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On the Structure of T-Closed Dual Rickart Modules

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Abstract. We introduce and study the notion of T-closed dual Rickart modules, which generalizes that of closed dual Rickart modules. A right R-module M is called T-closed dual Rickart if for every endomorphism f of M, the image $\operatorname{im}(f)$ is a T-closed submodule of M, where T is a fixed submodule of M. We establish fundamental properties of this class, including its behavior under direct summands, direct sums, and homomorphic images. Characterizations in terms of the endomorphism ring and several equivalent conditions are provided. We also explore connections with well-known classes of modules such as coquasi-Dedekind, continuous, and quasi-continuous modules. Examples are provided to illustrate and delimit the theory.

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Key Words and Phrases: Rickart module, dual Rickart module, T-closed submodule, endomorphism ring, module theory

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1. Introduction

The class of dual Rickart modules, introduced and systematically developed by Lee, Rizvi, and Roman in [1], consists of modules M for which the image $\operatorname{im}(f)$ is a direct summand of M for every endomorphism $f \in \operatorname{End}_R(M)$. This notion has proven to be a fruitful generalization of dual Baer modules, inspiring numerous extensions. Among these, Ghawi [2] replaced the direct summand condition by the weaker requirement that $\operatorname{im}(f)$ be a closed submodule, leading to the study of *closed dual Rickart modules*. More recently, Al-Bahrani and Rahman [3] introduced y-closed dual Rickart modules, while other variations such as C-co-epi-retractable modules [4] and strong dual Rickart modules [5] have also been investigated.

In this paper, we propose a unifying framework that encompasses and extends these existing notions: the T*-closed dual Rickart modules*. For a fixed submodule T of a right

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R-module M, we call M *T-closed dual Rickart* if, for every $f \in \mathrm{End}_R(M)$, the image $\mathrm{im}(f)$ is a T-closed submodule of M. The choice $T = 0$ recovers precisely Ghawi's closed dual Rickart modules [2], while other choices of T reveal new structural phenomena not captured by the classical theory.

Our objective is to develop a systematic study of this new class. We investigate their fundamental properties, provide characterizations, and analyze their behavior under standard module-theoretic constructions such as direct summands, finite direct sums, and homomorphic images. This approach not only unifies known results but also highlights the role of the distinguished submodule T in shaping the module structure.

The paper is organized as follows. Section 2 recalls the necessary background on dual Rickart modules and the notion of relative closedness (including T-essentiality and T-closed submodules). In Section 3, we establish the basic properties of T-closed dual Rickart modules and examine their stability under various operations. Section 4 explores connections with other classes of modules, such as T-extending modules and T-nonsingular modules. Finally, Section 5 investigates the behavior of this property under polynomial ring extensions.

Throughout, R denotes an associative ring with identity, and all modules are unitary right R-modules. For a module M, we write $N \leq M$, $N \leq_e M$, $N \leq_{T-e} M$, $N \leq_c M$, $N \leq_{T-c} M$, and $N \leq_{\oplus} M$ to denote that N is a submodule, an essential submodule, a T-essential submodule, a closed submodule, a T-closed submodule, and a direct summand of M, respectively. Here T always denotes a fixed submodule of M. For standard terminology and background, we refer to [6, 7].

2. Preliminaries

In this section, we recall the basic definitions and results that will be used throughout the paper. We begin by reviewing dual Rickart modules and the notions of essentiality and relative closedness.

Definition 1 ([1], [2]). *A right R-module M is called:*

(i) dual Rickart *(briefly, d-Rickart) if for every $f \in \mathrm{End}_R(M)$, the image* $\mathrm{im}(f)$ *is a direct summand of M;*

(ii) closed dual Rickart *(briefly, c-d-Rickart) if for every $f \in \mathrm{End}_R(M)$, the image* $\mathrm{im}(f)$ *is a closed submodule of M.*

Definition 2. *Let M be an R-module.*

(i) A submodule $K \leq M$ is said to be essential *in M, denoted $K \leq_e M$, if $K \cap L \neq 0$ for every nonzero submodule $L \leq M$ [6, 8].*

*(ii) Let $T \leq M$. A submodule $K \leq M$ is called T-*essential *in M, denoted $K \leq_{T-e} M$, if $K \nleq T$ and for every submodule $L \leq M$, the condition $K \cap L \leq T$ implies $L \leq T$ [9, 10].*

(iii) A submodule $K \leq M$ is called closed *in M, denoted $K \leq_c M$, if it has no proper essential extension in M; that is, whenever $K \leq_e L \leq M$, we have $L = K$ [7, 11].*

*(iv) A submodule $K \leq M$ is called T-*closed *in M, denoted $K \leq_{T-c} M$, if for every submodule $L \leq M$, the condition $K \leq_{T-e} L$ implies $L = K + T$ [10].*

Example 1 (Illustration with a lattice diagram)**.** *Consider the module $M = \mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}$ and let $T = \langle 3 \rangle = \{0, 3\}$. The lattice of submodules of M is shown in Figure 1. This diagram helps to visualize the different notions introduced in Definition 2.*

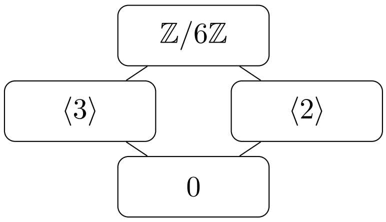

Figure 1: Lattice of submodules of $M = \mathbb{Z}/6\mathbb{Z}$ with $T = \langle 3 \rangle$. Essential submodules: $\langle 2 \rangle$ and M; closed submodules (0-closed): 0, $\langle 3 \rangle$, M; T-closed submodules: $\langle 3 \rangle$ and M.

For a submodule $K \leq M$, we denote by $\mathrm{cl}_T(K)$ the smallest T-closed submodule containing K, called the *T-closure* of K. This exists because the intersection of any family of T-closed submodules is T-closed (see Proposition 4).
Dual Rickart modules generalize dual Baer modules and have been extensively studied due to their strong structural and homological properties. The following characterization will be used frequently.

Proposition 1 ([1])**.** *For a right R-module M, the following statements are equivalent:*

(i) M is a dual Rickart module;

(ii) for every $f \in \mathrm{End}_R(M)$, there exists an idempotent $e \in \mathrm{End}_R(M)$ such that $\mathrm{im}(f) = eM$;

(iii) the kernel of every epimorphism from M is a direct summand of M.

Definition 3 (Coquasi-Dedekind module)**.** *A right R-module M is called* coquasi-Dedekind *if every nonzero endomorphism of M is an epimorphism [12].*

3. Some Properties of T-Closed Dual Rickart Modules

Definition 4 (T-Closed Dual Rickart Module)**.** *Let M be a right R-module and $T \leq M$ a fixed submodule. We say that M is a T-*closed dual Rickart module *(briefly, T-c-d-Rickart module) if for every endomorphism $f \in \mathrm{End}_R(M)$, the image $\mathrm{im}(f)$ is a T-closed submodule of M (see Definition 2 for the notion of T-closed submodules).*

Remark 1. *The case $T = 0$ coincides with Ghawi's notion of closed dual Rickart modules (c-d-Rickart). Hence, the class of T-closed dual Rickart modules properly generalizes the class of closed dual Rickart modules.*

Moreover, we have the following chain of implications:

$$\text{d-Rickart} \Rightarrow T\text{-closed d-Rickart} \Rightarrow \text{c-d-Rickart (for } T = 0).$$

Indeed, if M is dual Rickart, then every endomorphic image is a direct summand, and every direct summand is T-closed in M. The second implication follows immediately from the definitions. In general, none of these implications is reversible.

3.1. Examples and Non-trivial Cases

Example 2. *Let $M = \mathbb{Z}_4$ and $T = 2\mathbb{Z}_4 = \{0, 2\}$. The endomorphism ring of M consists of multiplications by $0, 1, 2,$ and 3.*

For each endomorphism $f \in \mathrm{End}_{\mathbb{Z}}(M)$, the image $\mathrm{im}(f)$ is one of the submodules $\{0\}$, T, or M. If $\mathrm{im}(f) \subseteq T$, then $\mathrm{im}(f)$ is vacuously T-closed. If $\mathrm{im}(f) = M$, then it is trivially T-closed.

Hence, every endomorphic image of M is a T-closed submodule, and therefore M is a T-closed dual Rickart module.

