline{\mathbb{A}}(\pi).$$

Moreover, for every $\phi \in \mathcal{H}$,

$$\mathcal{B}_\phi + L^\infty_{d-} \subseteq \mathcal{B}_\phi,$$

so

$$\mathcal{H}^o + L^\infty_{d-} \subseteq \cap_{\phi\in\mathcal{H}}\mathcal{B}_\phi = \mathcal{H}^o.$$

Hence $\mathcal{H}^o \in \underline{\mathbb{A}}_+(\pi)$.

(ii) From Proposition 6, we know that $\varphi_{\mathcal{A}^o} = \psi_{\pi,\mathcal{A}} \in \mathcal{A}^o$. Let $\phi \in \underline{\Phi}_\pi^+$ and $\varphi \in \mathcal{A}^o$ such that $\phi \leq \varphi$. Then, for all $h \in \mathcal{A}$,

$$\phi(h) \leq \varphi(h) \leq 0,$$

which implies $\phi \in \mathcal{A}^o$. Hence $\mathcal{A}^o$ is solid, and therefore $\mathcal{A}^o \in \underline{\mathbb{H}}(\pi)$.

We now state the bipolar theorem in both $\underline{\mathbb{A}}_+(\pi)$ and $\underline{\mathbb{H}}(\pi)$.

Proposition 8. *Let* $\mathcal{A} \in \underline{\mathbb{A}}_+^{ps}(\pi)$*. The following assertions are equivalent:*

(1) $\mathcal{A} \in \underline{\mathbb{A}}_+(\pi)$.

(2) $\mathcal{A} = \mathcal{A}^{oo} := (\mathcal{A}^o)^o$.

(3) *For all* $h \in L^\infty_d$ *such that* $h \notin \mathcal{A}$*, there exists* $\phi \in \mathcal{A}^o$ *such that* $\phi(h) > 0$.

Proof. (1) ⇒ (2) the inclusion $\mathcal{A} \subseteq \mathcal{A}^{oo}$ is immediate. Let $h \in \mathcal{A}^{oo}$. Since $\psi_{\pi,\mathcal{A}} \in \mathcal{A}^o$, we have $\psi_{\pi,\mathcal{A}}(h) \leq 0$, hence $h \in \mathcal{A}$.

(2) ⇒ (3) Let $h \notin \mathcal{A}$. Since $\mathcal{A} = \mathcal{A}^{oo}$, it follows that $h \notin \mathcal{A}^{oo}$, so there exists $\phi \in \mathcal{A}^o$ such that $\phi(h) > 0$.

(3) ⇒ (2) The inclusion $\mathcal{A} \subseteq \mathcal{A}^{oo}$ is trivial. Suppose there exists $h \in \mathcal{A}^{oo}$ such that $h \notin \mathcal{A}$. Then there exists $\phi \in \mathcal{A}^o$ such that $\phi(h) > 0$, which contradicts the fact that $\varphi(h) \leq 0$ for all $\varphi \in \mathcal{A}^o$.

(2) ⇒ (1) Since $\mathcal{A} = \mathcal{A}^{oo}$, we conclude from Proposition 7 that $\mathcal{A} \in \underline{\mathbb{A}}_+(\pi)$.

Proposition 9. *Let* $\mathcal{H} \in \underline{\mathbb{H}}^{ps}(\pi)$*. The following assertions are equivalent:*

(1) $\mathcal{H} \in \underline{\mathbb{H}}(\pi)$.

(2) $\mathcal{H} = \mathcal{H}^{oo} := (\mathcal{H}^o)^o$.

(3) *For all* $\phi \in \underline{\Phi}_\pi^+$ *such that* $\phi \notin \mathcal{H}$*, there exists* $h \in \mathcal{H}^o$ *such that* $\phi(h) > 0$.

