EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7557
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Directional Smarandache Curves of the Integral Curves in Euclidean 3−Space with Respect to the Flc Frame

Safaa Mosa[1,2], Mervat Elzawy[3,4*]

[1] *Mathematics Department, College of Science, University of Bisha, Bisha, Saudi Arabia*
[2] *Mathematics Department, Faculty of Science, Damanhour University, Damanhour, Egypt*
[3] *Mathematics Department, College of Science, Taibah University, Medina, Saudi Arabia*
[4] *Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt*

Abstract. This research examines directional Smarandache curves derived from spatial polynomial curves within the context of Flc-frame in three-dimensional Euclidean space E^3. The authors define four novel categories of directional Smarandache curves: $\mathbf{TD_2}, \mathbf{TD_1}, \mathbf{D_2D_1}$, and $\mathbf{TD_2D_1}$ based on specific combinations of the Flc frame vectors. For each type, the authors derive the complete Frenet apparatus (tangent, normal, and binormal vectors, curvature, and torsion) and the corresponding Flc apparatus. The theoretical findings are shown by a detailed analysis of a third-order polynomial curve, in which specific parameterizations for directional Smarandache curves are derived. Graphical comparisons of directional Smarandache curves created from the classical Frenet frame and the novel Flc frame are provided.

2020 Mathematics Subject Classifications: 53A04, 53A05

Key Words and Phrases: Flc-frame, Frenet frame, integral curves, directional Smarnadache curves, Euclidean 3−space.

---

1. Introduction

The geometric properties of curves in planar and Euclidean spaces are examined in the basic field of differential geometry. This area uses techniques from differential and integral calculus, such as derivatives and integrals, to study the intrinsic and extrinsic properties of curves, such as curvature, torsion, and arc length [7, 9].

Curves are utilized in industry and computer graphics, as well as in highways and railroads as deliberate bends and in canals to facilitate multifaceted direction changes. In order to avoid a sudden change of stage at the apex, they are also rolled in the vertical plane at every stage change [12].

*Corresponding Author.
DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7557

Email addresses: smosa@ub.edu.sa (S. Mosa), mervatelzawy@science.tanta.edu.eg, mrzawy@taibahu.edu.sa (M. Elzawy)

The frame of curves is crucial to examine their characterizations in learning curve theory. Among the several techniques used in curve analysis, the Frenet frame is one of the most widely used frameworks.[10, 11]. Dede recently defined a novel frame called the Frenet-like curve frame [6]. The cross product of the curve's first- and highest order derivatives can be used to compute the binormal vector of the curve, which yields this new frame. One advantage of this frame is that it reduces the number of undefinable single points in the Frenet frame. It also reduces the unwanted rotation around the tangent vector of the curve [10].

Examining the connections between various curves produced from a single basic curve is a particularly fascinating aspect of this study. Smarandache curves are unique among these derived curves because of their structure and characteristics. They are determined by specific combinations of the Frenet frame's tangent $\mathbf{T}$, normal $\mathbf{N}$, and binormal $\mathbf{B}$ vectors of the original curve [8, 10]. In [2], Ali presented the first research on this topic in Euclidean space. Smarandache curves have been examined in a number of settings, such as Minkowski space-time [13], Euclidean space's quasi-frame [5], and the Bishop frame [4]. Moreover, the Darboux frame [3] and other specialized frames [9] have been employed to investigate Euclidean Smarandache curves in the three-dimensional space.

In this research, the authors explore directional Smarandache curves in the three-dimensional Euclidean space associated with the Flc-frame. In particular, the authors study four different kinds of directional Smarandache curves: $\mathbf{TD_2}, \mathbf{TD_1}, \mathbf{D_2D_1}$, and $\mathbf{TD_2D_1}$-directional Smarandache curves. The whole Frenet apparatus (tangent, normal, and binormal vectors, curvature, and torsion) and the associated Flc apparatus are derived for each type.

The document is organized as outlined below: Section 2 reviews the fundamental concepts of the Frenet and Flc frames. Sections $3, 4, 5$, and 6 are dedicated to the detailed study of the directional Smarandache curves $\mathbf{TD_2}, \mathbf{TD_1}, \mathbf{D_2D_1}$, and $\mathbf{TD_2D_1}$, respectively. Section 7 presents an illustrative example and graphical comparisons, to confirm our theoretical conclusions. Lastly, the authors address the significance of their findings for the subject of differential geometry.

2. Basic Concepts

This section reviews the fundamental ideas referenced in the context of the study. Letting E^3 refer to Euclidean space in three dimensions with the standard metric:

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

such that (x, y, z) depicts an E^3 coordinate system. The representation of the inner product of two vectors $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ is represented by [7]:

$$\langle P, Q \rangle = p_1q_1 + p_2q_2 + p_3q_3. \tag{1}$$

Let $\alpha : I \to E^3$ be a curve of unit speed in E^3, where the interval I is on the real line $\mathbb{R}$. $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ represent the Frenet frame that moves, where $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ represent the curve's tangent, principal normal, and binormal vector fields of $\alpha(s)$, respectively. The Frenet formulas for the curve α are provided by [1, 9]:

$$\begin{aligned}
&\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s),\\
&\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s),\\
&\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s).
\end{aligned} \tag{2}$$

These vectors have the following definition:

$$\mathbf{T}(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|},\ \mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},\ \text{ and } \ \mathbf{B}(\mathbf{s}) = \mathbf{T}(s) \times \mathbf{N}(s), \tag{3}$$

and prime indicates differentiation in terms of s. In this instance, the curve's torsion and curvature are determined by using the following relationships:

$$\kappa(s) = \langle \mathbf{T}'(s), \mathbf{N}(s)\rangle,\ \tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle. \tag{4}$$

Let $\xi(s)$ be a degree n polynomial spatial curve. Assume that $\xi(s)$ is regular and $\xi'(s) \neq 0$. The following are the Flc frame's vector fields:

$$\mathbf{T} = \frac{\xi'}{\|\xi'\|},\ \mathbf{D_1} = \frac{\xi' \times \xi^{(n)}}{\|\xi' \times \xi^{(n)}\|},\ \text{ and } \ \mathbf{D_2} = \mathbf{D_1} \times \mathbf{T}, \tag{5}$$

where the unit tangent vector is represented by $\mathbf{T}$, the binormal-like vector by $\mathbf{D_1}$, and the principal normal-like vector by $\mathbf{D_2}$. The Flc frame d_1, d_2, and d_3 curvatures are provided by:

$$d_1 = \frac{1}{\vartheta}\langle \mathbf{T}', \mathbf{D_2}\rangle,\ d_2 = \frac{1}{\vartheta}\langle \mathbf{T}', \mathbf{D_1}\rangle,\ \text{ and } \ d_3 = \frac{1}{\vartheta}\langle \mathbf{D_2}', \mathbf{D_1}\rangle, \tag{6}$$

such that $\vartheta = \|\xi'\|$. In addition, the Frenet-like formulas are given in [10] as:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{D}'_{\mathbf{2}} \\ \mathbf{D}'_{\mathbf{1}} \end{pmatrix} = \vartheta \begin{pmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{D_2} \\ \mathbf{D_1} \end{pmatrix}. \tag{7}$$

The following are the relationships between the Frenet-like curve apparatus and the Frenet-curve apparatus:

$$\begin{aligned}
&d_1 = \kappa\ cos\phi,\\
&d_2 = -\kappa\ sin\phi,\\
&d_3 = d\phi + \tau,
\end{aligned} \tag{8}$$

and

$$\begin{cases} \mathbf{D_2} & = cos\phi\ \mathbf{N} + sin\phi\ \mathbf{B} \\ \mathbf{D_1} & = -sin\phi\ \mathbf{N} + cos\phi\ \mathbf{B} \end{cases} \ \text{ or } \ \begin{cases} \mathbf{N} & = cos\phi\ \mathbf{D_2} - sin\phi\ \mathbf{D_1} \\ \mathbf{B} & = sin\phi\ \mathbf{D_2} + cos\phi\ \mathbf{D_1}, \end{cases} \tag{9}$$

where ϕ is the angle formed by the principal normal vector $\mathbf{N}$ and the principal normal-like vector $\mathbf{D_2}$.