Example 3. *To illustrate that the implications in Remark 1 are not reversible in general:*

- *(i)* ***T-closed d-Rickart $\not\Rightarrow$ d-Rickart**: Take $M = \mathbb{Z}_4$ with $T = 2\mathbb{Z}_4$ as in Example 2. Then M is T-closed dual Rickart, but not dual Rickart, because $\mathrm{im}(f_2) = \{0, 2\}$ is not a direct summand of M.*
- *(ii)* ***c-d-Rickart $\not\Rightarrow$ T-closed d-Rickart for $T \neq 0$**: Consider $M = \mathbb{Z}_6$ with $T = 3\mathbb{Z}_6 = \{0, 3\}$. For the endomorphism $f(x) = 2x$, we have $\mathrm{im}(f) = \{0, 2, 4\}$. One can verify that this submodule is closed (0-closed) but not T-closed. Hence, M is c-d-Rickart but not T-closed dual Rickart.*

Example 4. *Let R be a discrete valuation ring with uniformizer π, and let $M = R \oplus R/(\pi^2)$. Take $T = R \oplus 0$. Then M is T-closed dual Rickart but not dual Rickart. Indeed, any endomorphism f of M either has image contained in T (hence T-closed) or has image containing a nonzero element of $R/(\pi^2)$. In the latter case, one checks that $\mathrm{im}(f)$ is T-essential in M, so its T-closure is M, hence T-closed. However, M is not dual Rickart because $R/(\pi^2)$ is not projective.*

Example 5. Composition may not preserve T-closedness. *Let $M = \mathbb{Z}_{12}$ and $T = 6\mathbb{Z}_{12} = \{0, 6\}$. Consider the endomorphisms f and g defined by multiplication by 2, i.e., $f(x) = 2x$ and $g(x) = 2x$ for all $x \in M$. Then $\mathrm{im}(f) = \mathrm{im}(g) = 2\mathbb{Z}_{12} = \{0, 2, 4, 6, 8, 10\}$. One checks that this submodule is T-closed because $2\mathbb{Z}_{12} + T = 2\mathbb{Z}_{12}$ and $2\mathbb{Z}_{12} \leq_{T-e} 2\mathbb{Z}_{12}$ (since $2\mathbb{Z}_{12} \nsubseteq T$ and for every $X \leq 2\mathbb{Z}_{12}$, $2\mathbb{Z}_{12} \cap X \subseteq T$ implies $X \subseteq T$). However, $(f \circ g)(x) = 4x$, so $\mathrm{im}(f \circ g) = 4\mathbb{Z}_{12} = \{0, 4, 8\}$. This submodule is not T-closed because $4\mathbb{Z}_{12} \leq_{T-e} 4\mathbb{Z}_{12}$ (as $4 \notin T$) but $4\mathbb{Z}_{12} + T = 2\mathbb{Z}_{12} \neq 4\mathbb{Z}_{12}$. Thus, the composition of two endomorphisms with T-closed images may have an image that is not T-closed.*

Remark 2. *Semisimple modules trivially satisfy the T-closed dual Rickart condition for any T, since all submodules are direct summands, hence T-closed. The interesting cases arise when the module is not semisimple. Example 2 demonstrates such a non-semisimple case: $\mathbb{Z}_4$ is not semisimple over $\mathbb{Z}$, yet it is T-closed dual Rickart for $T = 2\mathbb{Z}_4$.*

Remark 3. *In the proof of Proposition 2, the map $\tilde{f} : M \to M$ defined by $\tilde{f}(n+n') = f(n)$ for $n \in N$, $n' \in N'$ is indeed an R-endomorphism. It is the composition $f \circ \pi_N$, where $\pi_N : M \to N$ is the projection onto N (an R-homomorphism since N is a direct summand) and $f : N \to N$ is an R-endomorphism by hypothesis.*

Proposition 2. *Let M be a T-closed dual Rickart module and N a direct summand of M, so that $M = N \oplus N'$ for some submodule $N' \leq M$. If $T \subseteq N$, then N is also T-closed dual Rickart.*

Proof. Let $f \in \mathrm{End}_R(N)$. Define an extension $\tilde{f} \in \mathrm{End}_R(M)$ by

$$\tilde{f}(n + n') = f(n), \quad \text{for all } n \in N,\ n' \in N'.$$

Since f is an R-endomorphism of N and the projection $\pi_N : M \to N$ is an R-homomorphism, the composition $\pi_N \circ \tilde{f}$ is an R-homomorphism. In fact, $\tilde{f}$ itself is an R-homomorphism because it is defined componentwise. Hence $\tilde{f} \in \mathrm{End}_R(M)$.

Since M is T-closed dual Rickart, $\mathrm{im}(\tilde{f})$ is T-closed in M.

We claim that $\mathrm{im}(f)$ is T-closed in N. Let $L \leq N$ such that $\mathrm{im}(f) \leq_{T-e} L$ in N, i.e., $\mathrm{im}(f) \nsubseteq T$ and for all $X \leq L$, $\mathrm{im}(f) \cap X \subseteq T \implies X \subseteq T$.

Consider $L \oplus N' \leq M$. Observe that

$$\mathrm{im}(\tilde{f}) \cap (X \oplus X') = (\mathrm{im}(f) \cap X) \oplus 0$$

for any $X \leq L, X' \leq N'$. Hence, if $\mathrm{im}(\tilde{f}) \cap (X \oplus X') \subseteq T$, then $\mathrm{im}(f) \cap X \subseteq T$, which implies $X \subseteq T$. Also, since $T \subseteq N$ and $N \cap N' = 0$, we must have $X' = 0$. Therefore, any submodule $X \oplus X' \leq L \oplus N'$ satisfying $\mathrm{im}(\tilde{f}) \cap (X \oplus X') \subseteq T$ is contained in T. This verifies that $\mathrm{im}(\tilde{f}) \leq_{T-e} (L \oplus N')$.

By T-closedness of $\mathrm{im}(\tilde{f})$ in M, we have $L \oplus N' = \mathrm{im}(\tilde{f}) + T$. Projecting onto N gives $L = \mathrm{im}(f) + T$. Hence, $\mathrm{im}(f)$ is T-closed in N. Thus, N is T-closed dual Rickart, completing the proof.

Remark 4. *Proposition 2 ensures that the property of being T-closed dual Rickart is inherited by direct summands containing T. This result will be particularly useful when studying decompositions of non-semisimple modules and their T-closed endomorphic images.*

Theorem 1. *Let M be a T-closed dual Rickart module. Then:*

(i) The class of T-closed dual Rickart modules is closed under isomorphisms.

(ii) If $T = 0$ and M is indecomposable, then every nonzero endomorphism of M has essential image.

(iii) If N is a direct summand of M with $T \subseteq N$, then N is T-closed dual Rickart.

Proof. **(i) Closure under isomorphisms.** Let $\phi : M \to M'$ be an isomorphism. For any $f' \in \mathrm{End}_R(M')$, define $f = \phi^{-1} \circ f' \circ \phi \in \mathrm{End}_R(M)$. Since M is T-closed dual Rickart, $\mathrm{im}(f)$ is T-closed in M, i.e., for any $X \leq M$ with $\mathrm{im}(f) \cap X \subseteq T$, we have $X \subseteq \mathrm{im}(f) + T$. Applying ϕ, we get $\mathrm{im}(f') = \phi(\mathrm{im}(f))$ is $\phi(T)$-closed in M'.

(ii) Case $T = 0$ and M indecomposable. Here, M is closed dual Rickart (c-d-Rickart). Let $f \in \mathrm{End}_R(M)$ with $\mathrm{im}(f) \neq 0$. If $\mathrm{im}(f)$ were not essential, there exists $L \leq M$ with $\mathrm{im}(f) \cap L = 0$. Since $\mathrm{im}(f)$ is closed, it is a direct summand: $M = \mathrm{im}(f) \oplus K$ for some $K \neq 0$, contradicting the indecomposability of M. Thus, every nonzero endomorphism has essential image.

(iii) Direct summand containing T. Suppose $M = N \oplus N'$ with $T \subseteq N$, and let $f \in \mathrm{End}_R(N)$. Define $\tilde{f} \in \mathrm{End}_R(M)$ by $\tilde{f}(n + n') = f(n)$ for all $n \in N$, $n' \in N'$. Since M is T-closed dual Rickart, $\mathrm{im}(\tilde{f})$ is T-closed in M. Note that $\mathrm{im}(\tilde{f}) = \mathrm{im}(f) \subseteq N$.