Proof. (1) $\Rightarrow$ (2) The inclusion $\mathcal{H} \subseteq \mathcal{H}^{oo}$ is immediate. Let $\phi \in \mathcal{H}^{oo}$. For any $h \in L_d^\infty$ and $e \in \mathbf{e}(\pi)$, define

$$g := h - \varphi_{\mathcal{H}}(h)e \in \mathcal{H}^o.$$

Then $g \in \mathcal{H}^o$, so $\phi(g) \leq 0$, which implies $\phi(h) \leq \varphi_{\mathcal{H}}(h)$. Since $\varphi_{\mathcal{H}} \in \mathcal{H}$ and $\mathcal{H}$ is solid, we conclude that $\phi \in \mathcal{H}$.

(2) $\Rightarrow$ (3) Let $\phi \notin \mathcal{H}$. Since $\mathcal{H} = \mathcal{H}^{oo}$, it follows that $\phi \notin \mathcal{H}^{oo}$, hence there exists $h \in \mathcal{H}^o$ such that $\phi(h) > 0$.

(3) $\Rightarrow$ (2) The inclusion $\mathcal{H} \subseteq \mathcal{H}^{oo}$ is trivial. Suppose there exists $\phi \in \mathcal{H}^{oo}$ such that $\phi \notin \mathcal{H}$. Then there exists $h \in \mathcal{H}^o$ such that $\phi(h) > 0$, which contradicts the definition of $\mathcal{H}^o$.

(2) $\Rightarrow$ (1) Since $\mathcal{H} = \mathcal{H}^{oo}$, we conclude from Proposition 7 that $= \mathcal{H} \in \underline{\mathbb{H}}(\pi)$.

4.3. On representing multivariate scalar invariant maps.

In Proposition 6, it was shown for any map $\phi \in \underline{\Phi}_\pi^+$,

$$\phi = \psi_{\pi,\mathcal{B}_\phi} = \varphi_{\mathcal{B}_\phi^o} = \sup_{\varphi \in \mathcal{B}_\phi^o} \varphi,$$

where

$$\mathcal{B}_\phi^o = \{\psi \in \underline{\Phi}_\pi^+ : \ \psi \leq \phi\}.$$

Thus, the representation $\phi = \sup_{\varphi \in \mathcal{B}_\phi^o} \varphi$ is essentially tautological and does not provide additional structural insight. This naturally raises the following question: given a class $\Psi \subseteq \underline{\Phi}_\pi^+$, does there exist a strictly smaller subclass $\Psi' \subsetneq \Psi$ such that every $\phi \in \Psi$ can be represented as

$$\phi = \sup_{\varphi \in K} \varphi,$$

for some subset $K \subseteq \Psi'$?

To address this question, let $\mathcal{P}$, denote the collection of properties (P), that are satisfied by at least one element of $\underline{\Phi}_\pi^+$. For a given property $(P) \in \mathcal{P}$ and $\phi \in \underline{\Phi}_\pi^+$, define

$$\underline{\Phi}_\pi^+(P) := \{\varphi \in \underline{\Phi}_\pi^+ : \ \varphi \text{ satisfies } (P)\},$$

and

$$\Gamma(\phi, (P)) := \{\varphi \in \underline{\Phi}_\pi^+(P) : \ \varphi \leq \phi\}.$$

Definition 6. *Let $\phi \in \underline{\Phi}_\pi^+$ and $(P) \in \mathcal{P}$ such that $\Gamma(\phi, (P)) \neq \emptyset$. The $(\phi, (P))$-transformed of ϕ, denoted $T^{(P)}\phi$, is defined for $h \in L_d^\infty$ by*

$$T^{(P)}\phi(h) := \sup\{\varphi(h) : \ \varphi \in \Gamma(\phi, (P))\}.$$

We now collect some basic properties of the transformation $T^{(P)}$.

Theorem 5. *Let $\phi, \varphi \in \underline{\Phi}_\pi^+$ and $(P), (P') \in \mathcal{P}$. Then* *(i)* $T^{(P)}\phi \in \underline{\Phi}_\pi^+$.

(ii) $T^{(P)}\phi = \phi$ *if* ϕ *satisfies* (P).

(iii) *If* $\phi \leq \varphi$, *then* $T^{(P)}\phi \leq T^{(P)}\varphi$.