A curve whose position vector is a composite of the Frenet frame vectors from another regular curve is called a Smarandache curve [2]. The following Smarandache curves can be defined for a unit-speed curve $\alpha(s)$ with Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

$$\begin{aligned} \alpha_1(s) &= \frac{1}{\sqrt{2}}\Big(\mathbf{T}+\mathbf{N}\Big) && \mathbf{TN} - \text{Smarndache curve,}\\ \alpha_2(s) &= \frac{1}{\sqrt{2}}\Big(\mathbf{T}+\mathbf{B}\Big) && \mathbf{TB} - \text{Smarndache curve,}\\ \alpha_3(s) &= \frac{1}{\sqrt{2}}\Big(\mathbf{N}+\mathbf{B}\Big) && \mathbf{NB} - \text{Smarndache curve,}\\ \alpha_4(s) &= \frac{1}{\sqrt{3}}\Big(\mathbf{T}+\mathbf{N}+\mathbf{B}\Big) && \mathbf{TNB} - \text{Smarndache curve.} \end{aligned} \tag{10}$$

3. The Directional $\mathbf{TD_2}-$Smarandache curve

In this section, the authors will create the appropriate Flc apparatus of the directional $\mathbf{TD_2}-$Smarandache curve utilizing its Flc frame, as well as the Frenet apparatus, which includes tangent, normal, and binormal vectors, curvature, and torsion.

Definition 1. *Let $\xi(s)$ be a spatial polynomial curve in E^3 with the Flc frame $\{\mathbf{T(s)}, \mathbf{D_2(s)}, \mathbf{D_1(s)}\}$. The directional $\mathbf{TD_2}-$Smarandache curve associated with ξ, indicated by η, is the curve whose tangent vector field satisfies:*

$$\eta\,'(s) = \frac{1}{\sqrt{2}}\Big(\mathbf{T(s)}+\mathbf{D_2(s)}\Big). \tag{11}$$

Alternatively,

$$\eta(s) = \frac{1}{\sqrt{2}}\int\Big(\mathbf{T(s)}+\mathbf{D_2(s)}\Big)ds.$$

Theorem 1. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\eta(s)$ is the directional $\mathbf{TD_2}-$Smarandache curve of ξ, the curvature, torsion, and Frenet vector fields of η are then derived by:*

$$\begin{aligned} \mathbf{T}^\eta &= \frac{1}{\sqrt{2}}\Big(\mathbf{T(s)}+\mathbf{D_2(s)}\Big),\\ \mathbf{N}^\eta &= \frac{1}{\Delta}\Big(-d_1\mathbf{T(s)}+d_1\mathbf{D_2(s)}+\big(d_2+d_3\big)\mathbf{D_1(s)}\Big),\\ \mathbf{B}^\eta &= \frac{1}{\sqrt{2}\Delta}\Big(\big(d_2+d_3\big)\mathbf{T(s)}-\big(d_2+d_3\big)\mathbf{D_2(s)}+2d_1\mathbf{D_1(s)}\Big),\\ \kappa^\eta &= \frac{1}{\sqrt{2}}\vartheta\Delta,\ \ \textit{and}\ \ \tau^\eta = \frac{\sqrt{2}}{\Big(2+\big(\frac{d_2+d_3}{d_1}\big)^2\Big)}\frac{d}{ds}\Big(\frac{d_2+d_3}{d_1}\Big)+\frac{\vartheta}{\sqrt{2}}\big(d_3-d_2\big), \end{aligned}$$

such that $\Delta = \sqrt{2d_1^2 + \big(d_2+d_3\big)^2}$.

Proof. Let η be the directional $\mathbf{TD_2}$−Smarandache curve of ξ. Then from Eq.(11) and Eq.(7), we obtain

$$\eta\,''(s) = \frac{\vartheta}{\sqrt{2}}\Big(- d_1\mathbf{T}(s) + d_1\mathbf{D_2}(\mathbf{s}) + \big(d_2+d_3\big)\mathbf{D_1}(\mathbf{s})\Big).$$

Now using Eq.(3), we get

$$\begin{aligned}
\mathbf{T}^\eta &= \frac{1}{\sqrt{2}}\Big(\mathbf{T}(\mathbf{s}) + \mathbf{D_2}(\mathbf{s})\Big),\\
\mathbf{N}^\eta &= \frac{1}{\Delta}\Big(- d_1\mathbf{T}(\mathbf{s}) + d_1\mathbf{D_2}(\mathbf{s}) + \big(d_2+d_3\big)\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^\eta &= \frac{1}{\sqrt{2}\Delta}\Big(\big(d_2+d_3\big)\mathbf{T}(\mathbf{s}) - \big(d_2+d_3\big)\mathbf{D_2}(\mathbf{s}) + 2d_1\mathbf{D_1}(\mathbf{s})\Big).
\end{aligned} \tag{12}$$

From Eqs.(13) and (7), we reach

$$\begin{aligned}
\big(\mathbf{T}^\eta\big)\,' &= \frac{\vartheta}{\sqrt{2}}\Big(- d_1\mathbf{T}(\mathbf{s}) + d_1\mathbf{D_2}(\mathbf{s}) + \big(d_2+d_3\big)\mathbf{D_1}(\mathbf{s})\Big),\\
\big(\mathbf{N}^\eta\big)\,' &= \left(\frac{-1}{\Delta}\Big(d_1\,' + \vartheta d_1^2 + \vartheta d_2(d_2+d_3)\Big) + \frac{d_1\Delta\,'}{\Delta^2}\right)\mathbf{T}(s)\\
&\quad + \left(\frac{1}{\Delta}\Big(d_1\,' - \vartheta d_1^2 - \vartheta d_3(d_2+d_3)\Big) - \frac{d_1\Delta\,'}{\Delta^2}\right)\mathbf{D_2}(\mathbf{s})\\
&\quad + \left(\frac{1}{\Delta}\Big((d_2+d_3)\,' - \vartheta d_1 d_2 + \vartheta d_1 d_3\Big) - \frac{(d_2+d_3)\Delta\,'}{\Delta^2}\right)\mathbf{D_1}(\mathbf{s}).
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\kappa^\eta &= \langle \big(\mathbf{T}^\eta\big)\,', \mathbf{N}^\eta\rangle = \frac{1}{\sqrt{2}}\vartheta\Delta,\\
\tau^\eta &= \langle \big(\mathbf{N}^\eta\big)\,', \mathbf{B}^\eta\rangle = \frac{\sqrt{2}}{\left(2 + \left(\frac{d_2+d_3}{d_1}\right)^2\right)}\frac{d}{ds}\Big(\frac{d_2+d_3}{d_1}\Big) + \frac{\vartheta}{\sqrt{2}}\Big(d_3 - d_2\Big),
\end{aligned}$$

where $\Delta = \sqrt{2d_1^2 + \big(d_2+d_3\big)^2}$.