Now, let $L \leq N$ satisfy $\mathrm{im}(f) \leq_{T-e} L$ in N. Consider $L \oplus N' \leq M$. Take $X \leq L \oplus N'$ with $\mathrm{im}(\tilde{f}) \cap X \subseteq T$. Write $X = X_N \oplus X_{N'}$ with $X_N \leq L$, $X_{N'} \leq N'$. Then $\mathrm{im}(f) \cap X_N \subseteq T$, so $X_N \subseteq T$ since $\mathrm{im}(f) \leq_{T-e} L$. Also, $\mathrm{im}(\tilde{f}) \cap X_{N'} = 0 \subseteq T$, which implies $X_{N'} \subseteq T \cap N' = 0$ (because $T \subseteq N$ and $N \cap N' = 0$). Thus $X = X_N \oplus X_{N'} \subseteq T$, proving $\mathrm{im}(\tilde{f}) \leq_{T-e} (L \oplus N')$.

By T-closedness of $\mathrm{im}(\tilde{f})$ in M, we have $L \oplus N' = \mathrm{im}(\tilde{f}) + T = \mathrm{im}(f) + T$. Projecting onto N gives $L = \mathrm{im}(f) + T$, showing that $\mathrm{im}(f)$ is T-closed in N.

Proposition 3. *Let $M_1, M_2, \ldots, M_n$ be right R-modules and $T_i \leq M_i$ for each i. Then:*

(i) If each M_i is T_i-closed dual Rickart and $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$, then

$$\bigoplus_{i=1}^{n} M_i$$

is $\bigoplus_{i=1}^{n} T_i$-closed dual Rickart.

(ii) If $\bigoplus_{i=1}^{n} M_i$ is $\bigoplus_{i=1}^{n} T_i$-closed dual Rickart and the decomposition is fully orthogonal (i.e., $\mathrm{End}_R(M) \cong \prod_{i=1}^{n} \mathrm{End}_R(M_i)$), then each M_i is T_i-closed dual Rickart.

Proof. Let $M = \bigoplus_{i=1}^{n} M_i$ and $T = \bigoplus_{i=1}^{n} T_i$.

(i) Finite direct sum preserves T-closed dual Rickart. Assume each M_i is T_i-closed dual Rickart and $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$. Let $f \in \mathrm{End}_R(M)$. By orthogonality, $f = \bigoplus_{i=1}^{n} f_i$ with $f_i \in \mathrm{End}_R(M_i)$, so

$$\mathrm{im}(f) = \bigoplus_{i=1}^{n} \mathrm{im}(f_i).$$

Suppose $\mathrm{im}(f) \leq_{T-e} L \leq M$, with $L = \bigoplus_{i=1}^{n} L_i$, $L_i \leq M_i$. For each i, let $X_i \leq L_i$ with $\mathrm{im}(f_i) \cap X_i \subseteq T_i$ and set $X = X_i \oplus \left(\bigoplus_{j \neq i} 0\right) \leq L$. Then

$$\mathrm{im}(f) \cap X = (\mathrm{im}(f_i) \cap X_i) \oplus 0 \subseteq T_i \oplus 0 \subseteq T.$$

Since $\operatorname{im}(f) \leq_{T-e} L$, we get $X \subseteq T$, hence $X_i \subseteq T_i$. By T_i-closedness of $\operatorname{im}(f_i)$, we have $L_i = \operatorname{im}(f_i) + T_i$. Therefore,

$$L = \bigoplus_{i=1}^{n} L_i = \bigoplus_{i=1}^{n} (\operatorname{im}(f_i) + T_i) = \operatorname{im}(f) + T,$$

so $\operatorname{im}(f)$ is T-closed.

(ii) Direct summands of a fully orthogonal T-closed sum. Assume M is T-closed dual Rickart with $\operatorname{End}_R(M) \cong \prod_{i=1}^{n} \operatorname{End}_R(M_i)$. Fix j and let $f_j \in \operatorname{End}_R(M_j)$. Define $f \in \operatorname{End}_R(M)$ by $f = (0, \ldots, f_j, \ldots, 0)$ (only j-th component nonzero). Then $\operatorname{im}(f) \cong \operatorname{im}(f_j)$ is T-closed in M.

Suppose $\operatorname{im}(f_j) \leq_{T_j-e} L_j \leq M_j$ and set $L = L_j \oplus \left(\bigoplus_{i\neq j} M_i\right)$. Let $X \leq L$ with $\operatorname{im}(f) \cap X \subseteq T$. Writing $X = X_j \oplus \left(\bigoplus_{i\neq j} X_i\right)$ with $X_j \leq L_j$, $X_i \leq M_i$, we have

$$\operatorname{im}(f) \cap X = (\operatorname{im}(f_j) \cap X_j) \oplus 0 \subseteq T.$$

Thus $\operatorname{im}(f_j) \cap X_j \subseteq T_j$, and $X_j \subseteq T_j$ by T_j-essentiality. For $i \neq j$, $X_i \subseteq T_i$ because $\operatorname{im}(f) \cap X_i = 0 \subseteq T$ and $T = \bigoplus_i T_i$. Hence $X \subseteq T$, proving $\operatorname{im}(f) \leq_{T-e} L$.

By T-closedness of $\operatorname{im}(f)$ in M, we have $L = \operatorname{im}(f) + T$, so projecting onto M_j gives $L_j = \operatorname{im}(f_j) + T_j$. Therefore, $\operatorname{im}(f_j)$ is T_j-closed in M_j.

Theorem 2 (Stability of T-Closed Dual Rickart Modules under Quotients)**.** *Let M be a T-closed dual Rickart module and $N \leq M$ a fully invariant submodule that is T-closed in M. Assume that M has the T-**Closed Sum Property (CSP)**, i.e., the sum of any two T-closed submodules of M is again T-closed.*

Then the quotient module M/N is $\overline{T}$-closed dual Rickart, where $\overline{T} = (T + N)/N$.

Proof. Let $\pi : M \to M/N$ denote the canonical projection. Since N is fully invariant, every $\bar{f} \in \operatorname{End}_R(M/N)$ lifts to an endomorphism $f \in \operatorname{End}_R(M)$ satisfying $f(N) \subseteq N$ and $\bar{f} \circ \pi = \pi \circ f$. Hence,

$$\operatorname{im}(\bar{f}) = \pi(\operatorname{im}(f)) = (\operatorname{im}(f) + N)/N.$$

Suppose that $\operatorname{im}(\bar{f}) \leq_{\overline{T}-e} L \leq M/N$, and set $K = \pi^{-1}(L) \leq M$. Then

$$\operatorname{im}(f) + N \leq_{T-e} K \text{ in } M.$$

Since M is T-closed dual Rickart, $\operatorname{im}(f)$ is T-closed in M. By hypothesis, N is also T-closed. Using the T-CSP, the sum $\operatorname{im}(f) + N$ is therefore T-closed.

From $\operatorname{im}(f) + N \leq_{T-e} K$, we deduce

$$K \subseteq \operatorname{im}(f) + N + T.$$

Applying the projection π, we obtain

$$L = \pi(K) \subseteq \pi(\operatorname{im}(f) + N + T) = \operatorname{im}(\bar{f}) + \overline{T}.$$

On the other hand, since $\operatorname{im}(\bar{f}) \leq_{\overline{T}-e} L$, we also have

$$\operatorname{im}(\bar{f}) + \overline{T} \subseteq L.$$

Combining these inclusions, we conclude that

$$L = \operatorname{im}(\bar{f}) + \overline{T},$$

showing that $\operatorname{im}(\bar{f})$ is $\overline{T}$-closed in M/N.

Remark 5. *The T-CSP hypothesis is essential. If $N \subseteq T$, the sum $\operatorname{im}(f) + N$ is automatically T-closed, so the CSP condition is trivially satisfied in that case.*

Proposition 4. *Let M be a right R-module and $T \leq M$. The intersection of any finite family of T-closed submodules of M is T-closed.*

Proof. It suffices to prove the case of two submodules; the general finite case follows by induction.

Let $N_1, N_2 \leq M$ be T-closed submodules, and set $N = N_1 \cap N_2$. Suppose $N \leq_{T-e} L$ for some $L \leq M$. We need to show $L = N + T$.

Since $N \leq_{T-e} L$, for any $X \leq L$ with $N \cap X \subseteq T$, we have $X \subseteq T$. In particular, $N_i \cap X \subseteq T$ for $i = 1, 2$. But N_i is T-closed, so $L \subseteq N_i + T$ for each i. Therefore,

$$L \subseteq (N_1 + T) \cap (N_2 + T).$$

By the modular law (or standard lattice identity for submodules),

$$(N_1 + T) \cap (N_2 + T) = (N_1 \cap N_2) + T = N + T.$$

Thus we obtain $L \subseteq N + T$. Since $N \leq_{T-e} L$, it follows that $L = N + T$, proving that N is T-closed.