(iv) *If* $(P) \Rightarrow (P')$, *then* $T^{(P)}\phi \leq T^{(P')}\phi$.

(v) $T^{(P)}T^{(P)}\phi = T^{(P)}\phi$.

(vi) $T^{(P)}\phi \leq T^{(P')}\phi$ *if and only if* $T^{(P)}T^{(P')}\phi = T^{(P)}\phi$.

Proof. Assertions (1-4) follow directly from Definition 6.
For (5), observe that

$$T^{(P)}T^{(P)}\phi = \sup\{\varphi :\ \varphi \in \Gamma(T^{(P)}\phi\,; (P))\}.$$

Since $\Gamma(T^{(P)}\phi\,; (P)) = \Gamma(\phi\,; (P))$, it follows immediately that

$$T^{(P)}T^{(P)}\phi = T^{(P)}\phi.$$

For (6), assume first that $T^{(P)}\phi \leq T^{(P')}\phi$. Then

$$T^{(P)}\phi = T^{(P)}T^{(P)}\phi \leq T^{(P)}T^{(P')}\phi \leq T^{(P)}\phi,$$

which implies equality.
Conversely, if $T^{(P)}\phi = T^{(P)}T^{(P')}\phi$, then

$$T^{(P)}\phi = T^{(P)}T^{(P')}\phi \leq T^{(P')}\phi.$$

Definition 7. *A property* $(P) \in \mathcal{P}$ *is said to be consistent (with respect to the supremum operator) if for every* $\phi \in \underline{\Phi}_\pi^+$, *the transformed map* $T^{(P)}\phi$ *satisfies the property* (P). *We denote by* $\mathcal{P}_{cst}$, *the subset of* $\mathcal{P}$, *consisting of all consistent properties.*

When a property $(P) \in \mathcal{P}$ is consistent, the map $T^{(P)}\phi$ is called the (P)-hull of ϕ for $\phi \in \underline{\Phi}_\pi^+$.
A useful characterization of consistency is given in the following result.

Theorem 6. *A property (P) is consistent if and only if for every family* $\Theta \subseteq \underline{\Phi}_\pi^+(P)$ *such that*

$$\varphi_\Theta := \sup_{\varphi \in \Theta} \varphi,$$

is finitely valued, the map φ_Θ *satisfies the property (P).*

Proof. The converse implication is immediate by taking $\Theta := \Gamma(\phi\,;(P))$.

For the direct implication, let $\Theta \subseteq \underline{\Phi}_\pi^+(P)$ be such that φ_Θ is finitely valued. By definition of $T^{(P)}$, we have

$$T^{(P)}\varphi_\Theta \leq \varphi_\Theta.$$

On the other hand, since $\Theta \subseteq \Gamma(\varphi_\Theta\,;(P))$, it follows that

$$T^{(P)}\varphi_\Theta \geq \sup_{\varphi\in\Theta} \varphi = \varphi_\Theta.$$

Hence,

$$\varphi_\Theta = T^{(P)}\varphi_\Theta,$$

which shows that φ_Θ satisfies the property (P).

We now consider a property (P), which is not necessarily consistent. The next result shows that every such property admits a canonical extension to a consistent property, called its consistency hull.

Theorem 7. *Let $(P) \in \mathcal{P}$ and define a new property $(P^*) \in \mathcal{P}$ by*

$$\underline{\Phi}_\pi^+(P^*) := \{T^{(P)}\phi :\ \phi \in \underline{\Phi}_\pi^+\}.$$

Then the following assertions hold:

(i) $(P) \Rightarrow (P^*)$.

(ii) $T^{(P^*)} = T^{(P)}$.

(iii) $(P^*) \in \mathcal{P}_{cst}$.

(iv) If $(P') \in \mathcal{P}_{cst}$ *and* $(P) \Rightarrow (P')$, *then* $(P^*) \Rightarrow (P')$.

(v) (P) *and* (P^*) *are equivalent if and only if* (P) *is consistent.*

Proof. (1) Suppose that $\phi \in \underline{\Phi}_\pi^+(P)$. Since ϕ already satisfies (P), we have $\phi = T^{(P)}\phi$ and therefore $\phi \in \underline{\Phi}_\pi^+(P^*)$. Hence $(P) \Rightarrow (P^*)$.

