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Generalization of Ono's inequality

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Abstract. Let a, b, c be the side lengths of an acute triangle in the Euclidean plane, and let Δ denote its area. The classical inequality of Ono asserts that

$$27(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2(a^2+b^2-c^2)^2 \le (4\Delta)^6,$$

with equality if and only if the triangle is equilateral. The traditional proof relies on trigonometric identities and the AM–GM inequality.
In this paper, we provide a new analytic proof based on a Cartesian coordinate reduction and multivariable optimization techniques. By transforming the geometric inequality into a two-variable polynomial extremal problem, we establish the inequality using elementary tools from multivariable calculus. Furthermore, we derive a nontrivial two-parameter generalization valid for all acute triangles: for any $\alpha > 0$ and $\beta > 0$,

$$(4\Delta)^{2\alpha+\beta} \ge C_{\alpha,\beta}(b^2+c^2-a^2)^\alpha(c^2+a^2-b^2)^\alpha(a^2+b^2-c^2)^\beta,$$

where the optimal constant $C_{\alpha,\beta}$ is computed explicitly. The classical Ono inequality is recovered as the special case $\alpha = \beta = 2$. We also present an algebraic reformulation of the result and discuss the geometric structure underlying the equality cases.

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1. Introduction

In 1914, Tôda Ono conjectured an inequality for any triangle with side lengths a, b, and c, and area Δ, of the form

$$27\,(b^2+c^2-a^2)^2\,(c^2+a^2-b^2)^2\,(a^2+b^2-c^2)^2 \;\le\; (4\Delta)^6.$$

This statement, now known as Ono's inequality, was intended to hold universally for all triangles [1]. However, Quijano (1915) soon provided counterexamples demonstrating

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that the inequality does not hold in general. For instance, in the triangle with side lengths $a = \frac{3}{4}$, $b = \frac{1}{2}$, and $c = 1$, the inequality fails [2].

Indeed, for the triangle $a = \dfrac{3}{4}$, $b = \dfrac{1}{2}$, $c = 1$, we have

$$b^2 + c^2 - a^2 = \frac{11}{16}, \qquad c^2 + a^2 - b^2 = \frac{21}{16}, \qquad a^2 + b^2 - c^2 = -\frac{3}{16}.$$

Hence

$$\text{LHS} = 27\Big(\frac{11}{16}\Big)^2\Big(\frac{21}{16}\Big)^2\Big(\frac{3}{16}\Big)^2 = \frac{12966723}{256^3}.$$

By Heron's formula, $s = \frac{a+b+c}{2} = \frac{9}{8}$, so

$$\Delta^2 = s(s-a)(s-b)(s-c) = \frac{9}{8} \cdot \frac{3}{8} \cdot \frac{5}{8} \cdot \frac{1}{8} = \frac{135}{4096}, \qquad 4\Delta = \frac{3\sqrt{15}}{16},$$

and therefore

$$\text{RHS} = (4\Delta)^6 = \Big(\frac{3\sqrt{15}}{16}\Big)^6 = \frac{2460375}{256^3}.$$

Since $12966723 > 2460375$, we get LHS $>$ RHS, so Ono's inequality fails for this (obtuse) triangle.

Despite this counterexample, Balitrand (1916) proved that the inequality is indeed valid when restricted to the class of acute triangles [3]. The failure of Ono's inequality for obtuse triangles is intrinsic to its algebraic structure. If a triangle is obtuse with obtuse angle opposite the side c, then

$$a^2 + b^2 - c^2 < 0,$$

which measures the deviation from acuteness. In Ono's inequality this term appears, after squaring, as one of the three multiplicative factors on the left-hand side. As the triangle becomes more obtuse, the magnitude $|a^2 + b^2 - c^2|$ increases, and its square contributes a rapidly growing positive factor to the left-hand side. By contrast, the right-hand side $(4\Delta)^6$ depends only on the area and does not compensate for this growth, since the area does not increase proportionally with the obtuseness of the triangle. As a result, the balance between the two sides is lost, and the inequality can be violated. This shows that the restriction to acute triangles is not technical but necessary for the validity of Ono's inequality.

The classical proof of Ono's inequality relies on trigonometry and the AM-GM inequality. For completeness and comparative analysis of our new proof, we present the classical trigonometry based here. One can see [3] and ([4], p.241)

Theorem 1. *Let ABC be an acute triangle with side lengths a, b, c, angles A, B, C, and area Δ. Then*

$$27\,(b^2 + c^2 - a^2)^2(c^2 + a^2 - b^2)^2(a^2 + b^2 - c^2)^2 \le (4\Delta)^6, \tag{1}$$

with equality if and only if ABC is equilateral.

Proof. Since $a, b, c > 0$, dividing both sides of (1) by $64(abc)^4 > 0$ yields the equivalent form

$$27\,\frac{(b^2+c^2-a^2)^2}{4b^2c^2}\,\frac{(c^2+a^2-b^2)^2}{4a^2c^2}\,\frac{(a^2+b^2-c^2)^2}{4a^2b^2} \le \frac{4\Delta^2}{b^2c^2}\,\frac{4\Delta^2}{a^2c^2}\,\frac{4\Delta^2}{a^2b^2}. \tag{2}$$

By the cosine law,

$$a^2 = b^2 + c^2 - 2bc\cos A, \tag{3}$$

hence $b^2 + c^2 - a^2 = 2bc\cos A$, and therefore

$$\frac{(b^2+c^2-a^2)^2}{4b^2c^2} = \frac{(2bc\cos A)^2}{4b^2c^2} = \cos^2 A. \tag{4}$$

Similarly,

$$\frac{(c^2+a^2-b^2)^2}{4a^2c^2} = \cos^2 B, \qquad \frac{(a^2+b^2-c^2)^2}{4a^2b^2} = \cos^2 C. \tag{5}$$

Moreover, using the area formula $\Delta = \frac{1}{2}bc\sin A$, we obtain

$$\frac{4\Delta^2}{b^2c^2} = \frac{4\left(\frac{1}{2}bc\sin A\right)^2}{b^2c^2} = \sin^2 A, \tag{6}$$

and likewise

$$\frac{4\Delta^2}{a^2c^2} = \sin^2 B, \qquad \frac{4\Delta^2}{a^2b^2} = \sin^2 C. \tag{7}$$