Example 6. *Let $M = \mathbb{Z}_4$ as a $\mathbb{Z}$-module, $T = 2\mathbb{Z}_4 = \{0, 2\}$, and $N = \{0\}$. Then:*

- *M is T-closed dual Rickart, since all endomorphic images of M are $\{0\}$, T, or M, all of which are T-closed.*
- *N is fully invariant in M (trivially, as $N = \{0\}$).*
- *N is T-closed in M.*

Moreover, M has the T-CSP because the only T-closed submodules are $\{0\}, T, M$, and any sum of these submodules is again T-closed.

By Theorem 2, the quotient $M/N \cong \mathbb{Z}_4$ is $\overline{T}$-closed dual Rickart, with $\overline{T} = T$.

Corollary 1. *Let M be a T-closed dual Rickart module. Then:*

(i) The set of T-closed submodules of M forms an inf-semilattice; that is, finite intersections of T-closed submodules are T-closed.

*(ii) For any submodule $K \leq M$, there exists a unique smallest T-closed submodule containing K, called the T-*closure *of K and denoted by $\mathrm{cl}_T(K)$, which is given by the intersection of all T-closed submodules containing K.*

(iii) The T-closure operation is idempotent and monotonic:

- $\mathrm{cl}_T(\mathrm{cl}_T(K)) = \mathrm{cl}_T(K)$,
- *If $K \leq L$, then $\mathrm{cl}_T(K) \leq \mathrm{cl}_T(L)$.*

Proof. (i) This is immediate from Proposition 4, which states that the intersection of any finite family of T-closed submodules is T-closed.

(ii) Consider the collection of all T-closed submodules of M that contain K. This collection is non-empty since M itself is T-closed. Define

$$\mathrm{cl}_T(K) = \bigcap \{N \leq M \mid N \text{ is } T\text{-closed and } K \leq N\}.$$

We claim that $\mathrm{cl}_T(K)$ is T-closed. Let $L \leq M$ with $\mathrm{cl}_T(K) \leq_{T-e} L$. Then for each T-closed submodule N containing K, we have $\mathrm{cl}_T(K) \subseteq N$, and the T-closedness of N gives $L \subseteq N + T$. Since this holds for all such N, we conclude

$$L \subseteq \bigcap (N + T) = \mathrm{cl}_T(K) + T.$$

Thus, $\mathrm{cl}_T(K)$ is T-closed.

(iii) Idempotence follows immediately from the fact that $\mathrm{cl}_T(K)$ is T-closed. Monotonicity holds because if $K \leq L$, then every T-closed submodule containing L also contains K, so the intersection defining $\mathrm{cl}_T(K)$ is contained in that defining $\mathrm{cl}_T(L)$.

Proposition 5. *Let M be a T-closed dual Rickart module and $f, g \in \mathrm{End}_R(M)$. Then:*

(i) $\mathrm{im}(f) \cap \mathrm{im}(g)$ is T-closed.

(ii) More generally, for any finite family $f_1, \ldots, f_n \in \mathrm{End}_R(M)$, the intersection

$$\bigcap_{i=1}^{n} \mathrm{im}(f_i)$$

is T-closed.

(iii) The set

$$\mathcal{I} = \{\mathrm{im}(f) \mid f \in \mathrm{End}_R(M)\}$$

is a subset of the lattice of T-closed submodules of M, closed under finite intersections.

Proof. (i) Since M is T-closed dual Rickart, both $\operatorname{im}(f)$ and $\operatorname{im}(g)$ are T-closed submodules. By Proposition 4, the intersection $\operatorname{im}(f) \cap \operatorname{im}(g)$ is T-closed.

(ii) The result follows by induction on n. The base case $n = 2$ is exactly (i). Suppose the intersection of any $n-1$ images is T-closed. Then for $f_1, \dots, f_n$:

$$\bigcap_{i=1}^{n} \operatorname{im}(f_i) = \left(\bigcap_{i=1}^{n-1} \operatorname{im}(f_i) \right) \cap \operatorname{im}(f_n),$$

which is an intersection of two T-closed submodules, hence T-closed.

(iii) Immediate from (ii). Indeed, the intersection of finitely many elements of $\mathcal{I}$ is T-closed, so $\mathcal{I}$ is closed under finite intersections when viewed as a subset of the lattice of T-closed submodules. Note that the intersection may not itself be the image of a single endomorphism, but it still belongs to the lattice of T-closed submodules.

Theorem 3. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of right R-modules, and let $T = \bigoplus_{i \in I} T_i$ with $T_i \leq M_i$. Assume that*

$$\operatorname{Hom}_R(M_i, M_j) = 0 \quad \text{for all } i \neq j.$$

Let $\{N_j\}_{j \in J}$ be a family of submodules of M such that each N_j decomposes as

$$N_j = \bigoplus_{i \in I} (N_j \cap M_i),$$

i.e., each N_j is a direct sum of its intersections with the M_i. Then

$$\bigcap_{j \in J} N_j = \bigoplus_{i \in I} \left(\bigcap_{j \in J} (N_j \cap M_i) \right).$$

If, in addition, each M_i is T_i-closed dual Rickart and each N_j is T-closed, then for each $i \in I$, the intersection

$$\bigcap_{j \in J} (N_j \cap M_i)$$

is T_i-closed in M_i, provided either

- *J is finite, or*
- *M_i satisfies the descending chain condition on T_i-closed submodules.*

Proof. First, by the decomposition of each N_j:

$$\bigcap_{j \in J} N_j = \bigcap_{j \in J} \bigoplus_{i \in I} (N_j \cap M_i).$$

For each $i \in I$, the component of any element in $\bigcap_{j \in J} N_j$ in M_i must belong to $\bigcap_{j \in J} (N_j \cap M_i)$. Conversely, any element in $\bigoplus_i \left(\bigcap_{j \in J} (N_j \cap M_i) \right)$ clearly belongs to each N_j. The assumption $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$ ensures that there is no "mixing" between components, hence the equality holds.

Next, consider the T_i-closedness. Since each N_j is T-closed and $T = \bigoplus_i T_i$, it follows that

$$N_j \cap M_i$$

is T_i-closed in M_i for every $i \in I$ and $j \in J$. Indeed, if $N_j \cap M_i \leq_{T_i - e} L \leq M_i$, then extending L trivially to the other components yields a T-essential extension of N_j in M, which forces $L \subseteq N_j \cap M_i + T_i$, by T-closedness of N_j.

Finally, the intersection $\bigcap_{j\in J}(N_j \cap M_i)$ is an intersection of T_i-closed submodules of M_i. If J is finite, Proposition 4 ensures the intersection is T_i-closed. If J is infinite but M_i satisfies the descending chain condition on T_i-closed submodules, the intersection equals a finite subintersection, which is again T_i-closed.

Remark 6. *In Theorem 3, we work with direct sums $\bigoplus$ rather than unions. This is natural in the context of direct sum decompositions. The condition $N_j = \bigoplus_{i\in I}(N_j \cap M_i)$ ensures each N_j is compatible with the decomposition $M = \bigoplus_{i\in I} M_i$, which is essential for the componentwise analysis.*

Remark 7. *One might ask whether a more general version of Theorem 3 holds using only the T-closedness in M rather than the componentwise T_i-closedness. In general, if M is T-closed dual Rickart with $T = \bigoplus_i T_i$, it does not necessarily imply that each M_i is T_i-closed dual Rickart unless additional conditions hold (such as the orthogonality condition $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$). The componentwise condition is essential to control the behavior of endomorphisms on each summand.*

Remark 8. *The condition $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$ is essential. It guarantees that the decomposition $M = \bigoplus_i M_i$ is **fully invariant**, i.e., every endomorphism of M preserves each summand. Consequently, any submodule that is a direct sum of its intersections with the M_i has its intersection with another such submodule computed componentwise. Without this condition, an element could have contributions from multiple components under some endomorphism, and intersections would not necessarily decompose componentwise.*

Theorem 4. *Let M be a T-closed dual Rickart module. Then the following are equivalent for a submodule $N \leq M$:*

(i) N is a direct summand of M.

(ii) N is T-closed and there exists an idempotent $e \in \mathrm{End}_R(M)$ with $\mathrm{im}(e) = N$.

Moreover, if $\mathrm{End}_R(M)$ is a commutative ring, then the set of direct summands of M forms a sublattice of the lattice of T-closed submodules.