Theorem 2. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\eta(s)$ is the directional* $\mathbf{TD_2}-$*Smarandache curve of ξ, then the Flc apparatus of η is obtained by:*

$$\mathbf{T}^\eta = \frac{1}{\sqrt{2}}\Big(\mathbf{T(s)} + \mathbf{D_2(s)}\Big),$$

$$\mathbf{D_2}^\eta = \frac{1}{\sqrt{2}\Delta}\left(\Big((d_2+d_3)sin\phi^\eta - \sqrt{2}d_1cos\phi^\eta\Big)\mathbf{T}(s) + \Big(\sqrt{2}d_1cos\phi^\eta - (d_2+d_3)sin\phi^\eta\Big)\mathbf{D_2}(s) + \Big(2d_1sin\phi^\eta + \sqrt{2}(d_2+d_3)cos\phi^\eta\Big)\mathbf{D_1}(s)\right),$$

$$\mathbf{D_1}^\eta = \frac{1}{\sqrt{2}\Delta}\left(\Big((d_2+d_3)cos\phi^\eta + \sqrt{2}d_1sin\phi^\eta\Big)\mathbf{T}(s) - \Big(\sqrt{2}d_1sin\phi^\eta + (d_2+d_3)cos\phi^\eta\Big)\mathbf{D_2}(s) + \Big(2d_1cos\phi^\eta - \sqrt{2}(d_2+d_3)sin\phi^\eta\Big)\mathbf{D_1}(s)\right),$$

$$d_3^\eta = \frac{\sqrt{2}}{\left(2+\left(\frac{d_2+d_3}{d_1}\right)^2\right)}\frac{d}{ds}\Big(\frac{d_2+d_3}{d_1}\Big) + \frac{\vartheta}{\sqrt{2}}\Big(d_3-d_2\Big) + \frac{d_2^\eta\big(d_1^\eta\big)' - d_1^\eta\big(d_2^\eta\big)'}{\big(d_1^\eta\big)^2 + \big(d_2^\eta\big)^2},$$

$$d_1^\eta = \frac{1}{\sqrt{2}}\vartheta\Delta\ cos\phi^\eta,\ \ and\ d_2^\eta = \frac{-1}{\sqrt{2}}\vartheta\Delta\ sin\phi^\eta,$$

where ϕ^η is the angle between $\mathbf{D_2}^\eta$ and $\mathbf{N}^\eta$.

Proof. Using Theorem (1) and Eqs.(8) and (9), the results can be directly derived.

4. The Directional $\mathbf{TD_1}-$Smarandache curve

In this part, the authors will establish the tangent, normal, and binormal vectors in the Frenet apparatus, curvature, and torsion, and the corresponding Flc apparatus of the directional $\mathbf{TD_1}-$Smarandache curve by using its Flc frame.

Definition 2. *Let $\xi(s)$ be a spatial polynomial curve in E^3 with the Flc frame $\{\mathbf{T(s)}, \mathbf{D_2(s)}, \mathbf{D_1(s)}\}$. The directional $\mathbf{TD_1}-$Smarandache curve associated with ξ, denoted by β, is defined as the curve whose tangent vector field satisfies:*

$$\beta\,'(s) = \frac{1}{\sqrt{2}}\Big(\mathbf{T(s)} + \mathbf{D_1(s)}\Big). \tag{13}$$

In an equivalent manner,

$$\beta(s) = \frac{1}{\sqrt{2}}\int\Big(\mathbf{T(s)} + \mathbf{D_1(s)}\Big)ds.$$

Theorem 3. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\beta(s)$ is the directional* $\mathbf{TD_1}$*−Smarandache curve of ξ, the curvature, torsion, and Frenet vector fields of β are obtained by:*

$$
\begin{aligned}
\mathbf{T}^{\beta} &= \frac{1}{\sqrt{2}}\Big(\mathbf{T}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{N}^{\beta} &= \frac{1}{\Delta_1}\Big(- d_2\mathbf{T}(\mathbf{s}) + \big(d_1 - d_3\big)\mathbf{D_2}(\mathbf{s}) + d_2\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^{\beta} &= \frac{1}{\sqrt{2}\Delta_1}\Big(\big(d_3 - d_1\big)\mathbf{T}(\mathbf{s}) - 2d_2\mathbf{D_2}(\mathbf{s}) + \big(d_1 - d_3\big)\mathbf{D_1}(\mathbf{s})\Big),\\
\kappa^{\beta} &= \frac{1}{\sqrt{2}}\vartheta\Delta_1, \text{ and } \tau^{\beta} = \frac{-\sqrt{2}}{\Big(2 + \big(\frac{d_1 - d_3}{d_2}\big)^2\Big)}\frac{d}{ds}\Big(\frac{d_1 - d_3}{d_2}\Big) + \frac{\vartheta}{\sqrt{2}}\big(d_3 + d_1\big),
\end{aligned}
$$

such that $\Delta_1 = \sqrt{2d_2^2 + \big(d_1 - d_3\big)^2}$.

Proof. Let β be the directional $\mathbf{TD_1}$−Smarandache curve of ξ. Then from Eq.(13) and Eq.(7), we obtain

$$\beta\,''(s) = \frac{\vartheta}{\sqrt{2}}\Big(- d_2\mathbf{T}(s) + \big(d_1 - d_3\big)\mathbf{D_2}(\mathbf{s}) + d_2\mathbf{D_1}(\mathbf{s})\Big).$$