Substituting (4)–(7) into (2) gives

$$27\,(\cos A\cos B\cos C)^2 \le (\sin A\sin B\sin C)^2. \tag{8}$$

Since the triangle is acute, $\cos A, \cos B, \cos C > 0$, and dividing (8) by $(\cos A\cos B\cos C)^2$ yields

$$27 \le (\tan A\,\tan B\,\tan C)^2, \qquad \text{equivalently} \qquad \tan A\,\tan B\,\tan C \ge 3\sqrt{3}. \tag{9}$$

For angles A, B, C satisfying $A + B + C = \pi$, one has the identity

$$\tan A + \tan B + \tan C = \tan A\,\tan B\,\tan C. \tag{10}$$

We introduce the positive quantities,

$$p = \tan A, \qquad q = \tan B, \qquad r = \tan C, \tag{11}$$

so that (10) becomes

$$p + q + r = pqr. \tag{12}$$

By the AM–GM inequality,

$$\frac{p+q+r}{3} \ge \sqrt[3]{pqr}. \tag{13}$$

Cubing (13) gives

$$(p+q+r)^3 \geq 27\,pqr. \tag{14}$$

Using (12) in (14) yields

$$(pqr)^3 \geq 27\,pqr. \tag{15}$$

Since $pqr > 0$, dividing (15) by pqr we obtain

$$(pqr)^2 \geq 27, \qquad \text{hence} \qquad pqr \geq 3\sqrt{3}. \tag{16}$$

By (11), inequality (16) is precisely (9), and therefore (1) follows.

Finally, equality holds in (1) if and only if equality holds in AM–GM (13), that is, if and only if $p = q = r$. Then $\tan A = \tan B = \tan C$ and with $A, B, C \in (0, \pi/2)$ and $A + B + C = \pi$, we get $A = B = C = \pi/3$, so the triangle is equilateral. Conversely, for an equilateral triangle one checks that equality holds in (1).

In this paper, we present an analytical proof of Ono's inequality and extend it to two parameters. The main difficulty addressed in this work lies in replacing traditional trigonometric and inequality-based arguments with a fully analytic, coordinate based optimization approach. Establishing Ono's inequality through multivariable calculus requires the reduction of a three variable geometric inequality to a two variable polynomial system, identifying and analyzing all interior and boundary critical configurations, and proving that the equilateral triangle uniquely minimizes the associated functional. Extending this method to a two–parameter generalization introduces further complexity, including balancing mixed homogeneity, handling multiple derivative branches, and ensuring positivity of the functional across the entire acute region.

We need the second partial derivative test from multivariable calculus stated below.

Proposition 1. *[5] Let $g(u, v)$ be a twice continuously differentiable real-valued function on an open set in $\mathbb{R}^2$. Suppose (u_0, v_0) is a critical point of g, that is,*

$$g_u(u_0, v_0) = 0, \quad g_v(u_0, v_0) = 0.$$

Define the Hessian determinant at (u_0, v_0) by

$$D \;=\; g_{uu}(u_0, v_0)\, g_{vv}(u_0, v_0) \;-\; \big(g_{uv}(u_0, v_0)\big)^2.$$

Then:

- *(i) If $D > 0$ and $g_{uu}(u_0, v_0) > 0$, then g has a local minimum at (u_0, v_0).*
- *(ii) If $D > 0$ and $g_{uu}(u_0, v_0) < 0$, then g has a local maximum at (u_0, v_0).*
- *(iii) If $D < 0$, then (u_0, v_0) is a saddle point of g.*
- *(iv) If $D = 0$, the test is inconclusive, and no determination can be made using only second derivatives.*

2. Main Results

Our new proof of Ono's inequality proceeds by a Cartesian-coordinate reduction and an analysis of a single two-variable polynomial using multivariable calculus.

Theorem 2. *Let a, b, c be the side lengths of an acute triangle in the Euclidean plane and let Δ denote its area. Then*

$$27\,(b^2+c^2-a^2)^2\,(c^2+a^2-b^2)^2\,(a^2+b^2-c^2)^2 \;\le\; (4\Delta)^6.$$

Equality holds if and only if the triangle is equilateral.

Proof. Both sides of the Ono's inequality are homogeneous in the side lengths. Each of the three factors of the left-hand side, e.g. $b^2+c^2-a^2$, is homogeneous of degree 2 in (a, b, c), hence the left-hand side is homogeneous of degree $2 \cdot 2 \cdot 3 = 12$, the right-hand side $(4\Delta)^6$ has degree $6 \cdot 2 = 12$ because area Δ is homogeneous of degree 2 in the side lengths. Consequently, if we multiply all side lengths by a common factor $\lambda > 0$ both sides are multiplied by λ^{12}, so the inequality is scale-invariant.

Therefore we may fix the scale. Put $c = 1$, it suffices to prove the inequality under this normalization, the general case follows by rescaling.

With this normalization, we may place the triangle in the Cartesian plane with vertices

$$A = (0,0), \qquad B = (1,0), \qquad C = (x,y),$$

where $0 < x < 1$ and $y > 0$. Then the squared side lengths are

$$b^2 = x^2+y^2, \qquad a^2 = (x-1)^2+y^2, \qquad c^2 = 1. \tag{17}$$

It follows that

$$b^2+c^2-a^2 = (x^2+y^2)+1-\big((x-1)^2+y^2\big) = 2x, \tag{18}$$

$$c^2+a^2-b^2 = 1+\big((x-1)^2+y^2\big)-(x^2+y^2) = 2(1-x), \tag{19}$$

$$a^2+b^2-c^2 = (x-1)^2+y^2+(x^2+y^2)-1 = 2(x^2-x+y^2). \tag{20}$$

Hence the product appearing on the left-hand side on Ono's inequality becomes

$$(b^2+c^2-a^2)^2\,(c^2+a^2-b^2)^2\,(a^2+b^2-c^2)^2 = 2^6\,x^2(1-x)^2(x^2-x+y^2)^2. \tag{21}$$