Proof. (i) $\Rightarrow$ (ii): Assume N is a direct summand, so $M = N \oplus K$ for some submodule $K \leq M$. Let $e : M \to M$ be the canonical projection onto N. Then $e^2 = e$ and $\mathrm{im}(e) = N$. Since M is T-closed dual Rickart, the image of any endomorphism is T-closed, hence N is T-closed.

(ii) $\Rightarrow$ (i): If $e^2 = e$ and $\mathrm{im}(e) = N$, then

$$M = \mathrm{im}(e) \oplus \mathrm{im}(1-e) = N \oplus \mathrm{im}(1-e),$$

so N is a direct summand.

For the lattice property under commutativity: let $N_1, N_2 \leq M$ be direct summands with corresponding idempotents $e_1, e_2 \in \mathrm{End}_R(M)$. If $\mathrm{End}_R(M)$ is commutative, then $e_1e_2 = e_2e_1$, and both

$$e_1e_2 \quad \text{and} \quad e_1 + e_2 - e_1e_2$$

are idempotents. Their images satisfy

$$\mathrm{im}(e_1e_2) = N_1 \cap N_2, \quad \mathrm{im}(e_1 + e_2 - e_1e_2) = N_1 + N_2,$$

showing that $N_1 \cap N_2$ and $N_1 + N_2$ are again direct summands. Hence the set of direct summands forms a sublattice of the lattice of T-closed submodules.

Proposition 6. *Let M be a right module of finite composition length that is T-closed dual Rickart. Assume further that M is* extending *(i.e., every closed submodule is a direct summand). Then:*

(i) M decomposes into a finite direct sum of indecomposable T-closed dual Rickart modules.

(ii) For each indecomposable summand U, the endomorphism ring $\mathrm{End}_R(U)$ is local.

(iii) The Krull–Remak–Schmidt theorem holds for such decompositions: the indecomposable summands are uniquely determined up to isomorphism and permutation.

Proof. (i) Since M has finite length, it satisfies both the ascending and descending chain conditions on submodules. By the extending property, every T-closed submodule (in particular every image of an endomorphism) is a direct summand. Hence, M is a *dual Rickart module* in the usual sense. A standard argument (using the finiteness of length) then yields that M is a finite direct sum of indecomposable submodules, each of which inherits the T-closed dual Rickart property.

(ii) Let U be an indecomposable summand. Since U is T-closed dual Rickart, every endomorphism $f \in \mathrm{End}_R(U)$ has T-closed image. If f is not an isomorphism, then $\mathrm{im}(f)$ is a proper T-closed submodule of U; by the extending property, $\mathrm{im}(f)$ is a direct summand of U. Because U is indecomposable, $\mathrm{im}(f) = 0$ or $\mathrm{im}(f) = U$. Since f is not onto, $\mathrm{im}(f) = 0$, i.e., f is nilpotent (in fact zero, because the kernel of a non-zero map would give a non-trivial direct summand). Consequently, every non-invertible endomorphism of U is nilpotent, which implies that $\mathrm{End}_R(U)$ is a local ring (see, e.g., [13, Lemma 4.1]).

(iii) The existence of a decomposition into indecomposables with local endomorphism rings allows us to apply the Azumaya–Krull–Schmidt theorem [14, Theorem 2.12]. Thus the decomposition is unique up to isomorphism and permutation of the summands.

Proposition 7. *Let M be a T-closed dual Rickart module and $f, g \in \mathrm{End}_R(M)$. Then:*

(i) If g is an isomorphism, then $\mathrm{im}(f \circ g)$ is T-closed.

(ii) If f is an isomorphism, then $\mathrm{im}(f \circ g)$ is T-closed.

(iii) If f and g are idempotents, then $\mathrm{im}(f \circ g)$ is T-closed.

(iv) In general, $\mathrm{im}(f \circ g)$ may not be T-closed, even if M is T-closed dual Rickart.

Proof. (i) If g is an isomorphism, then $\mathrm{im}(f \circ g) = \mathrm{im}(f)$, which is T-closed by hypothesis.

(ii) If f is an isomorphism, then $\mathrm{im}(f \circ g) = f(\mathrm{im}(g))$. Since f is an isomorphism, it preserves T-closedness, so $\mathrm{im}(f \circ g)$ is T-closed.

(iii) If f and g are idempotents, denote $e = f$ and $h = g$. Then

$$\mathrm{im}(e \circ h) = e(h(M)) \subseteq e(M) = \mathrm{im}(e).$$

Since $\mathrm{im}(e)$ is a direct summand (hence T-closed), it suffices to show that $\mathrm{im}(e \circ h)$ is T-closed in $\mathrm{im}(e)$. Now, $\mathrm{im}(e)$ is itself a T-closed dual Rickart module (as a direct summand of M), and $e \circ h|_{\mathrm{im}(e)}$ is an endomorphism of $\mathrm{im}(e)$. Therefore, $\mathrm{im}(e \circ h)$ is T-closed in $\mathrm{im}(e)$, and hence also in M.

(iv) For a general counterexample, consider $M = \mathbb{Z}_4$ as a $\mathbb{Z}$-module and $T = 2\mathbb{Z}_4 = \{0, 2\}$. Define $f(x) = 2x$ and $g(x) = 2x$. Then $\mathrm{im}(f) = \mathrm{im}(g) = T$, which is T-closed. But

$$\mathrm{im}(f \circ g) = f(g(M)) = 4M = \{0\},$$

which is not T-closed. Indeed, $\{0\} \leq_{T-e} \mathbb{Z}_4$, but $\{0\} + T = T \neq \mathbb{Z}_4$, so $\mathrm{im}(f \circ g)$ is not T-closed.

Remark 9. *The examples in Proposition 7 show that the composition of two T-closed endomorphisms may fail to be T-closed. A sufficient condition for the composition $g \circ f$ to be T-closed is that $\mathrm{im}(f)$ be a direct summand (or more generally, a T-closed submodule that is also T-complemented) and that g be a monomorphism when restricted to $\mathrm{im}(f)$. In the counterexample, $\mathrm{im}(f) = T$ is T-closed but not a direct summand, and g is not injective on T.*

Proposition 8. *Let M be a T-closed dual Rickart module. The set*

$$\mathcal{I} = \{\mathrm{im}(f) \mid f \in \mathrm{End}_R(M)\}$$

of images of endomorphisms satisfies the following properties:

(i) $\mathcal{I}$ contains 0 and M.

(ii) If $N \in \mathcal{I}$, then N is T-closed.

(iii) Finite intersections of elements of $\mathcal{I}$ are T-closed (but not necessarily in $\mathcal{I}$).

(iv) If $N \in \mathcal{I}$ is a direct summand of M, then every direct summand of N belongs to $\mathcal{I}$.

Moreover, if M is retractable and $T = 0$, then $\mathcal{I}$ coincides with the set of direct summands of M.

Proof. (i) Clearly, $0 = \mathrm{im}(0)$ and $M = \mathrm{im}(\mathrm{id}_M)$.

(ii) Immediate from the definition of a T-closed dual Rickart module.

(iii) Let $N_1 = \mathrm{im}(f_1), \ldots, N_k = \mathrm{im}(f_k)$. By Proposition 4,

$$\bigcap_{i=1}^{k} N_i$$

is T-closed. However, this intersection is not necessarily the image of a single endomorphism of M.

(iv) Let $N = \mathrm{im}(f)$ be a direct summand, so $M = N \oplus K$. Let L be a direct summand of N, then $N = L \oplus L'$. The projection

$$\pi_L : M \to L$$

obtained by composing the projection onto N with the projection onto L, is an endomorphism of M whose image is L. Thus $L \in \mathcal{I}$.

For the last statement: if $T = 0$, a T-closed submodule is exactly a direct summand. If M is retractable, every direct summand is the image of an idempotent, hence belongs to $\mathcal{I}$. Conversely, every element of $\mathcal{I}$ is a direct summand by (ii). Thus $\mathcal{I}$ coincides with the set of direct summands.

Remark 10. *An attempt to define a semigroup structure on $\mathcal{I}$ by composition fails for two reasons:*

- *The operation $\mathrm{im}(f) \cdot \mathrm{im}(g) := \mathrm{im}(f \circ g)$ is not well-defined: different endomorphisms having the same image may yield different composed images.*
- *Even when choosing representatives, the set $\mathcal{I}$ is not closed under this operation in general. Proposition 7 shows that the composition of two endomorphisms may have an image not in $\mathcal{I}$, even if M is T-closed dual Rickart.*

Thus, $\mathcal{I}$ does not carry a natural semigroup structure induced by endomorphism composition.