Now using Eq.(3), we get

$$
\begin{aligned}
\mathbf{T}^{\beta} &= \frac{1}{\sqrt{2}}\Big(\mathbf{T}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{N}^{\beta} &= \frac{1}{\Delta_1}\Big(- d_2\mathbf{T}(\mathbf{s}) + \big(d_1 - d_3\big)\mathbf{D_2}(\mathbf{s}) + d_2\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^{\beta} &= \frac{1}{\sqrt{2}\Delta_1}\Big(\big(d_3 - d_1\big)\mathbf{T}(\mathbf{s}) - 2d_2\mathbf{D_2}(\mathbf{s}) + \big(d_1 - d_3\big)\mathbf{D_1}(\mathbf{s})\Big),
\end{aligned}
\tag{14}
$$

where $\Delta_1 = \sqrt{2d_2^2 + \big(d_1 - d_3\big)^2}$. From Eqs.(14) and (7), we obtain

$$
\begin{aligned}
\big(\mathbf{T}^{\beta}\big)\,' &= \frac{\vartheta}{\sqrt{2}}\Big(- d_2\mathbf{T}(\mathbf{s}) + \big(d_1 - d_3\big)\mathbf{D_2}(\mathbf{s}) + d_2\mathbf{D_1}(\mathbf{s})\Big),\\
\big(\mathbf{N}^{\beta}\big)\,' &= \left(\frac{1}{\Delta_1}\Big(- d_2\,' - \vartheta d_1\big(d_1 - d_3\big) - \vartheta d_2^2\Big) + \frac{d_2\Delta_1\,'}{\Delta_1^2}\right)\mathbf{T}(s)\\
&\quad + \left(\frac{1}{\Delta_1}\Big(\big(d_1 - d_3\big)\,' - \vartheta d_1 d_2 - \vartheta d_2 d_3\Big) - \frac{\big(d_1 - d_3\big)\Delta_1\,'}{\Delta_1^2}\right)\mathbf{D_2}(\mathbf{s})\\
&\quad + \left(\frac{1}{\Delta_1}\Big(d_2\,' - \vartheta d_2^2 + \vartheta d_3\big(d_1 - d_3\big)\Big) - \frac{d_2\Delta_1\,'}{\Delta_1^2}\right)\mathbf{D_1}(\mathbf{s}).
\end{aligned}
$$

Therefore, we deduce that

$$\kappa^{\beta} = \langle \big(\mathbf{T}^{\beta}\big)\,', \mathbf{N}^{\beta} \rangle = \frac{1}{\sqrt{2}}\vartheta \Delta_1,$$

$$\tau^{\beta} = \langle \big(\mathbf{N}^{\beta}\big)\,', \mathbf{B}^{\beta} \rangle = \frac{-\sqrt{2}}{\Big(2+\big(\frac{d_1-d_3}{d_2}\big)^2\Big)}\frac{d}{ds}\Big(\frac{d_1-d_3}{d_2}\Big) + \frac{\vartheta}{\sqrt{2}}\big(d_3+d_1\big),$$

where $\Delta_1 = \sqrt{2d_2^2 + \big(d_1-d_3\big)^2}$.

Theorem 4. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\beta(s)$ is the directional $\mathbf{TD_1}-$Smarandache curve of ξ, then the Flc apparatus of β is obtained by:*

$$\mathbf{T}^{\beta} = \frac{1}{\sqrt{2}}\Big(\mathbf{T(s)} + \mathbf{D_1(s)}\Big),$$

$$\begin{aligned}\mathbf{D_2}^{\beta} = \frac{1}{\sqrt{2}\Delta_1}\Bigg(&\Big(-\sqrt{2}d_2 cos\phi^{\beta} + (d_3-d_1)sin\phi^{\beta}\Big)\mathbf{T}(s) + \Big(\sqrt{2}\big(d_1-d_3\big)cos\phi^{\beta} - 2d_2 sin\phi^{\beta}\Big)\mathbf{D_2}(s)\\ &+ \Big(\sqrt{2}d_2 cos\phi^{\beta} + \big(d_1-d_3\big)sin\phi^{\beta}\Big)\mathbf{D_1}(s)\Bigg),\end{aligned}$$

$$\begin{aligned}\mathbf{D_1}^{\beta} = \frac{1}{\sqrt{2}\Delta_1}\Bigg(&\Big(\sqrt{2}d_2 sin\phi^{\beta} + (d_3-d_1)cos\phi^{\beta}\Big)\mathbf{T}(s) - \Big(\sqrt{2}\big(d_1-d_3\big)sin\phi^{\beta} + 2d_2 cos\phi^{\beta}\Big)\mathbf{D_2}(s)\\ &+ \Big(\big(d_1-d_3\big)cos\phi^{\beta} - \sqrt{2}d_2 sin\phi^{\beta}\Big)\mathbf{D_1}(s)\Bigg),\end{aligned}$$

$$d_3^{\beta} = \frac{d_2^{\beta}\big(d_1^{\beta}\big)\,' - d_1^{\beta}\big(d_2^{\beta}\big)\,'}{\big(d_1^{\beta}\big)^2 + \big(d_2^{\beta}\big)^2} - \frac{\sqrt{2}}{\Bigg(2+\Big(\frac{d_1-d_3}{d_2}\Big)^2\Bigg)}\frac{d}{ds}\Big(\frac{d_1-d_3}{d_2}\Big) + \frac{\vartheta}{\sqrt{2}}\Big(d_3+d_1\Big),$$

$$d_1^{\beta} = \frac{1}{\sqrt{2}}\vartheta\Delta_1\ cos\phi^{\beta},\ \textit{and}\ d_2^{\beta} = \frac{-1}{\sqrt{2}}\vartheta\Delta_1\ sin\phi^{\beta},$$

where ϕ^{β} is the angle between $\mathbf{D_2}^{\beta}$ and $\mathbf{N}^{\beta}$.

Proof. Using Theorem (3) and Eqs.(8) and (9), the results can be directly derived.

5. The Directional $\mathbf{D_2D_1}-$Smarandache curve

Here, for the directional $\mathbf{D_2D_1}-$Smarandache curve, the authors will determine the Frenet apparatus as well as the Flc apparatus of this curve.

Definition 3. *Let $\xi(s)$ be a spatial polynomial curve in E^3 with the Flc frame $\{\mathbf{T(s)}, \mathbf{D_2(s)}, \mathbf{D_1(s)}\}$. The directional $\mathbf{D_2D_1}-$Smarandache curve associated with ξ, indicated by γ, is the curve* *whose tangent vector field meets the following criteria:*

$$\gamma\,'(s) = \frac{1}{\sqrt{2}}\Big(\mathbf{D_2}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big). \tag{15}$$

Similarly,

$$\gamma(s) = \frac{1}{\sqrt{2}}\int \Big(\mathbf{D_2}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big)ds.$$

Theorem 5. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\gamma(s)$ is the directional $\mathbf{D_2D_1}-$Smarandache curve of ξ, then Frenet vector fields, the curvature, and the torsion of γ are obtained by:*

$$\begin{aligned}
\mathbf{T}^\gamma &= \frac{1}{\sqrt{2}}\Big(\mathbf{D_2}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{N}^\gamma &= \frac{1}{\Delta_2}\Big(-\big(d_1+d_2\big)\mathbf{T}(\mathbf{s}) - d_3\mathbf{D_2}(\mathbf{s}) + d_3\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^\gamma &= \frac{1}{\sqrt{2}\Delta_2}\Big(2d_3\mathbf{T}(\mathbf{s}) - \big(d_1+d_2\big)\mathbf{D_2}(\mathbf{s}) + \big(d_1+d_2\big)\mathbf{D_1}(\mathbf{s})\Big),\\
\kappa^\gamma &= \frac{1}{\sqrt{2}}\vartheta\Delta_2, \ \ \textit{and} \ \ \tau^\gamma = \frac{-\sqrt{2}}{\Big(2+\big(\frac{d_1+d_2}{d_3}\big)^2\Big)}\frac{d}{ds}\Big(\frac{d_1+d_2}{d_3}\Big) + \frac{\vartheta}{\sqrt{2}}\big(d_1-d_2\big),
\end{aligned}$$

such that $\Delta_2 = \sqrt{2d_3^2 + \big(d_1+d_2\big)^2}$.