The area of the triangle is

$$\Delta = \frac{1}{2}\cdot c \cdot y = \frac{y}{2}, \tag{22}$$

so that

$$(4\Delta)^6 = (2y)^6 = 2^6 y^6. \tag{23}$$

Dividing (23) by 2^6 and comparing with (21), the inequality is equivalent to

$$G(x,y) = y^6 - 27\,x^2(1-x)^2(x^2-x+y^2)^2 \;\ge\; 0, \tag{24}$$

for all admissible (x, y) corresponding to an acute triangle, that is, $0 < x < 1$, $y > 0$, and $x^2 - x + y^2 > 0$. Now, we will apply Proposition 1 to the function G. Differentiating (24), one obtains

$$G_x(x, y) = 54\,x(1-x)(2x-1)(x^2 - x + y^2)(2x^2 - 2x + y^2), \tag{25}$$

$$G_y(x, y) = 6y\big(y^4 - 18x^2(1-x)^2(x^2 - x + y^2)\big). \tag{26}$$

Interior critical points must satisfy $G_x = 0$ and $G_y = 0$. From (25) either

$$x = \tfrac{1}{2}, \tag{27}$$

or

$$2x^2 - 2x + y^2 = 0 \quad \Longleftrightarrow \quad y^2 = 2x(1-x). \tag{28}$$

If $x = \frac{1}{2}$, substituting into (26) yields

$$G_y\big(\tfrac{1}{2}, y\big) = \tfrac{3y}{16}\,(4y^2 - 3)(8y^2 - 3).$$

Thus $y = \sqrt{3}/2$ or $y = \sqrt{3/8}$ are critical values, giving candidate points

$$\Big(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\Big), \qquad \Big(\tfrac{1}{2}, \sqrt{\tfrac{3}{8}}\Big).$$

If instead $y^2 = 2x(1-x)$, then $x^2 - x + y^2 = x(1-x)$. Substituting into (26) gives

$$G_y = 6y\,x^2(1-x)^2\big(4 - 18x(1-x)\big) = 0,$$

so that $x(1-x) = \frac{2}{9}$, hence $x = \frac{1}{3}$ or $x = \frac{2}{3}$, with $y^2 = \frac{4}{9}$. This yields the points

$$\Big(\tfrac{1}{3}, \tfrac{2}{3}\Big), \qquad \Big(\tfrac{2}{3}, \tfrac{2}{3}\Big).$$

Evaluating G at the critical points:
1. At $\big(\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$ we have $y^2 = \frac{3}{4}$, $x^2 - x + y^2 = \frac{1}{2}$, hence

$$G = \Big(\tfrac{3}{4}\Big)^3 - 27 \cdot \Big(\tfrac{1}{2}\Big)^2 \cdot \Big(\tfrac{1}{2}\Big)^2 = \frac{27}{64} - \frac{27}{64} = 0. \tag{29}$$

2. At $\big(\frac{1}{2}, \sqrt{\frac{3}{8}}\big)$ we have $y^2 = \frac{3}{8}$, $x^2 - x + y^2 = \frac{1}{8}$, hence

$$G = \Big(\tfrac{3}{8}\Big)^3 - 27 \cdot \Big(\tfrac{1}{4}\Big) \cdot \Big(\tfrac{1}{8}\Big)^2 = \frac{27}{512} - \frac{27}{1024} = \frac{27}{1024} > 0. \tag{30}$$

3. At $\big(\frac{1}{3}, \frac{2}{3}\big)$ or $\big(\frac{2}{3}, \frac{2}{3}\big)$ we have $x(1-x) = \frac{2}{9}$, so $x^2 - x + y^2 = \frac{2}{9}$, $y^2 = \frac{4}{9}$, and

$$G = \Big(\tfrac{2}{3}\Big)^6 - 27\Big(\tfrac{2}{9}\Big)^4 = \frac{64}{729} - \frac{48}{729} = \frac{16}{729} > 0. \tag{31}$$

Thus among interior candidates, $G \geq 0$ with equality only at $\big(\frac{1}{2}, \frac{\sqrt{3}}{2}\big)$.

At this point, evaluating second derivatives gives

$$G_{xx}\Big(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\Big) = \frac{27}{8}, \qquad G_{yy}\Big(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\Big) = \frac{27}{8}, \qquad G_{xy}\Big(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\Big) = 0,$$

so the Hessian is positive definite, proving that this is a strict local minimum.

On the boundary as $y \to \infty$, we have $G \sim y^6 \to +\infty$. On the arc $x^2 - x + y^2 = 0$ corresponding to a right angle, $G = y^6 > 0$. As $x \to 0^+$ or $x \to 1^-$ with $y > 0$, the factor $x^2(1-x)^2 \to 0$, so $G \to y^6 \geq 0$. Therefore G attains its global minimum at the equilateral configuration (29).

Undoing the normalization, the inequality (24) implies

$$27\,(b^2 + c^2 - a^2)^2\,(c^2 + a^2 - b^2)^2\,(a^2 + b^2 - c^2)^2 \;\leq\; (4\Delta)^6,$$

with equality if and only if the triangle is equilateral.

From a purely algebraic point of view, Ono's inequality involves the symmetric product

$$\big(b^2 + c^2 - a^2\big)^2\big(c^2 + a^2 - b^2\big)^2\big(a^2 + b^2 - c^2\big)^2,$$

where the fixed exponent 2 appears in each factor. A natural question is whether this exponent may be replaced by higher powers, leading to a broader family of sharp algebraic inequalities.

In particular, one may ask whether an inequality of comparable form persists when the above product is replaced by

$$\big(b^2 + c^2 - a^2\big)^{\alpha}\big(c^2 + a^2 - b^2\big)^{\beta}\big(a^2 + b^2 - c^2\big)^{\gamma}, \qquad \alpha, \beta, \gamma > 0,$$

and whether the similar symmetry and extremal behavior of the classical inequality are preserved. Motivated by this observation, we focus on the case $\alpha = \beta > 0$. Under this parametric deformation, the inequality remains valid in sharp form, and the equilateral configuration continues to be the unique equality case. More precisely, we have the following generalization.