Proposition 9. *Let M be a T-closed dual Rickart module, and let $f_1, f_2, \ldots, f_n \in \mathrm{End}_R(M)$. Then:*

(i) If each f_i is injective, then $\mathrm{im}(f_1 \circ \cdots \circ f_n)$ is T-closed.

(ii) If each f_i is surjective, then $\mathrm{im}(f_1 \circ \cdots \circ f_n) = M$, which is T-closed.

(iii) In general, $\mathrm{im}(f_1 \circ \cdots \circ f_n)$ need not be T-closed, even if M is T-closed dual Rickart.

(iv) The chain

$$\mathrm{im}(f_1) \supseteq \mathrm{im}(f_1 \circ f_2) \supseteq \cdots \supseteq \mathrm{im}(f_1 \circ \cdots \circ f_n)$$

always holds, but intermediate images may fail to be T-closed.

Proof. (i) If each f_i is injective, the composition $f = f_1 \circ \cdots \circ f_n$ is injective, so $\mathrm{im}(f) \cong M$. Since M is T-closed and isomorphisms preserve T-closedness, $\mathrm{im}(f)$ is T-closed.

(ii) If each f_i is surjective, then the composition is surjective and its image is M, which is T-closed.

(iii) Consider $M = \mathbb{Z}_4$ as a $\mathbb{Z}$-module with $T = 2\mathbb{Z}_4 = \{0, 2\}$. Then M is T-closed dual Rickart (the images of all endomorphisms are $\{0\}, T, M$, all T-closed). Define $f_1(x) = f_2(x) = 2x$. Then

$$(f_1 \circ f_2)(x) = 4x = 0, \quad \mathrm{im}(f_1 \circ f_2) = \{0\}.$$

The zero submodule $\{0\}$ is not T-closed: take $L = M$; we have $\{0\} \leq_{T-e} L$ but $\{0\} + T = T \neq L$. Thus $\mathrm{im}(f_1 \circ f_2)$ is not T-closed.

(iv) The inclusions $\mathrm{im}(f_1 \circ \cdots \circ f_{k+1}) \subseteq \mathrm{im}(f_1 \circ \cdots \circ f_k)$ are always true. Point (iii) shows that intermediate images may not be T-closed.

Proposition 10. *Let M be a T-closed dual Rickart module. For any $f, g \in \mathrm{End}_R(M)$, we have the isomorphism:*

$$\mathrm{im}(f \circ g) \cong \mathrm{im}(g)/(\ker(f) \cap \mathrm{im}(g)).$$

However, this quotient does not necessarily preserve the T-closed property, even when $\ker(f) \cap \mathrm{im}(g)$ *is T-closed in* $\mathrm{im}(g)$.

Proof. The isomorphism follows from the first isomorphism theorem applied to the restriction $f|_{\mathrm{im}(g)} : \mathrm{im}(g) \to M$.

For the failure of T-closed preservation: consider $M = \mathbb{Z}_4$ as a $\mathbb{Z}$-module, $T = 2\mathbb{Z}_4 = \{0, 2\}$. Then M is T-closed dual Rickart. Define $g(x) = 2x$, $f(x) = 2x$. Then: - $\mathrm{im}(g) = T$ is T-closed. - $\ker(f) \cap \mathrm{im}(g) = T \cap T = T$, which is T-closed in $\mathrm{im}(g)$. - The quotient $\mathrm{im}(g)/(\ker(f) \cap \mathrm{im}(g)) = T/T \cong 0$. - But $\mathrm{im}(f \circ g) = \{0\}$, which is not T-closed in M (as shown in previous examples).

Thus, even when both $\mathrm{im}(g)$ and $\ker(f) \cap \mathrm{im}(g)$ are T-closed, the quotient (hence $\mathrm{im}(f \circ g)$) may fail to be T-closed.

Lemma 1. *Let M be an indecomposable T-c-d-Rickart module. Then every nonzero T-closed submodule of M equals M.*

Proof. Let N be a nonzero T-closed submodule of M. Suppose $N \neq M$. Then N is a proper T-closed submodule. Since M is T-c-d-Rickart, there exists an endomorphism f of M such that $N = \mathrm{im}(f)$ (for example, the inclusion map if N is a direct summand, but we do not assume that). However, we can use the property that N is T-closed. Consider

$N \cap T$, which is T-closed by Proposition 4. Because M is indecomposable, the only T-closed submodules are 0 and M. Since N is proper, we must have $N \cap T = 0$ (otherwise, if $N \cap T = N$, then $N \subseteq T$, and by T-closedness of N, we would have $N = N + T = T$, and then T is a proper nonzero T-closed submodule, which would lead to a contradiction as in the theorem below).

Now we show that $N \oplus T = M$. Let $L = N + T$. We prove that $N \leq_{T-e} L$. Indeed, $N \not\subseteq T$ because $N \cap T = 0$ and $N \neq 0$. Let $X \leq L$ such that $N \cap X \subseteq T$. Then $N \cap X \subseteq N \cap T = 0$, so $N \cap X = 0$. Write any element of L as $a + b$ with $a \in N$, $b \in T$. If $x = a + b \in X$, then $a = x - b \in X + T \subseteq L$ and $a \in N \cap X = 0$, hence $x = b \in T$. Thus, $X \subseteq T$. Therefore, $N \leq_{T-e} L$. By T-closedness of N, we have $L = N + T$. Since $N \cap T = 0$, the sum is direct, so $M = N \oplus T$. This contradicts the indecomposability of M. Hence, $N = M$.

Theorem 5. *Let M be an indecomposable module and $T \leq M$. The following statements are equivalent:*

(i) M is T-closed dual Rickart.

(ii) Every nonzero endomorphism of M is an epimorphism (i.e., M is coquasi-Dedekind) and $T = 0$ or $T = M$.

(iii) $\mathrm{End}_R(M)$ is a domain and $T = 0$ or $T = M$.

In particular, if $T = 0$, then M is simple; if $T = M$, then M is coquasi-Dedekind.

Proof. We prove the implications step by step.
(1) ⇒ (2). Assume that M is indecomposable and T-c-d-Rickart. Let $0 \neq f \in \mathrm{End}_R(M)$. Then $\mathrm{im}(f)$ is a nonzero T-closed submodule. By Lemma 1, $\mathrm{im}(f) = M$. Hence, every nonzero endomorphism is an epimorphism, so M is coquasi-Dedekind.

Now we show that $T = 0$ or $T = M$. Suppose, for a contradiction, that $0 \neq T \neq M$. Then T is a proper nonzero submodule. Since M is coquasi-Dedekind, every nonzero endomorphism is surjective. Consider the identity map id_M. Its image M is T-closed. Now, T itself is T-closed (as it is a submodule, and one can check that it satisfies the definition). But then T is a proper nonzero T-closed submodule, which contradicts Lemma 1 because M is indecomposable. Therefore, T must be either 0 or M.
(2) ⇒ (3). Suppose that every nonzero endomorphism of M is an epimorphism and $T = 0$ or $T = M$. Let $f, g \in \mathrm{End}_R(M)$ with $f \circ g = 0$. If $g \neq 0$, then g is an epimorphism, so $f = 0$. Thus, $\mathrm{End}_R(M)$ has no zero divisors; it is a domain.
(3) ⇒ (1). Assume that $\mathrm{End}_R(M)$ is a domain and $T = 0$ or $T = M$. Let $f \in \mathrm{End}_R(M)$. If $f = 0$, then $\mathrm{im}(f) = 0$. We show that 0 is T-closed. If $0 \leq_{T-e} L$, then by definition $L \subseteq T$ (because for every $X \leq L$, $0 \cap X \subseteq T$ implies $X \subseteq T$), so $L = 0 + T = T$. Hence, 0 is T-closed.

If $f \neq 0$, then since $\mathrm{End}_R(M)$ is a domain, f is injective. Thus $\mathrm{im}(f) \cong M$. Because $T = 0$ or $T = M$, the only T-closed submodules are 0 and M (if $T = 0$, a submodule is T-closed if and only if it is closed in the usual sense; in an indecomposable module, the only closed submodules are 0 and M; if $T = M$, every submodule is T-closed). Therefore, $\mathrm{im}(f) = M$ is T-closed. Consequently, M is T-c-d-Rickart.

Last assertions. If $T = 0$, then every nonzero endomorphism is both an epimorphism and a monomorphism (since $\mathrm{End}_R(M)$ is a domain), hence an isomorphism. It follows that M is simple. If $T = M$, the T-c-d-Rickart condition reduces to requiring that every nonzero endomorphism be surjective, i.e., M is coquasi-Dedekind.