Proof. Let γ be the directional $\mathbf{D_2D_1}-$Smarandache curve of ξ. Then from Eq.(15) and Eq.(7), we obtain

$$\gamma\,''(s) = \frac{\vartheta}{\sqrt{2}}\Big(-\big(d_1+d_2\big)\mathbf{T}(s) - d_3\mathbf{D_2}(\mathbf{s}) + d_3\mathbf{D_1}(\mathbf{s})\Big).$$

Now using Eq.(3), we get

$$\begin{aligned}
\mathbf{T}^\gamma &= \frac{1}{\sqrt{2}}\Big(\mathbf{D_2}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{N}^\gamma &= \frac{1}{\Delta_2}\Big(-\big(d_1+d_2\big)\mathbf{T}(\mathbf{s}) - d_3\mathbf{D_2}(\mathbf{s}) + d_3\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^\gamma &= \frac{1}{\sqrt{2}\Delta_2}\Big(2d_3\mathbf{T}(\mathbf{s}) - \big(d_1+d_2\big)\mathbf{D_2}(\mathbf{s}) + \big(d_1+d_2\big)\mathbf{D_1}(\mathbf{s})\Big),
\end{aligned} \tag{16}$$

where $\Delta_2 = \sqrt{2d_3^2 + \big(d_1 + d_2\big)^2}$. From Eqs.(16) and (7), we obtain

$$\begin{aligned}
\big(\mathbf{T}^\gamma\big)' &= \frac{\vartheta}{\sqrt{2}}\Big(-(d_1+d_2)\mathbf{T}(\mathbf{s}) - d_3\mathbf{D_2}(\mathbf{s}) + d_3\mathbf{D_1}(\mathbf{s})\Big),\\
\big(\mathbf{N}^\gamma\big)' &= \left(\frac{1}{\Delta_2}\Big(-(d_1+d_2)' + \vartheta d_1 d_3 - \vartheta d_2 d_3\Big) + \frac{(d_1+d_2)\Delta_2'}{\Delta_2^2}\right)\mathbf{T}(s)\\
&\quad + \left(\frac{1}{\Delta_2}\Big(-d_3' - \vartheta d_1(d_1+d_2) - \vartheta d_3^2\Big) + \frac{d_3\Delta_2'}{\Delta_2^2}\right)\mathbf{D_2}(\mathbf{s})\\
&\quad + \left(\frac{1}{\Delta_2}\Big(d_3' - \vartheta d_3^2 - \vartheta d_2\big(d_1+d_2\big)\Big) - \frac{d_3\Delta_2'}{\Delta_2^2}\right)\mathbf{D_1}(\mathbf{s}).
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\kappa^\gamma &= \langle \big(\mathbf{T}^\gamma\big)', \mathbf{N}^\gamma\rangle = \frac{1}{\sqrt{2}}\vartheta\Delta_2,\\
\tau^\gamma &= \langle \big(\mathbf{N}^\gamma\big)', \mathbf{B}^\gamma\rangle = \frac{-\sqrt{2}}{\Big(2 + \big(\frac{d_1+d_2}{d_3}\big)^2\Big)}\frac{d}{ds}\Big(\frac{d_1+d_2}{d_3}\Big) + \frac{\vartheta}{\sqrt{2}}\big(d_1 - d_2\big),
\end{aligned}$$

where $\Delta_2 = \sqrt{2d_3^2 + \big(d_1 + d_2\big)^2}$.

Theorem 6. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\gamma(s)$ is the directional $\mathbf{D_2D_1}-$Smarandache curve of ξ, then the Flc apparatus of γ is obtained by:*

$$\begin{aligned}
\mathbf{T}^\gamma &= \frac{1}{\sqrt{2}}\Big(\mathbf{D_2}(\mathbf{s}) + \mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{D_2}^\gamma &= \frac{1}{\sqrt{2}\Delta_2}\Bigg(\Big(-\sqrt{2}(d_1+d_2)cos\phi^\gamma + 2d_3 sin\phi^\gamma\Big)\mathbf{T}(s) + \Big(-\sqrt{2}d_3 cos\phi^\gamma - (d_1+d_2)sin\phi^\gamma\Big)\mathbf{D_2}(s)\\
&\quad + \Big(\sqrt{2}d_3 cos\phi^\gamma + \big(d_1+d_2\big)sin\phi^\gamma\Big)\mathbf{D_1}(s)\Bigg),\\
\mathbf{D_1}^\gamma &= \frac{1}{\sqrt{2}\Delta_1}\Bigg(\Big(\sqrt{2}d_2 sin\phi^\gamma + (d_3 - d_1)cos\phi^\gamma\Big)\mathbf{T}(s) - \Big(\sqrt{2}\big(d_1 - d_3\big)sin\phi^\gamma + 2d_2 cos\phi^\gamma\Big)\mathbf{D_2}(s)\\
&\quad + \Big(\big(d_1 - d_3\big)cos\phi^\gamma - \sqrt{2}d_2 sin\phi^\gamma\Big)\mathbf{D_1}(s)\Bigg),\\
d_3^\gamma &= \frac{d_2^\gamma\big(d_1^\gamma\big)' - d_1^\gamma\big(d_2^\gamma\big)'}{\big(d_1^\gamma\big)^2 + \big(d_2^\gamma\big)^2} - \frac{\sqrt{2}}{\left(2 + \left(\frac{d_1+d_2}{d_3}\right)^2\right)}\frac{d}{ds}\Big(\frac{d_1+d_2}{d_3}\Big) + \frac{\vartheta}{\sqrt{2}}\Big(d_1 - d_2\Big),\\
d_1^\gamma &= \frac{1}{\sqrt{2}}\vartheta\Delta_2\ cos\phi^\gamma,\ \textbf{\textit{and}}\ d_2^\gamma = \frac{-1}{\sqrt{2}}\vartheta\Delta_2\ sin\phi^\gamma,
\end{aligned}$$

where ϕ^{γ} is the angle between $\mathbf{D_2}^{\gamma}$ and $\mathbf{N}^{\gamma}$.

Proof. Using Theorem (5) and Eqs.(8) and (9), the results can be directly derived.

6. The Directional $\mathbf{TD_2D_1}-$Smarandache curve

In this section, the Frenet apparatus and the Flc equipment of the directional $\mathbf{TD_2D_1}-$Smarandache curve are presented.