Theorem 3. *Let ABC be an acute triangle with side lengths a, b, c and area Δ. Let $\alpha > 0$ and $\beta > 0$.*

(i-1) *If $\alpha > \beta/2$, then for every acute triangle,*

$$(4\Delta)^{2\alpha+\beta} \;\geq\; C_{\alpha,\beta}\,(b^2 + c^2 - a^2)^{\alpha}(c^2 + a^2 - b^2)^{\alpha}(a^2 + b^2 - c^2)^{\beta}, \tag{32}$$

where the best constant is

$$C_{\alpha,\beta} = 2^{\beta}\,\frac{\big(\alpha + \frac{\beta}{2}\big)^{\alpha+\beta/2}}{\beta^{\beta}\big(\alpha - \frac{\beta}{2}\big)^{\alpha-\beta/2}}. \tag{33}$$

(i-2) *Moreover, the minimizing configuration (hence the equality case in* (32)*) is unique up to similarity and occurs for an acute isosceles triangle with $a = b$.*

(ii) *If* $\alpha = \beta/2$, *then* (32) *holds for all acute triangles with best constant* $C_{\alpha,\beta} = 2^{\beta}$, *but equality is not attained (it is approached as an acute triangle tends to a right triangle).*

(iii) *If* $\alpha < \beta/2$, *then there is no positive constant* $C_{\alpha,\beta}$ *for which* (32) *holds for all acute triangles (the best constant is* 0*).*

In particular, choosing $\alpha = \beta = 2$ *in* (32) *yields the classical Ono inequality with sharp constant* 27.

Proof. Both sides of (32) are homogeneous of degree $2(2\alpha + \beta)$ in (a, b, c). Hence we may normalize $c = 1$. We may place the triangle in the plane as

$$A = (0, 0), \qquad B = (1, 0), \qquad C = (x, y),$$

with $0 < x < 1$ and $y > 0$. Then

$$\Delta = \frac{y}{2}, \qquad b^2+c^2-a^2 = 2x, \qquad c^2+a^2-b^2 = 2(1-x), \qquad a^2+b^2-c^2 = 2(x^2-x+y^2). \tag{34}$$

We set

$$f(x) = x(1-x), \qquad g(x, y) = x^2 - x + y^2 = y^2 - f(x). \tag{35}$$

Since ABC is acute, each of the three expressions in (34) is positive, so in particular

$$0 < x < 1, \qquad y > 0, \qquad g(x, y) > 0. \tag{36}$$

Substituting (34) into (32) and cancelling the common factor $2^{2\alpha+\beta} > 0$ yields the equivalent inequality

$$y^{2\alpha+\beta} \ \geq\ C_{\alpha,\beta}\, f(x)^{\alpha}\, g(x, y)^{\beta}, \qquad (x, y) \text{ satisfying (36)}. \tag{37}$$

Hence the largest admissible constant equals the infimum

$$C_{\alpha,\beta} = \inf_{(x,y)\ \text{satisfying (36)}} \frac{y^{2\alpha+\beta}}{f(x)^{\alpha} g(x, y)^{\beta}}. \tag{38}$$

We define

$$R(x, y) = \frac{y^{2\alpha+\beta}}{f(x)^{\alpha} g(x, y)^{\beta}} \quad \text{and} \quad J(x, y) = \log R(x, y).$$

Since log is strictly increasing, R attains its minimum exactly where J does, and (38) becomes

$$C_{\alpha,\beta} = \inf R = \exp\Big(\inf J\Big). \tag{39}$$

Writing $y^{2\alpha+\beta} = (y^2)^{\alpha+\beta/2}$, we have

$$J(x, y) = \left(\alpha + \frac{\beta}{2}\right) \log(y^2) - \alpha \log f(x) - \beta \log g(x, y). \tag{40}$$

On the open domain (36), J is twice continuously differentiable.

We will compute critical points of J. We differentiate (40). Using $f'(x) = 1 - 2x$ and $g_x(x, y) = 2x - 1$, $g_y(x, y) = 2y$, we get

$$J_x(x, y) = -\alpha \frac{f'(x)}{f(x)} - \beta \frac{g_x(x, y)}{g(x, y)} = -(1 - 2x) \left(\frac{\alpha}{f(x)} - \frac{\beta}{g(x, y)} \right), \tag{41}$$

$$J_y(x, y) = \left(\alpha + \frac{\beta}{2} \right) \frac{2y}{y^2} - \beta \frac{2y}{g(x, y)} = 2y \left(\frac{\alpha + \beta/2}{y^2} - \frac{\beta}{g(x, y)} \right). \tag{42}$$

At an interior critical point (x_0, y_0) we have $J_x = J_y = 0$. Since $y_0 > 0$, equation $J_y = 0$ gives

$$\frac{\alpha + \beta/2}{y_0^2} = \frac{\beta}{g(x_0, y_0)} \quad \Longleftrightarrow \quad g(x_0, y_0) = \frac{\beta}{\alpha + \beta/2} y_0^2. \tag{43}$$

Using $g = y^2 - f$, relation (43) becomes

$$y_0^2 - f(x_0) = \frac{\beta}{\alpha + \beta/2} y_0^2,$$

hence

$$y_0^2 \left(1 - \frac{\beta}{\alpha + \beta/2} \right) = f(x_0) \quad \Longleftrightarrow \quad y_0^2 \frac{\alpha - \beta/2}{\alpha + \beta/2} = f(x_0). \tag{44}$$

Thus an interior critical point can exist only if $\alpha - \beta/2 > 0$, i.e. $\alpha > \beta/2$.

Now, we analyze three cases $\alpha > \beta/2, \alpha = \beta/2$ and $\alpha < \beta/2$.

Case 1: $\alpha > \beta/2$**.** This case is non-trivial, therefore we divide the proof into many steps.
Step 1: In this step, we will compute critical points of J. Assume from now on $\alpha > \beta/2$ and set

$$A = \alpha + \frac{\beta}{2} > 0, \qquad B = \alpha - \frac{\beta}{2} > 0. \tag{45}$$

Then (44) becomes

$$y_0^2 = \frac{A}{B} f(x_0), \qquad g(x_0, y_0) = y_0^2 - f(x_0) = \left(\frac{A}{B} - 1 \right) f(x_0) = \frac{\beta}{B} f(x_0). \tag{46}$$

Next, equation $J_x = 0$ from (41) gives

$$(1 - 2x_0) \left(\frac{\alpha}{f(x_0)} - \frac{\beta}{g(x_0, y_0)} \right) = 0. \tag{47}$$

If $1 - 2x_0 \neq 0$, then $\alpha/f = \beta/g$, i.e. $g = (\beta/\alpha)f$. Comparing with (46) gives $\beta/\alpha = \beta/B$, hence $\alpha = B = \alpha - \beta/2$, impossible since $\beta > 0$. Therefore $1 - 2x_0 = 0$, so

$$x_0 = \frac{1}{2}. \tag{48}$$

Substituting $x_0 = 1/2$ into (46) yields $f(x_0) = \frac{1}{4}$ and thus the unique interior critical point is

$$(x_0, y_0) = \left(\frac{1}{2},\ \frac{1}{2}\sqrt{\frac{A}{B}}\right) = \left(\frac{1}{2},\ \frac{1}{2}\sqrt{\frac{\alpha+\beta/2}{\alpha-\beta/2}}\right). \tag{49}$$

Step 2: In this step, we will compute the Hessian of J at (x_0, y_0) and apply Proposition 1.