Corollary 2. *Let M be an indecomposable T-closed dual Rickart module. Then*

$$T = 0 \quad or \quad T = M.$$

In particular, there exists no indecomposable T-closed dual Rickart module with $0 \neq T \neq M$.

Proof. This follows immediately from Theorem 5. Indeed, for an indecomposable module M, the theorem shows that any nonzero endomorphism is surjective (so M is coquasi-Dedekind) and that T must be either 0 or M. Hence, no proper nonzero submodule T can occur.

Proposition 11. *Let M be a T-closed dual Rickart module, and let $f, g \in \mathrm{End}_R(M)$.*

(i) If f is monic, then $\mathrm{im}(f \circ g) \cong \mathrm{im}(g)$. In particular, $\mathrm{im}(f \circ g)$ is T-closed.

(ii) If g is epic, then $\mathrm{im}(f \circ g) = \mathrm{im}(f)$, which is T-closed.

(iii) If both f and g are isomorphisms, then $\mathrm{im}(f \circ g) = M$, which is T-closed.

Proof. (i) If f is monic, the restriction $f|_{\mathrm{im}(g)} : \mathrm{im}(g) \to \mathrm{im}(f \circ g)$ is an isomorphism. Since isomorphisms preserve T-closedness and M is T-closed dual Rickart, $\mathrm{im}(f \circ g)$ is T-closed.

(ii) If g is epic, then $g(M) = M$, hence $\mathrm{im}(f \circ g) = f(M) = \mathrm{im}(f)$, which is T-closed.

(iii) Immediate from (i) and (ii), since an isomorphism is both monic and epic, giving $\mathrm{im}(f \circ g) = M$.

Remark 11. *This proposition clarifies the mechanism behind Proposition 7: in a T-closed dual Rickart module, compositions preserve T-closedness whenever one factor is either monic or epic. The failure of T-closedness for general compositions arises only when neither factor has these properties.*

Theorem 6. *Let R be a ring. The following conditions are equivalent:*

(i) Every right R-module is T-closed dual Rickart for every *submodule $T \leq M$.*

(ii) R is a semisimple Artinian ring.

Proof. (2) $\Rightarrow$ (1): If R is semisimple Artinian, every right R-module is semisimple. In a semisimple module, every submodule is a direct summand, hence T-closed for any choice of $T \leq M$. Consequently, every module is T-closed dual Rickart.

(1) $\Rightarrow$ (2): Assume that every right R-module is T-closed dual Rickart for all T. In particular, taking $T = 0$, every module is dual Rickart in the usual sense (every endomorphic image is a direct summand). By [1, Theorem 4.13], this implies that R is a semisimple Artinian ring. (One may apply the cited theorem to the module R_R.)

Proposition 12. *Let R be a ring. Consider the following conditions:*

(i) Every right R-module is dual Rickart (i.e., T-closed dual Rickart for $T = 0$).

(ii) Every injective right R-module is dual Rickart.

(iii) R is a semisimple Artinian ring.

Then (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii), but (ii) $\Rightarrow$ (iii) does not hold in general.

Proof. (i) $\Leftrightarrow$ (iii) is classical: a ring is semisimple Artinian if and only if every right module is dual Rickart (see [1, Theorem 4.13]).

(iii) $\Rightarrow$ (ii) is immediate.

To see that (ii) $\Rightarrow$ (iii) fails, take $R = \mathbb{Z}$. Every injective $\mathbb{Z}$-module is a direct sum of copies of $\mathbb{Q}$ and Prüfer groups $\mathbb{Z}_{p^\infty}$, which are injective. For each such module, every endomorphic image is either 0 or the whole module, hence dual Rickart. Yet $\mathbb{Z}$ is not semisimple Artinian.

Proposition 13. *Let M be any right module and set $T = M$. Then M is T-closed dual Rickart if and only if every nonzero endomorphism of M is surjective. In particular, if M is coquasi-Dedekind, then M is M-closed dual Rickart.*

Proof. Let $T = M$. Recall that $N \leq M$ is M-closed if for every L with $N \subseteq L \subseteq M$, one has $L = N + M = M$. Hence the only M-closed submodule of M is M itself.

Therefore, M is M-closed dual Rickart if and only if $\mathrm{im}(f) = M$ for every nonzero $f \in \mathrm{End}_R(M)$. This is precisely the condition that every nonzero endomorphism of M is surjective, i.e., that M is coquasi-Dedekind.

Theorem 7. *Let M be a right module and $T \leq M$. The following implications hold:*

M is dual Rickart $\Longrightarrow$ M is T-closed dual Rickart $\Longrightarrow$ M is closed dual Rickart (i.e., $T = 0$).

In detail:

- *M is* dual Rickart *if for every $f \in \mathrm{End}_R(M)$, $\mathrm{im}(f)$ is a direct summand of M.*
- *M is* T-closed dual Rickart *if for every $f \in \mathrm{End}_R(M)$, $\mathrm{im}(f)$ is a T-closed submodule of M.*

- *M is* closed dual Rickart *(Ghawi, 2017) if it is T-closed dual Rickart with $T = 0$; equivalently, every endomorphic image is closed in the usual sense.*

Proof. First implication: assume M is dual Rickart. For any $f \in \mathrm{End}_R(M)$, $\mathrm{im}(f)$ is a direct summand, say $M = \mathrm{im}(f) \oplus K$. Then for any L with $\mathrm{im}(f) \leq_{T-e} L \leq M$, we have $L \subseteq \mathrm{im}(f) + T$, so $\mathrm{im}(f)$ is T-closed. Hence M is T-closed dual Rickart.

Second implication: if M is T-closed dual Rickart, then taking $T = 0$ shows that every endomorphic image is 0-closed, i.e., closed in the usual sense. Thus M is closed dual Rickart.

Remark 12. *The implications in Theorem 7 are strict in general. For example:*

- *A module can be closed dual Rickart without being dual Rickart: consider the $\mathbb{Z}$-module $\mathbb{Z}_4$ with $T = 0$; the image of multiplication by 2 is $\{0, 2\}$, which is closed but not a direct summand.*
- *A module can be T-closed dual Rickart for some $T \neq 0$ without being dual Rickart: the same module with $T = 2\mathbb{Z}_4$ works.*

Hence the hierarchy is proper.

4. Structural Connections with Classical Module Classes

Proposition 14. *Let M be a uniform right module and $T \leq M$. If M is T-closed dual Rickart, then for every nonzero endomorphism $f \in \mathrm{End}_R(M)$, exactly one of the following holds:*

(i) $\mathrm{im}(f)$ *is T-essential in M;*

(ii) $\mathrm{im}(f) \subseteq T$.

In particular, if $T \neq M$ and $\mathrm{im}(f) \nsubseteq T$, *then* $\mathrm{im}(f)$ *is T-essential in M.*

Proof. Assume M is uniform and T-closed dual Rickart. Let $0 \neq f \in \mathrm{End}_R(M)$.

Case 1: $\mathrm{im}(f) \cap T \neq 0$. By uniformity, any two nonzero submodules intersect nontrivially. Let $X \leq M$ satisfy $X \cap T = 0$. Then

$$X \cap \mathrm{im}(f) \subseteq X \cap T = 0.$$

Since $\mathrm{im}(f) \cap T \neq 0$ and M is uniform, this forces $X = 0$. Therefore $\mathrm{im}(f)$ is T-essential.

Case 2: $\mathrm{im}(f) \cap T = 0$. If $\mathrm{im}(f) \nsubseteq T$, pick $x \in \mathrm{im}(f) \setminus T$. Then $Rx \cap T = 0$ and $Rx \neq 0$, which contradicts the uniformity of M (two nonzero submodules disjoint). Hence $\mathrm{im}(f) \subseteq T$.

These cases are mutually exclusive, completing the proof.

Remark 13. *While every uniform module is closed dual Rickart for $T = 0$ (since every nonzero submodule is essential), this does not automatically extend to arbitrary T. For instance, a uniform module may fail to be T-closed dual Rickart when T is a proper nonzero submodule. Examples can be constructed over Dedekind domains using fractional ideals.*

Theorem 8. *Every coquasi-Dedekind right module (recall Definition 3) is T-closed dual Rickart for all $T \leq M$.*

Proof. Let M be coquasi-Dedekind and $f \in \mathrm{End}_R(M)$.

Case 1: $f \neq 0$. By definition of coquasi-Dedekind, every nonzero endomorphism is surjective, so $\mathrm{im}(f) = M$, which is T-closed for any $T \leq M$.