Definition 4. *Let $\xi(s)$ be a spatial polynomial curve in E^3 with the Flc frame $\{\mathbf{T(s)}, \mathbf{D_2(s)}, \mathbf{D_1(s)}\}$. The directional $\mathbf{TD_2D_1}-$Smarandache curve associated with ξ, indicated by δ, is the curve whose tangent vector field meets the following criteria:*

$$\delta\,'(s) = \frac{1}{\sqrt{3}}\Big(\mathbf{T(s)} + \mathbf{D_2(s)} + \mathbf{D_1(s)}\Big). \tag{17}$$

In the same way,

$$\delta(s) = \frac{1}{\sqrt{3}}\int\Big(\mathbf{T(s)} + \mathbf{D_2(s)} + \mathbf{D_1(s)}\Big)ds.$$

Theorem 7. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\delta(s)$ is the directional $\mathbf{TD_2D_1}-$Smarandache curve of ξ, the curvature, torsion, and Frenet vector fields of γ are then derived by:*

$$\begin{aligned}
\mathbf{T}^{\delta} &= \frac{1}{\sqrt{3}}\Big(\mathbf{T(s)} + \mathbf{D_2(s)} + \mathbf{D_1(s)}\Big),\\
\mathbf{N}^{\delta} &= \frac{1}{\Delta_3}\Big(-\big(d_1+d_2\big)\mathbf{T(s)} + \big(d_1-d_3\big)\mathbf{D_2(s)} + \big(d_2+d_3\big)\mathbf{D_1(s)}\Big),\\
\mathbf{B}^{\delta} &= \frac{1}{\sqrt{3}\Delta_3}\Big(\big(2d_3+d_2+d_1\big)\mathbf{T(s)} - \big(2d_2+d_1+d_3\big)\mathbf{D_2(s)} + \big(2d_1+d_2-d_3\big)\mathbf{D_1(s)}\Big),\\
\tau^{\delta} &= \frac{1}{\sqrt{3}\Delta_3^2}\Big(-3d_1\,'\big(d_3+d_2\big) + 3d_2\,'\big(d_1-d_3\big) + 3d_3\,'\big(d_2+d_1\big) + 2\vartheta\big(d_1^2-d_2^2+d_3^2+3d_1d_2d_3\big)\Big),\\
\kappa^{\delta} &= \frac{1}{\sqrt{3}}\vartheta\Delta_3,
\end{aligned}$$

such that $\Delta_3 = \sqrt{\big(d_1+d_2\big)^2 + \big(d_1-d_3\big)^2 + \big(d_2+d_3\big)^2}$.

Proof. Let δ be the directional $\mathbf{TD_2D_1}-$Smarandache curve of ξ. Then from Eq.(17) and Eq.(7), we obtain

$$\delta\,''(s) = \frac{\vartheta}{\sqrt{3}}\Big(-\big(d_1+d_2\big)\mathbf{T(s)} + \big(d_1-d_3\big)\mathbf{D_2(s)} + \big(d_2+d_3\big)\mathbf{D_1(s)}\Big).$$

Now using Eq.(3), we get

$$\begin{aligned}
\mathbf{T}^{\delta} &= \frac{1}{\sqrt{3}}\Big(\mathbf{T}(\mathbf{s})+\mathbf{D_2}(\mathbf{s})+\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{N}^{\delta} &= \frac{1}{\Delta_3}\Big(-\big(d_1+d_2\big)\mathbf{T}(\mathbf{s})+\big(d_1-d_3\big)\mathbf{D_2}(\mathbf{s})+\big(d_2+d_3\big)\mathbf{D_1}(\mathbf{s})\Big),\\
\mathbf{B}^{\delta} &= \frac{1}{\sqrt{3}\Delta_3}\Big(\big(2d_3+d_2+d_1\big)\mathbf{T}(\mathbf{s})-\big(2d_2+d_1+d_3\big)\mathbf{D_2}(\mathbf{s})+\big(2d_1+d_2-d_3\big)\mathbf{D_1}(\mathbf{s})\Big),
\end{aligned} \tag{18}$$

where $\Delta_3=\sqrt{\big(d_1+d_2\big)^2+\big(d_1-d_3\big)^2+\big(d_2+d_3\big)^2}$. From Eqs.(18) and (7), we obtain

$$\big(\mathbf{T}^{\delta}\big)'=\frac{\vartheta}{\sqrt{3}}\Big(-\big(d_1+d_2\big)\mathbf{T}(\mathbf{s})+\big(d_1-d_3\big)\mathbf{D_2}(\mathbf{s})+\big(d_2+d_3\big)\mathbf{D_1}(\mathbf{s})\Big),$$

$$\begin{aligned}
\big(\mathbf{N}^{\delta}\big)' =& \left(\frac{-1}{\Delta_3}\Big((d_1+d_2)'+\vartheta d_1(d_1-d_3)+\vartheta d_2(d_2+d_3)\Big)+\frac{(d_1+d_2)\Delta_3{}'}{\Delta_3^2}\right)\mathbf{T}(s)\\
&+\left(\frac{1}{\Delta_3}\Big((d_1-d_3)'-\vartheta d_1(d_1+d_2)-\vartheta d_3(d_2+d_3)\Big)-\frac{(d_1-d_3)\Delta_3{}'}{\Delta_3^2}\right)\mathbf{D_2}(\mathbf{s})\\
&+\left(\frac{1}{\Delta_3}\Big((d_2+d_3)'+\vartheta d_3(d_1-d_3)-\vartheta d_2\big(d_1+d_2\big)\Big)-\frac{(d_2+d_3)\Delta_3{}'}{\Delta_3^2}\right)\mathbf{D_1}(\mathbf{s}).
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\kappa^{\delta} &= \frac{1}{\sqrt{3}}\vartheta\Delta_3,\\
\tau^{\delta} &= \frac{1}{\sqrt{3}\Delta_3^2}\Big(-3d_1{}'\big(d_3+d_2\big)+3d_2{}'\big(d_1-d_3\big)+3d_3{}'\big(d_2+d_1\big)+2\vartheta\big(d_1^2-d_2^2+d_3^2+3d_1d_2d_3\big)\Big),
\end{aligned}$$

where $\Delta_3=\sqrt{\big(d_1+d_2\big)^2+\big(d_1-d_3\big)^2+\big(d_2+d_3\big)^2}$.

Theorem 8. *Let $\xi(s)$ be a spatial polynomial curve in E^3. If $\delta(s)$ is the directional*

$\mathbf{TD_2D_1}-$*Smarandache curve of* ξ*, then the Flc apparatus of* δ *is obtained by:*

$$\mathbf{T}^{\delta} = \frac{1}{\sqrt{3}}\Big(\mathbf{T(s)} + \mathbf{D_2(s)} + \mathbf{D_1(s)}\Big),$$

$$\mathbf{D_2}^{\delta} = \frac{1}{\sqrt{3}\Delta_3}\Bigg[\Big(-\sqrt{3}(d_1+d_2)cos\phi^{\delta} + \big(2d_3+d_2-d_1\big)sin\phi^{\delta}\Big)\mathbf{T}(s) - \Big(\sqrt{3}\big(d_1-d_3\big)cos\phi^{\delta} + (2d_2+d_1+d_3)sin\phi^{\delta}\Big)\mathbf{D_2}(s) + \Big(\sqrt{3}\big(d_2+d_3\big)cos\phi^{\delta} + \big(2d_1+d_2-d_3\big)sin\phi^{\delta}\Big)\mathbf{D_1}(s)\Bigg],$$