From (41) we have

$$J_x(x,y) = (2x-1)\left(\frac{\alpha}{f(x)} - \frac{\beta}{g(x,y)}\right).$$

Differentiate with respect to x, we obtain

$$J_{xx}(x,y) = 2\left(\frac{\alpha}{f(x)} - \frac{\beta}{g(x,y)}\right) + (2x-1)\frac{d}{dx}\left(\frac{\alpha}{f(x)} - \frac{\beta}{g(x,y)}\right).$$

At $x_0 = \frac{1}{2}$ we have $2x_0 - 1 = 0$, hence

$$J_{xx}(x_0,y_0) = 2\left(\frac{\alpha}{f(x_0)} - \frac{\beta}{g(x_0,y_0)}\right). \tag{50}$$

Using $f(x_0) = \frac{1}{4}$ and $g(x_0,y_0) = \frac{\beta}{B}\cdot\frac{1}{4} = \frac{\beta}{4B}$ from (46), we compute

$$\frac{\alpha}{f(x_0)} = 4\alpha, \qquad \frac{\beta}{g(x_0,y_0)} = \frac{\beta}{\beta/(4B)} = 4B = 4\alpha - 2\beta.$$

Substituting into (50) gives

$$J_{xx}(x_0,y_0) = 2\big(4\alpha - (4\alpha - 2\beta)\big) = 4\beta > 0. \tag{51}$$

Next, differentiate (41) with respect to y. Only g depends on y, and since $\partial_y(1/g) = -(g_y/g^2) = -(2y/g^2)$, we obtain

$$J_{xy}(x,y) = (2x-1)\,(-\beta)\left(-\frac{2y}{g(x,y)^2}\right) = (2x-1)\frac{2\beta y}{g(x,y)^2}.$$

At $x_0 = 1/2$ this gives

$$J_{xy}(x_0,y_0) = 0. \tag{52}$$

Finally, from (42) we can write

$$J_y(x,y) = \frac{2A}{y} - \frac{2\beta y}{g(x,y)} \qquad (A = \alpha + \beta/2).$$

Differentiate with respect to y, we obtain

$$J_{yy}(x,y) = -\frac{2A}{y^2} - 2\beta\,\frac{d}{dy}\left(\frac{y}{g}\right).$$

Since $\dfrac{d}{dy}\left(\dfrac{y}{g}\right) = \dfrac{g - yg_y}{g^2} = \dfrac{g-2y^2}{g^2}$, we get

$$J_{yy}(x,y) = -\frac{2A}{y^2} - 2\beta\,\frac{g(x,y)-2y^2}{g(x,y)^2}. \tag{53}$$

At the critical point, (43) gives $g(x_0,y_0) = \dfrac{\beta}{A}y_0^2$. Hence

$$g(x_0,y_0) - 2y_0^2 = y_0^2\left(\frac{\beta}{A} - 2\right) = y_0^2\left(\frac{\beta-2A}{A}\right) = y_0^2\left(\frac{-2\alpha}{A}\right) = -\frac{2\alpha}{A}\,y_0^2.$$

Substituting this and $g(x_0,y_0) = \dfrac{\beta}{A}y_0^2$ into (53) yields

$$\begin{aligned} J_{yy}(x_0,y_0) &= -\frac{2A}{y_0^2} - 2\beta\,\frac{-\frac{2\alpha}{A}y_0^2}{\left(\frac{\beta}{A}y_0^2\right)^2} = -\frac{2A}{y_0^2} + \frac{4\alpha\beta}{A}\cdot\frac{A^2}{\beta^2}\cdot\frac{1}{y_0^2} \\ &= \left(-2A + \frac{4\alpha A}{\beta}\right)\frac{1}{y_0^2} = \frac{2A(2\alpha-\beta)}{\beta\,y_0^2}. \end{aligned} \tag{54}$$

Since $\alpha > \beta/2$, we have $2\alpha - \beta > 0$, hence $J_{yy}(x_0,y_0) > 0$.

Now the Hessian determinant at (x_0,y_0) equals

$$D = J_{xx}(x_0,y_0)\,J_{yy}(x_0,y_0) - J_{xy}(x_0,y_0)^2 = (4\beta)\left(\frac{2A(2\alpha-\beta)}{\beta\,y_0^2}\right) - 0 = \frac{8A(2\alpha-\beta)}{y_0^2} > 0. \tag{55}$$

By Proposition 1, J has a strict local minimum at the critical point (49).