(2) Since $(P) \Rightarrow (P^*)$, assertion (4) of Theorem 5 yields

$$T^{(P)}\phi \leq T^{(P^*)}\phi.$$

Conversely,

$$T^{(P^*)}\phi = \sup\{\psi :\ \psi \in \underline{\Phi}_\pi^+(P^*),\ \psi \leq \phi\}.$$

By definition of (P^*), every element of $\varphi \in \underline{\Phi}_\pi^+(P^*)$ is of the form $T^{(P)}\varphi$ for $\varphi \in \varphi \in \underline{\Phi}_\pi^+$. Hence

$$T^{(P^*)}\phi = \sup\{T^{(P)}\varphi :\ \varphi \in \underline{\Phi}_\pi^+,\ T^{(P)}\varphi \leq \phi\}.$$

Using the idempotence property from Theorem 5,

$$T^{(P)}T^{(P)}\varphi = T^{(P)}\varphi,$$

we obtain

$$T^{(P^*)}\phi \leq T^{(P)}\phi.$$

Therefore

$$T^{(P^*)}\phi = T^{(P)}\phi.$$

(3) By assertion (2),

$$T^{(P^*)}\phi = T^{(P)}\phi,$$

and by definition $T^{(P)}\phi \in \underline{\Phi}_\pi^+(P^*)$. Hence $T^{(P^*)}\phi$ satisfies (P^*) for every $\phi \in \underline{\Phi}_\pi^+$, proving that (P^*) is consistent.

(4) Assume that $(P') \in \mathcal{P}_{cst}$ and that $(P) \Rightarrow (P')$. Then by Theorem 5,

$$T^{(P)}\phi \leq T^{(P')}\phi.$$

Let $\phi \in \underline{\Phi}_\pi^+(P^*)$. By definition

$$\phi = T^{(P^*)}\phi = T^{(P)}\phi.$$

Therefore

$$\phi = T^{(P)}\phi \leq T^{(P')}\phi \leq \phi,$$

which implies

$$\phi = T^{(P')}\phi.$$

Since (P') is consistent, we conclude that $\phi \in \underline{\Phi}_\pi^+(P')$. Hence

$$(P^*) \Rightarrow (P').$$

(5) If (P) and (P^*) are equivalent, then (P) is consistent because (P^*) is consistent by assertion (3).

Conversely, assume that (P) is consistent. Since $(P) \Rightarrow (P^*)$ by assertion (1), it remains to prove the converse implication. Applying assertion (4) with $(P') = (P)$, we obtain

$$(P^*) \Rightarrow (P).$$

Definition 8. *The property (P^*), introduced in Theorem 7 for a property $(P) \in \mathcal{P}$, is called the consistency hull of (P).*

We conclude this section with two illustrative examples of the consistency hull construction.

Example 6. *The coherency property, introduced in Definition 4, is the consistency hull of the property (Q), where a map $\phi \in \underline{\Phi}_\pi^+$ is said to satisfy the property (Q) if there exists a vector probability measure $\mathbb{Q}$ such that*

$$\phi(h) = \pi(\mathbb{E}^{\mathbb{Q}}(h)), \;\; h \in L_d^\infty.$$

Indeed, Theorem 2 shows that every coherent map can be represented as the supremum of maps satisfying property (Q). Consequently, coherency is precisely the smallest consistent property, which is implied by (Q).

Example 7. *Assume that Ω is finite. Then the joint property $((SA),(PH))$, namely positive homogeneity and subadditivity, is the consistency hull of the linearity property (L).*

Indeed, every positively homogeneous and subadditive map is convex. Moreover, by the Legendre-Fenchel representation theorem, every positively homogeneous convex map can be expressed as the supremum of a family of linear maps.

We now establish a dual representation for maps satisfying the consistency hull property associated with a given property $(P) \in \mathcal{P}$.