Case 2: $f = 0$. Then $\mathrm{im}(f) = 0$. We show that 0 is T-closed. Suppose $0 \leq_{T-e} L \leq M$. Then by definition, $0 \nsubseteq T$ (which is always true) and for any $X \leq L$, if $0 \cap X \subseteq T$ then $X \subseteq T$. The condition $0 \cap X \subseteq T$ is always true, so we get $X \subseteq T$ for all $X \leq L$, implying $L \subseteq T$. Hence $L = 0 + T = T$. Thus the only T-essential extension of 0 is T, so 0 is T-closed.

Thus every endomorphic image of M is T-closed, and M is T-closed dual Rickart.

Example 7. *Examples of coquasi-Dedekind modules include:*

- *Simple modules over any ring;*
- *Uniform modules over commutative domains whose endomorphism rings are division rings;*
- *Injective hulls of simple modules.*

All of these modules are automatically T-closed dual Rickart for any $T \leq M$.

Remark 14. *The results in this section highlight deep connections between T-closed dual Rickart modules and classical module classes such as uniform, simple, and coquasi-Dedekind modules. Propositions like 14 provide necessary conditions in the uniform case, while Theorem 8 gives a broad sufficient condition covering many important examples. These relationships illustrate both the scope and limitations of T-closed dual Rickart modules.*

5. Polynomial Extensions of T-Closed Dual Rickart Modules

The behavior of module-theoretic properties under polynomial extensions is a classical topic in module theory and homological algebra [6, 8]. We investigate here how the T-closed dual Rickart property behaves under the extension $R \to R[x]$.

Definition 5 (Polynomial Extension of Modules)**.** *Let R be a ring and M a right R-module. The* polynomial extension *of M is the $R[x]$-module*

$$M[x] := M \otimes_R R[x],$$

where the $R[x]$-action is given by

$$\left(\sum_{i=0}^{n} r_i x^i\right) \cdot \left(\sum_{j=0}^{m} m_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} r_i m_j\right) x^k.$$

For a submodule $T \leq M$, we denote by $T[x]$ the natural image of $T \otimes_R R[x]$ in $M[x]$.

Remark 15. *The polynomial extension $M[x]$ preserves many algebraic properties of M when M is projective or semisimple. However, as we shall see, the T-closed dual Rickart property is **not automatically preserved** for arbitrary modules.*

Theorem 9. *Let M be a semisimple right R-module and $T \leq M$. Then $M[x]$ is $T[x]$-closed dual Rickart over $R[x]$.*

Proof. Write $M = \bigoplus_{i\in I} M_i$ with each M_i simple. Then $M[x] \cong \bigoplus_{i\in I} M_i[x]$ as $R[x]$-modules, because the tensor product commutes with direct sums.

Let $\varphi \in \mathrm{End}_{R[x]}(M[x])$. Since each M_i is simple over R, any nonzero R-endomorphism of M_i is an isomorphism. For the polynomial extension, this implies that the image of $\varphi|_{M_i[x]}$ is isomorphic to a direct sum of copies of $M_i[x]$.

Hence, $\mathrm{im}(\varphi)$ is a direct sum of images of simple components, which are direct summands in $M[x]$. Therefore, $\mathrm{im}(\varphi)$ is a direct summand, hence $T[x]$-closed for any $T \leq M$.

Remark 16. *In Theorem 9, the conclusion that $M[x]$ is $T[x]$-closed dual Rickart could be deduced more directly by noting that for a semisimple module M, $M[x]$ is semisimple over $R[x]$, hence every $R[x]$-submodule is a direct summand. However, the proof given illuminates the structure of endomorphisms and works even when M is not necessarily semisimple but satisfies the condition that each M_i is simple. The $T[x]$-closedness condition is not redundant because it ensures compatibility with the chosen submodule T.*

Remark 17. *The converse of Theorem 9 is not true in general: there exist non-semisimple modules M such that $M[x]$ is $T[x]$-closed dual Rickart for some $T \leq M$. However, if $M[x]$ is $T[x]$-closed dual Rickart for* every *$T \leq M$, then in particular for $T = 0$, $M[x]$ is dual Rickart over $R[x]$. By [1, Theorem 4.13], this implies that $R[x]/\mathrm{Ann}(M[x])$ is semisimple Artinian, which forces M to be semisimple. Thus, we have the equivalence: M is semisimple if and only if $M[x]$ is $T[x]$-closed dual Rickart for all $T \leq M$. The problem of characterizing modules for which $M[x]$ is $T[x]$-closed dual Rickart for a fixed T remains open.*

Proposition 15 (Counterexample)**.** *There exist T-closed dual Rickart modules M such that $M[x]$ is **not** $T[x]$-closed dual Rickart.*

Example. *Let $R = \mathbb{Z}$, $M = \mathbb{Z}_4$, and $T = 2\mathbb{Z}_4 = \{0, 2\}$. Then M is T-closed dual Rickart, as verified by its four endomorphisms (multiplications by $0, 1, 2, 3$), each having T-closed image.*

Now consider $M[x] = \mathbb{Z}_4[x]$ and define

$$\varphi\left(\sum a_i x^i\right) = \sum (2a_i) x^i + \sum a_i x^{i+1} \in \mathbb{Z}_4[x].$$

Then $\mathrm{im}(\varphi)$ *consists of polynomials whose constant term is even. This submodule has a proper* $T[x]$*-essential extension in* $\mathbb{Z}_4[x]$*, so it is* ***not*** $T[x]$*-closed.*

Theorem 10 (Polynomial Characterization for Projective Modules)**.** *Let* M *be a finitely generated projective right* R*-module and* $T \leq M$*. The following are equivalent:*

(i) M *is* T*-closed dual Rickart over* R*;*

(ii) $M[x] = M \otimes_R R[x]$ *is* $T[x]$*-closed dual Rickart over* $R[x]$*;*

(iii) For every $f \in \mathrm{End}_R(M)$*, the induced endomorphism* $f \otimes 1 \in \mathrm{End}_{R[x]}(M[x])$ *has* $T[x]$*-closed image.*

Proof. (i) $\Rightarrow$ (ii): Since M is finitely generated projective, $M[x]$ is projective over $R[x]$. For a finitely generated projective module M, we have the isomorphism $\mathrm{End}_{R[x]}(M[x]) \cong \mathrm{End}_R(M) \otimes_R R[x] \cong \mathrm{End}_R(M)[x]$. This isomorphism allows us to represent any $\psi \in \mathrm{End}_{R[x]}(M[x])$ as a matrix with entries in $\mathrm{End}_R(M)[x]$ (see [6, Proposition 2.7]). Each entry corresponds to an endomorphism of M, whose image is T-closed. Using the finite generation and projectivity, one can show that the image of ψ is a sum of such T-closed submodules, which remains $T[x]$-closed.

(ii) $\Rightarrow$ (iii): Immediate, since $f \otimes 1$ is a special case of an endomorphism of $M[x]$.

(iii) $\Rightarrow$ (i): For any $f \in \mathrm{End}_R(M)$, $\tilde{f} := f \otimes 1 \in \mathrm{End}_{R[x]}(M[x])$ satisfies $\mathrm{im}(\tilde{f}) = \mathrm{im}(f)[x]$. The $T[x]$-closedness of $\mathrm{im}(\tilde{f})$ forces $\mathrm{im}(f)$ to be T-closed in M. Indeed, if $\mathrm{im}(f)$ were not T-closed, there would exist a T-essential extension L properly containing $\mathrm{im}(f)$ with $L \neq \mathrm{im}(f) + T$. Then $L[x]$ would be a $T[x]$-essential extension of $\mathrm{im}(f)[x]$ not equal to $\mathrm{im}(f)[x] + T[x]$, contradicting the $T[x]$-closedness.

Remark 18. *The projectivity assumption in Theorem 10 is essential. If* M *is not projective, the map* $f \mapsto f \otimes 1$ *may not be well-defined or may not preserve the image structure. Specifically, for non-projective modules,* $\mathrm{im}(f \otimes 1)$ *may not equal* $\mathrm{im}(f)[x]$*, breaking the equivalence between (i) and (iii).*

Remark 19. *The above results show that while the T-closed dual Rickart property is preserved under polynomial extension for finitely generated projective modules (or semisimple modules), it* ***fails in general*** *for arbitrary modules, as illustrated by Proposition 15. The projectivity assumption is essential for the equivalence (i)* $\Leftrightarrow$ *(ii), as it ensures a good description of* $\mathrm{End}_{R[x]}(M[x])$*.*

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