Step 3: In this step, we will show that this local minimum is global. From (40) we have the following observations:

a. As $x \to 0^+$ or $x \to 1^-$, we have $f(x) = x(1-x) \to 0^+$, hence $-\alpha \log f(x) \to +\infty$, so $J(x,y) \to +\infty$.

b. As $g(x,y) \to 0^+$ (the right–triangle boundary), $-\beta \log g(x,y) \to +\infty$, so $J(x,y) \to +\infty$.

c. Let $x \in (0,1)$ be fixed and let $y \to \infty$. Recall that

$$g(x,y) = y^2 - f(x), \qquad f(x) = x(1-x) \in \left(0, \frac{1}{4}\right].$$

Choose $Y > 0$ such that $y^2 \geq 2f(x)$ for all $y \geq Y$. Then for $y \geq Y$ we have

$$\frac{1}{2}\,y^2 \leq g(x,y) \leq y^2.$$

Consequently,

$$\log g(x,y) \le \log(y^2), \qquad \log g(x,y) \ge \log\Big(\tfrac{1}{2}\,y^2\Big) = \log(y^2) - \log 2.$$

Recall that

$$J(x,y) = A\log(y^2) - \alpha\log f(x) - \beta\log g(x,y), \qquad A = \alpha + \frac{\beta}{2}.$$

Using the upper bound for $\log g(x,y)$, we obtain for all $y \ge Y$,

$$\begin{aligned} J(x,y) &\ge A\log(y^2) - \alpha\log f(x) - \beta\log(y^2) \\ &= (A-\beta)\log(y^2) - \alpha\log f(x) \\ &= \left(\alpha - \frac{\beta}{2}\right)\log(y^2) - \alpha\log f(x). \end{aligned}$$

Since $\alpha - \beta/2 > 0$ and $f(x) > 0$ is fixed, the right–hand side tends to $+\infty$ as $y \to \infty$. Hence

$$\lim_{y\to\infty} J(x,y) = +\infty \qquad \text{for every fixed } x \in (0,1).$$

Therefore J tends to $+\infty$ on every approach to the boundary and at infinity, so the unique interior critical point (49) is the global minimizer of J and hence of R.

Step 4: In this step, we will evaluate the sharp constant $C_{\alpha,\beta}$. At the minimizer, we have $x_0 = \frac{1}{2}$, hence $f(x_0) = \frac{1}{4}$. From (46) we also have

$$y_0^2 = \frac{A}{B}\cdot\frac{1}{4}, \qquad g(x_0,y_0) = y_0^2 - \frac{1}{4} = \frac{\beta}{B}\cdot\frac{1}{4}.$$

Thus

$$\begin{aligned} C_{\alpha,\beta} = \min R = \frac{y_0^{2\alpha+\beta}}{f(x_0)^\alpha g(x_0,y_0)^\beta} &= \frac{(y_0^2)^{\alpha+\beta/2}}{(1/4)^\alpha\left(\frac{\beta}{4B}\right)^\beta} \\ &= \frac{\left(\frac{A}{4B}\right)^{\alpha+\beta/2}}{4^{-\alpha}\,\beta^\beta\,4^{-\beta}\,B^{-\beta}} = 4^{\alpha+\beta}\left(\frac{A}{4B}\right)^{\alpha+\beta/2}\frac{B^\beta}{\beta^\beta} \\ &= 4^{\beta/2}\,\frac{A^{\alpha+\beta/2}}{\beta^\beta\,B^{\alpha-\beta/2}} = 2^\beta\,\frac{\left(\alpha+\frac{\beta}{2}\right)^{\alpha+\beta/2}}{\beta^\beta\left(\alpha-\frac{\beta}{2}\right)^{\alpha-\beta/2}}, \end{aligned} \tag{56}$$

which is (33). This proves part (i-1) of the theorem, including sharpness.

Step 5: In this step, we will establish part (i-2) of the theorem.

Assume $\alpha > \beta/2$ so that the sharp constant $C_{\alpha,\beta}$ is given by (33). In the proof, the sharp constant arises as

$$C_{\alpha,\beta} = \min_{(x,y)\in\mathcal{D}} R(x,y), \qquad R(x,y) = \frac{y^{2\alpha+\beta}}{f(x)^{\alpha} g(x,y)^{\beta}},$$

where $\mathcal{D} = \{(x,y) : 0 < x < 1,\ y > 0,\ g(x,y) > 0\}$, $f(x) = x(1-x)$ and $g(x,y) = x^2 - x + y^2$. We showed that R (equivalently $J = \log R$) has a unique global minimizer

$$(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\sqrt{\frac{\alpha+\beta/2}{\alpha-\beta/2}}\right). \tag{57}$$

Therefore, in the normalized model $c = 1$, equality in (32) holds if and only if $(x,y) = (x_0, y_0)$.

We now translate (57) into a geometric condition on the triangle. Since $x_0 = \frac{1}{2}$, we have

$$a^2 = |BC|^2 = (x_0-1)^2 + y_0^2 = \frac{1}{4} + y_0^2, \qquad b^2 = |AC|^2 = x_0^2 + y_0^2 = \frac{1}{4} + y_0^2,$$

hence $a = b$. Recalling that we normalized $c = 1$, we conclude that the equality triangle is isosceles with

$$a = b, \qquad c = 1, \qquad y_0^2 = \frac{1}{4}\cdot\frac{\alpha+\beta/2}{\alpha-\beta/2}. \tag{58}$$

In particular,

$$\frac{a^2}{c^2} = a^2 = \frac{1}{4} + y_0^2 = \frac{1}{4} + \frac{1}{4}\cdot\frac{\alpha+\beta/2}{\alpha-\beta/2} = \frac{\alpha}{2(\alpha-\beta/2)}.$$

Since this ratio is scale-invariant, removing the normalization $c = 1$ yields the following intrinsic characterization,

$$\text{Equality in (32) holds} \iff ABC \text{ is acute and isosceles with } a = b \text{ and } \frac{a^2}{c^2} = \frac{\alpha}{2(\alpha-\beta/2)}. \tag{59}$$

Equivalently, the equality condition can be written as

$$\text{Equality holds} \iff ABC \text{ is acute, } a = b, \text{ and } a^2 = \frac{\alpha}{2(\alpha-\beta/2)}\,c^2. \tag{60}$$

One may of course write the same condition with any pair of equal sides, depending on the labeling of the triangle, the condition selects a unique similarity class.

Case 2: $\alpha = \beta/2$**.** Here $A = \alpha+\beta/2 = \beta$ and $B = \alpha-\beta/2 = 0$, so the interior critical point computation above shows there is no interior minimizer. Using $R(x,y) = y^{2\beta}/(f^{\beta/2}g^{\beta})$ and the substitution $g = y^2 - f$, one checks that along $x = 1/2$,

$$R\left(\tfrac{1}{2}, y\right) = \frac{y^{2\beta}}{(1/4)^{\beta/2}(y^2-1/4)^{\beta}} = 2^{\beta}\left(\frac{y^2}{y^2-1/4}\right)^{\beta},$$

which decreases to 2^β as $y \to \infty$. Hence $\inf R = 2^\beta$, i.e. $C_{\alpha,\beta} = 2^\beta$, and it is not attained (this corresponds to acute triangles approaching a right triangle), and hence (32) is strict for every acute triangle. This proves (ii).