Theorem 8. *Let (P) be a non consistent property and let (P^*) denote its consistency hull. Then a map $\phi \in \underline{\Phi}_\pi^+$ satisfies the property (P^*) if and only if there exists a subset $K \subseteq \underline{\Phi}_\pi^+(P)$ such that*

$$\phi = \sup_{\varphi \in K} \varphi.$$

Proof. Suppose first that $\phi \in \underline{\Phi}_\pi^+(P^*)$. Then

$$\phi = T^{(P^*)}\phi = T^{(P)}\phi = \sup_{\varphi \in \Gamma(\phi;(P))} \varphi,$$

where

$$\Gamma(\phi; (P)) \subseteq \underline{\Phi}_\pi^+(P).$$

Hence ϕ is the supremum of a family of maps satisfying the property (P).

Conversely, assume that

$$\phi = \varphi_K := \sup_{\varphi \in K} \varphi,$$

for some $K \subseteq \underline{\Phi}_\pi^+(P)$. Since

$$(P) \Rightarrow (P^*),$$

we have $K \subseteq \underline{\Phi}_\pi^+(P^*)$. As (P^*) is consistent, Theorem 6 implies that the supremum of any finitely valued family of maps satisfying (P^*)) again satisfies (P^*). Therefore,

$$\phi = \varphi_K \in \underline{\Phi}_\pi^+(P^*).$$

We conclude by giving a further characterization of the consistency hull property.

Theorem 9. *Let $\phi \in \underline{\Phi}_\pi^+$, let $(P) \in \mathcal{P}$ and let (P^*) denote its consistency hull. Then the following assertions are equivalent:*

(i) *ϕ satisfies (P^*).*

(ii) *For every $h \in L_d^\infty$ such that $\phi(h) > 0$, there exists $\varphi \in \Gamma(\phi, (P))$ such that $\varphi(h) > 0$.*

(iii) *For every $h \in L_d^\infty$ and $y \in \mathbb{R}$ such that $\phi(h) > y$, there exists $\varphi \in \Gamma(\phi, (P))$ such that $\varphi(h) > y$.*

Proof. (1) $\Rightarrow$ (2) Let $\Theta := \Gamma(\phi, (P))$. Since ϕ satisfies (P^*), Theorem 8 yields

$$\phi = \sup_{\varphi \in \Theta} \varphi.$$

Equivalently

$$\mathcal{B}_\phi = \cap_{\varphi \in \Theta} \mathcal{B}_\varphi.$$

If $\phi(h) > 0$, then $h \notin \mathcal{B}_\phi$, hence $h \notin \mathcal{B}_\varphi$ for some $\varphi \in \Theta$. Therefore $\varphi(h) > 0$.

(2) $\Rightarrow$ (3) Let $h \in L_d^\infty$ and $y \in \mathbb{R}$ such that $\phi(h) > y$. For $e \in \mathbf{e}(\pi)$,

$$\phi(h - ye) = \phi(h) - y > 0.$$

By assertion (2), there exists $\varphi \in \Gamma(\phi, (P))$ such that $\varphi(h - ye) > 0$, which implies $\varphi(h) - y = \varphi(h - ye) > 0$ and then $\varphi(h) > y$.

(3) $\Rightarrow$ (2) This follows immediately by taking $y = 0$.

(2) $\Rightarrow$ (1) By definition

$$T^{(P)}\phi \leq \phi.$$

Suppose, by contradiction, that there exists $h \in L_d^\infty$ such that

$$\phi(h) > T^{(P)}\phi(h).$$

Let $e \in \mathbf{e}(\pi)$ and define $h' := h - T^{(P)}\phi(h)e$. Then $\phi(h') > 0$. By assertion (2), there exists $\varphi \in \Gamma(\phi, (P))$ such that $\varphi(h') > 0$. Hence

$$\varphi(h) - T^{(P)}\phi(h) > 0,$$

that is

$$\varphi(h) > T^{(P)}\phi(h),$$

which contradicts the definition of $T^{(P)}\phi$ as the supremum over all elements of $\Gamma(\phi, (P))$. Therefore,

$$\phi = T^{(P)}\phi,$$

which means that ϕ satisfies (P^*).

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