$$\mathbf{D_1}^{\delta} = \frac{1}{\sqrt{3}\Delta_3}\Bigg[\Big(\sqrt{3}\big(d_1+d_2\big)sin\phi^{\delta} + \big(2d_3+d_2-d_1\big)cos\phi^{\delta}\Big)\mathbf{T}(s) - \Big(\sqrt{3}\big(d_1-d_3\big)sin\phi^{\delta} + \big(2d_2+d_1+d_3\big)cos\phi^{\delta}\Big)\mathbf{D_2}(s) + \Big(\big(2d_1+d_2-d_3\big)cos\phi^{\delta} - \sqrt{3}\big(d_2+d_3\big)sin\phi^{\delta}\Big)\mathbf{D_1}(s)\Bigg],$$

$$d_3^{\delta} = \frac{d_2^{\delta}\big(d_1^{\delta}\big)' - d_1^{\delta}\big(d_2^{\delta}\big)'}{\big(d_1^{\delta}\big)^2 + \big(d_2^{\delta}\big)^2} + \frac{1}{\sqrt{3}\Delta_3^2}\Big(-3d_1'\big(d_3+d_2\big) + 3d_2'\big(d_1-d_3\big) + 3d_3'\big(d_2+d_1\big) + 2\vartheta\big(d_1^2-d_2^2+d_3^2+3d_1d_2d_3\big)\Big),$$

$$d_1^{\delta} = \frac{1}{\sqrt{3}}\vartheta\Delta_3\ cos\phi^{\delta},\ \text{and}\ d_2^{\delta} = \frac{-1}{\sqrt{3}}\vartheta\Delta_3\ sin\phi^{\delta}.$$

where ϕ^{δ} *is the angle between* $\mathbf{D_2}^{\delta}$ *and* $\mathbf{N}^{\delta}$*.*

Proof. Using Theorem (7) and Eqs.(8) and (9), the results can be directly derived.

Example 1. *Let us define* $\xi(s)$ *as a third-order polynomial curve characterized by*

$$\xi(s) = \Big(2s, s^2, \frac{1}{3}s^3\Big).$$

The associated Frenet apparatus of this curve can be obtained as:

$$\mathbf{T}(s) = \Big(\frac{2}{2+s^2}, \frac{2s}{2+s^2}, \frac{s^2}{2+s^2}\Big),$$

$$\mathbf{N}(s) = \Big(0, \frac{1}{\sqrt{1+s^2}}, \frac{s}{\sqrt{1+s^2}}\Big),$$

$$\mathbf{B}(s) = \Big(\frac{s^2}{\sqrt{1+s^2}\big(2+s^2\big)}, \frac{-2s}{\sqrt{1+s^2}\big(2+s^2\big)}, \frac{2}{\sqrt{1+s^2}\big(2+s^2\big)}\Big),$$

$$\kappa(s) = \langle \mathbf{T}'(s), \mathbf{N}(s)\rangle = \frac{2}{\sqrt{1+s^2}\big(2+s^2\big)}, \quad \text{and} \quad \tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle = \frac{2}{\big(1+s^2\big)\big(2+s^2\big)}.$$

Moreover, the associated Flc apparatus of $\xi(s)$ *is*

$$\begin{aligned}
\mathbf{T}(s) &= \Big(\frac{2}{2+s^2}, \frac{2s}{2+s^2}, \frac{s^2}{2+s^2}\Big),\\
\mathbf{D_2}(s) &= \Big(\frac{-s^2}{\sqrt{1+s^2}\big(2+s^2\big)}, \frac{-s^3}{\sqrt{1+s^2}\big(2+s^2\big)}, \frac{2\sqrt{1+s^2}}{2+s^2}\Big),\\
\mathbf{D_1}(s) &= \Big(\frac{s}{\sqrt{1+s^2}}, \frac{-1}{\sqrt{1+s^2}}, 0\Big),\\
d_1 &= \frac{1}{\vartheta}\langle \mathbf{T}\,'(s), \mathbf{D_2}(s)\rangle = \frac{2s}{\sqrt{1+s^2}\big(2+s^2\big)^3}, \quad d_2 = \frac{1}{\vartheta}\langle \mathbf{T}\,'(s), \mathbf{D_1}(s)\rangle = \frac{-2}{\sqrt{1+s^2}\big(2+s^2\big)^2},\\
d_3 &= \frac{1}{\vartheta}\langle \mathbf{D_2}\,'(s), \mathbf{D_1}(s)\rangle = \frac{s^2}{\big(1+s^2\big)\big(2+s^2\big)^2}.
\end{aligned}$$

To contrast the two frames, namely the Frenet-like frame and the Frenet frame, we label η_1 *and* η_2 *as the directional* $\mathbf{TN}-$*Smarandache curve and the directional* $\mathbf{TD_2}-$*Smarandache curve, respectively, and describe them as follows:*

$$\begin{aligned}
\eta_1(s) &= \frac{1}{\sqrt{2}}\int \Big(\mathbf{T(s)} + \mathbf{N(s)}\Big)ds\\
&= \left(tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + c_1, \frac{1}{\sqrt{2}}ln(2+s^2) + \frac{1}{\sqrt{2}}ln\big(\sqrt{s^2+1}+s\big) + c_2, \frac{s}{\sqrt{2}} - tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + \frac{1}{\sqrt{2}}\sqrt{1+s^2} + c_3\right),\\
\eta_2(s) &= \frac{1}{\sqrt{2}}\int \Big(\mathbf{T(s)} + \mathbf{D_2(s)}\Big)ds\\
&= \left(tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) - \frac{1}{\sqrt{2}}ln\big(\sqrt{1+s^2}-s\big) + \frac{1}{2}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big) + c_4,\right.\\
&\quad \frac{1}{\sqrt{2}}\Big(ln(s^2+2) - \sqrt{1+s^2} + 2tan^{-1}\big(\sqrt{1+s^2}\big)\Big) + c_5,\\
&\quad \left.\frac{s}{\sqrt{2}} - tan^{-1}\Big(\frac{s}{\sqrt{2}} + \sqrt{2}ln\big(\sqrt{s^2+1}+s\big) - \frac{1}{2}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big) + c_6\right),
\end{aligned}$$

where c_i*,* $i = 1, \ldots, 6$ *are integral constants, and Figure 1 displays the graphs of these curves. Subsequently, we designate* $\beta_1(s)$ *and* $\beta_2(s)$ *as the directional* $\mathbf{TB}-$*Smarandache curve and the directional* $\mathbf{TD_1}-$*Smarandache curve, respectively, and define them as follows:*

$$\begin{aligned}
\beta_1(s) &= \frac{1}{\sqrt{2}}\int \Big(\mathbf{T(s)} + \mathbf{B(s)}\Big)ds\\
&= \left(tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + \frac{1}{\sqrt{2}}ln\big(\sqrt{1+s^2}+s\big) - \frac{1}{2}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big) + b_1,\right.\\
&\quad \left.\frac{1}{\sqrt{2}}ln(2+s^2) - \sqrt{2}tan^{-1}\big(\sqrt{1+s^2}\big) + b_2, \frac{s}{\sqrt{2}} - tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + \frac{1}{2}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big) + b_3\right),
\end{aligned}$$