Case 3: $\alpha < \beta/2$**.** Then $\alpha - \beta/2 < 0$. Taking $x = 1/2$ and letting $y \to \infty$, we have $g = y^2 - 1/4 \sim y^2$, and therefore

$$R\Big(\tfrac{1}{2}, y\Big) = \frac{y^{2\alpha+\beta}}{(1/4)^\alpha (y^2 - 1/4)^\beta} \sim 4^\alpha\, y^{2\alpha-\beta} \;\longrightarrow\; 0,$$

so $\inf R = 0$ and no positive constant can satisfy (32), and hence there is no nontrivial equality case. This proves (iii).

The following corollary shows that the classical Ono's inequality is a special case of Theorem 3.

Corollary 1. *Let ABC be an acute triangle. Taking $\alpha = \beta = 2$ in* (32) *and using* (33) *gives*

$$C_{2,2} = 2^2\, \frac{(2+1)^{2+1}}{2^2 (2-1)^{2-1}} = 4 \cdot \frac{27}{4} = 27.$$

Therefore (32) *becomes*

$$(4\Delta)^6 \ge 27\, (b^2 + c^2 - a^2)^2 (c^2 + a^2 - b^2)^2 (a^2 + b^2 - c^2)^2,$$

which is precisely Ono's inequality for acute triangles.

Moreover, from (59) *with $\alpha = \beta = 2$ we obtain*

$$\frac{a^2}{c^2} = \frac{2}{2(2-1)} = 1,$$

so $a = c$ and, together with $a = b$, we get $a = b = c$. Hence, in the acute setting,

Equality in Ono's inequality holds $\iff$ ABC is equilateral.

There is active research on inequalities involving several real variables. We present the algebraic equivalent reformulation of Theorem 2.2 and of the classical Ono inequality as follows.

Theorem 4. *Let the real numbers $a, b, c > 0$ satisfy the triangle inequalities*

$$a < b + c, \qquad b < c + a, \qquad c < a + b, \tag{61}$$

and the acute conditions

$$b^2 + c^2 - a^2 > 0, \qquad c^2 + a^2 - b^2 > 0, \qquad a^2 + b^2 - c^2 > 0. \tag{62}$$

Define

$$s = \frac{a + b + c}{2}, \qquad \Delta = \sqrt{s(s-a)(s-b)(s-c)}. \tag{63}$$

Let $\alpha > 0$ and $\beta > 0$.

(i) *If $\alpha > \beta/2$, then for all triples (a, b, c) satisfying (61)–(62) we have*

$$(4\Delta)^{2\alpha+\beta} \;\geq\; C_{\alpha,\beta}\,(b^2+c^2-a^2)^{\alpha}(c^2+a^2-b^2)^{\alpha}(a^2+b^2-c^2)^{\beta}, \tag{64}$$

where the constant is best possible and equals

$$C_{\alpha,\beta} = 2^{\beta}\,\frac{\left(\alpha+\frac{\beta}{2}\right)^{\alpha+\beta/2}}{\beta^{\beta}\left(\alpha-\frac{\beta}{2}\right)^{\alpha-\beta/2}}. \tag{65}$$

Equality in (64) holds if and only if (a, b, c) satisfies, up to a permutation of the variables,

$$a = b, \qquad \frac{a^2}{c^2} = \frac{\alpha}{2(\alpha-\beta/2)} \qquad \Big(\textit{equivalently } c^2 = \frac{2(\alpha-\beta/2)}{\alpha}\,a^2\Big), \tag{66}$$

and the triple is acute (which is automatic under (66) when $\alpha > \beta/2$).

(ii) *If $\alpha = \beta/2$, then (64) holds for all triples satisfying (61)–(62) with best constant $C_{\alpha,\beta} = 2^{\beta}$, and equality is not attained.*

(iii) *If $\alpha < \beta/2$, then there is no $C_{\alpha,\beta} > 0$ for which (64) holds for all triples satisfying (61)–(62) (the best constant is $C_{\alpha,\beta} = 0$).*

In particular, taking $\alpha = \beta = 2$ gives

$$27\,(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2(a^2+b^2-c^2)^2 \leq (4\Delta)^6, \tag{67}$$

and equality in (67) holds if and only if $a = b = c$.

Few applications of the Theorem 4 to the symmetric algebraic inequalities are given.

Corollary 2. *Let $x, y, z > 0$ and define*

$$a = y+z, \qquad b = z+x, \qquad c = x+y.$$

Assume the resulting triangle is acute (equivalently, the three quantities in (62) are positive). Let

$$s = \frac{a+b+c}{2} = x+y+z, \qquad \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Then for $\alpha > 0$, $\beta > 0$ with $\alpha > \beta/2$ we have

$$\begin{aligned}\Big(4\sqrt{(x+y+z)\,xyz}\Big)^{2\alpha+\beta} &\geq C_{\alpha,\beta}\Big(2x(x+y+z)-2yz\Big)^{\alpha}\Big(2y(x+y+z)-2xz\Big)^{\alpha}\\ &\quad\times\Big(2z(x+y+z)-2xy\Big)^{\beta},\end{aligned} \tag{68}$$

where $C_{\alpha,\beta}$ is given by (65). Moreover, equality in (68) holds if and only if, up to a permutation of x, y, z,

$$x = y \quad \textit{and} \quad \frac{y+z}{x+y} = \sqrt{\frac{\alpha}{2(\alpha-\beta/2)}}.$$

Proof. We verify the hypotheses of Theorem 4 for the triple

$$(a, b, c) = (y + z,\ z + x,\ x + y).$$

Since $x, y, z > 0$, we have

$$b + c = (z + x) + (x + y) = 2x + y + z > a = y + z,$$

and similarly $c + a > a + b$ and $a + b > c$. Hence (61) holds.

We compute the three expressions in (62):

$$\begin{aligned} b^2 + c^2 - a^2 &= (x + z)^2 + (x + y)^2 - (y + z)^2 \\ &= x^2 + 2xz + z^2 + x^2 + 2xy + y^2 - (y^2 + 2yz + z^2) \\ &= 2x^2 + 2x(y + z) - 2yz = 2\big(x(x + y + z) - yz\big), \\ c^2 + a^2 - b^2 &= (x + y)^2 + (y + z)^2 - (x + z)^2 = 2\big(y(x + y + z) - xz\big), \\ a^2 + b^2 - c^2 &= (y + z)^2 + (x + z)^2 - (x + y)^2 = 2\big(z(x + y + z) - xy\big). \end{aligned}$$

Thus the acute conditions (62) become precisely the positivity of the three factors

$$2x(x + y + z) - 2yz, \qquad 2y(x + y + z) - 2xz, \qquad 2z(x + y + z) - 2xy,$$

which is exactly the assumption that the induced triangle is acute.

From $a = y + z$, $b = z + x$, $c = x + y$ we obtain

$$s = \frac{a + b + c}{2} = \frac{(y + z) + (z + x) + (x + y)}{2} = x + y + z.$$

Moreover

$$s-a = (x+y+z)-(y+z) = x, \quad s-b = (x+y+z)-(z+x) = y, \quad s-c = (x+y+z)-(x+y) = z.$$

Therefore by (63),

$$\Delta = \sqrt{s(s - a)(s - b)(s - c)} = \sqrt{(x + y + z)\, xyz}.$$

Substituting the above expressions for Δ and the three acute factors into (64) yields (68).

Equality in Theorem 4 holds (up to permutation of a, b, c) when $a = b$ and $a^2/c^2 = \alpha/(2(\alpha - \beta/2))$. With $a = y + z$, $b = z + x$, the condition $a = b$ gives $y + z = z + x$, hence $x = y$. Then $c = x + y = 2x$ and $a = y + z = x + z$, so

$$\frac{a^2}{c^2} = \frac{(x + z)^2}{(2x)^2} = \frac{(x + z)^2}{4x^2} = \frac{\alpha}{2(\alpha - \beta/2)}.$$

This is equivalent to $\frac{x+z}{2x} = \sqrt{\frac{\alpha}{2(\alpha-\beta/2)}}$, i.e. $\frac{y+z}{x+y} = \sqrt{\frac{\alpha}{2(\alpha-\beta/2)}}$ under $x = y$.

Corollary 3. *Let $x, y, z > 0$ and define*

$$a = \sqrt{y+z}, \qquad b = \sqrt{z+x}, \qquad c = \sqrt{x+y}.$$

Assume that (a, b, c) satisfies the triangle inequalities (61) and the acute conditions (62). Set

$$S = \frac{a+b+c}{2} = \frac{\sqrt{y+z} + \sqrt{z+x} + \sqrt{x+y}}{2}.$$

Then for $\alpha > 0$ and $\beta > 0$ with $\alpha > \beta/2$, we have the symmetric inequality

$$\left(4\sqrt{S\big(S - \sqrt{y+z}\big)\big(S - \sqrt{z+x}\big)\big(S - \sqrt{x+y}\big)}\right)^{2\alpha+\beta} \ \geq\ C_{\alpha,\beta}\, 2^{2\alpha+\beta}\, x^{\alpha} y^{\alpha} z^{\beta}, \tag{69}$$

where $C_{\alpha,\beta}$ is given by (65).

Moreover, equality in (69) holds if and only if, up to a permutation of x, y, z,

$$x = y \quad \text{and} \quad 2x = \frac{2(\alpha - \beta/2)}{\alpha}\,(x+z) \qquad \left(\text{equivalently } \frac{x+z}{2x} = \frac{\alpha - \beta/2}{\alpha}\right).$$

Proof. We verify the hypotheses of Theorem 4 for the triple

$$(a, b, c) = \big(\sqrt{y+z},\ \sqrt{z+x},\ \sqrt{x+y}\big).$$

By assumption, (61) and (62) hold for (a, b, c), hence Theorem 4 applies.

Since $a^2 = y+z$, $b^2 = z+x$, $c^2 = x+y$, we obtain

$$b^2 + c^2 - a^2 = (z+x) + (x+y) - (y+z) = 2x,$$
$$c^2 + a^2 - b^2 = (x+y) + (y+z) - (z+x) = 2y,$$
$$a^2 + b^2 - c^2 = (y+z) + (z+x) - (x+y) = 2z.$$

Therefore the right-hand side of (64) becomes

$$C_{\alpha,\beta}\,(2x)^{\alpha}(2y)^{\alpha}(2z)^{\beta} = C_{\alpha,\beta}\, 2^{2\alpha+\beta}\, x^{\alpha} y^{\alpha} z^{\beta}.$$

By (63), the area of the triangle with side lengths a, b, c satisfies

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}.$$

Substituting $a = \sqrt{y+z}$, $b = \sqrt{z+x}$, $c = \sqrt{x+y}$ gives the explicit expression

$$\Delta = \sqrt{S\big(S - \sqrt{y+z}\big)\big(S - \sqrt{z+x}\big)\big(S - \sqrt{x+y}\big)}.$$

Hence the left-hand side of (64) is exactly the left-hand side of (69).

Substituting the expressions from Steps 1–2 into (64) yields (69).

Equality in Theorem 4 holds (up to permutation of a, b, c) when $a = b$ and $c^2 = \frac{2(\alpha-\beta/2)}{\alpha}\, a^2$. Here $a = b$ means $y + z = z + x$, hence $x = y$. Also $a^2 = y + z = x + z$ and $c^2 = x + y = 2x$, so the ratio condition becomes

$$2x = \frac{2(\alpha - \beta/2)}{\alpha}\,(x+z),$$

which is equivalent to $\frac{x+z}{2x} = \frac{\alpha-\beta/2}{\alpha}$.

3. Conclusion

In this paper, we provided a new analytic proof of Ono's classical inequality for acute triangles using a Cartesian coordinate reduction and a two-variable optimization argument. Unlike the traditional trigonometric approach, our method relies on elementary tools from multivariable calculus and shows directly that the equilateral triangle is the unique extremal configuration.

We also established a sharp two-parameter generalization and computed the optimal constant explicitly. The classical Ono inequality appears as a special case of this broader result. We have also provided application of generalization to the algebraic inequalities.

The approach developed here suggests that other geometric inequalities may be studied effectively through similar analytic methods.

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