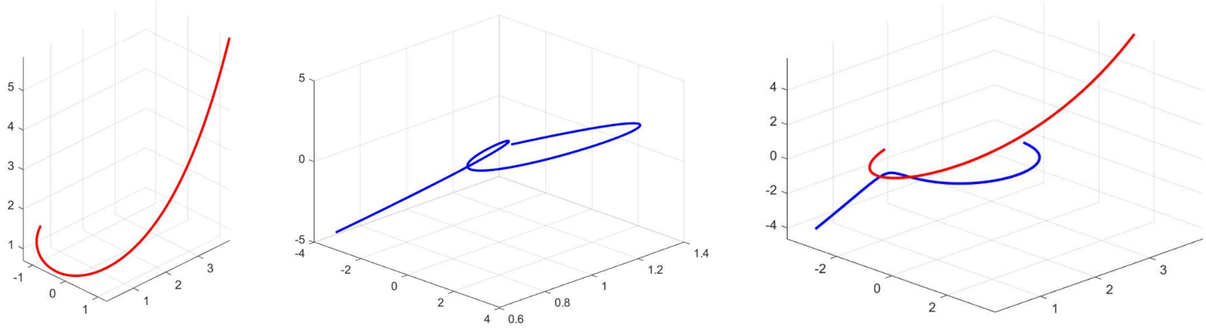

Figure 1: $\mathbf{TN}$ compared to $\mathbf{TD_2}-$ directional Smarandache Curves.

$$\beta_2 = \frac{1}{\sqrt{2}}\int \Big(\mathbf{T(s)} + \mathbf{D_1(s)}\Big)ds$$
$$= \left(tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + \frac{1}{\sqrt{2}}\sqrt{1+s^2} + b_4, \frac{1}{\sqrt{2}}\Big(ln(s^2+2) - sinh^{-1}(s)\Big) + b_5, \frac{s}{\sqrt{2}} - tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big) + b_6\right),$$

such that b_i*, for* $i = 1, \ldots, 6$ *are integral constants and the graphs of these curves can be seen in Figure 2.*

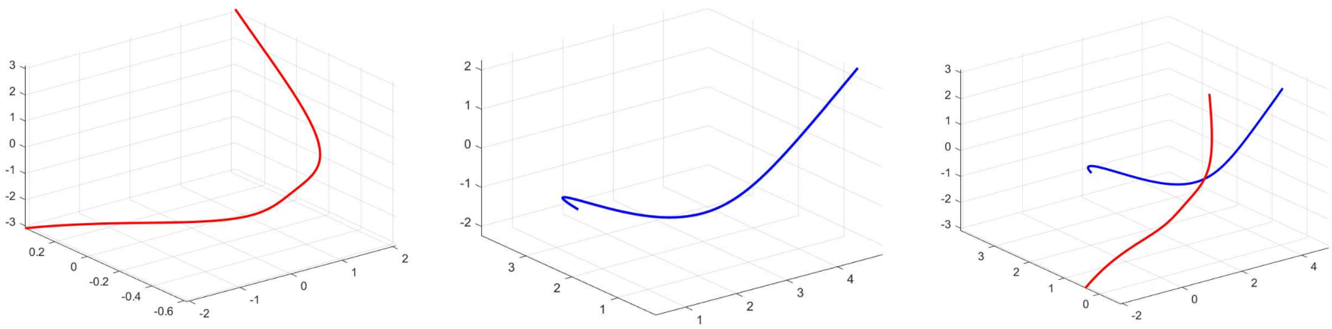

Figure 2: $\mathbf{TB}$ compared to $\mathbf{TD_1}-$ directional Smarandache Curves.

We will hereafter refer to $\gamma_1(s)$ *and* $\gamma_2(s)$ *as the directional* $\mathbf{NB}-$*Smarandache curve* *and the directional* $\mathbf{D_2D_1}-$*Smarandache curve, respectively, and define them accordingly:*

$$\begin{aligned}
\gamma_1(s) &= \frac{1}{\sqrt{2}}\int \Big(\mathbf{N(s)}+\mathbf{B(s)}\Big)ds \\
&= \Bigg(\frac{1}{\sqrt{2}}\ln\left(\sqrt{1+s^2}+s\right)-\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\right)+d_1, \\
&\quad \frac{1}{\sqrt{2}}\sinh^{-1}(s)-\sqrt{2}\ \tan^{-1}\left(\sqrt{1+s^2}\right)+d_2, \\
&\quad \frac{1}{\sqrt{2}}\sqrt{1+s^2}+\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\right)+d_3\Bigg), \\
\gamma_2 &= \frac{1}{\sqrt{2}}\int \Big(\mathbf{D_2(s)}+\mathbf{D_1(s)}\Big)ds \\
&= \Bigg(\frac{1}{\sqrt{2}}\sqrt{1+s^2}-\frac{1}{\sqrt{2}}\ln\left(\sqrt{1+s^2}+s\right)+\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\right)+d_4, \\
&\quad -\frac{1}{\sqrt{2}}\sinh^{-1}(s)-\frac{1}{\sqrt{2}}\sqrt{1+s^2}+\sqrt{2}\ \tan^{-1}\left(\sqrt{1+s^2}\right)+d_5, \\
&\quad \sqrt{2}\ \ln\left(\sqrt{1+s^2}+s\right)-\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\right)+d_6\Bigg).
\end{aligned}$$

in which d_i, $i=1,\dots,6$ *are integral constants and the graphs of these curves can be shown in Figure 3. If* $\delta_1(s)$ *and* $\delta_1(s)$ *are defined as the directional* $\mathbf{TNB}-$*Smarandache curve*

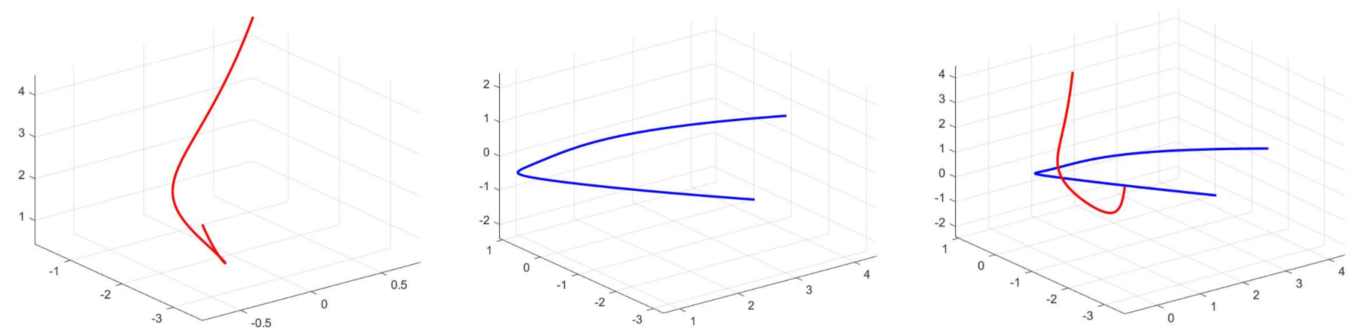

Figure 3: $\mathbf{NB}$ compared to $\mathbf{D_2D_1}-$ directional Smarandache Curves.

and the directional $\mathbf{TD_2D_1}-$*Smarandache curve, respectively, we obtain*

$$\begin{aligned}
\delta_1(s) &= \frac{1}{\sqrt{3}}\int \Big(\mathbf{T}(\mathbf{s})+\mathbf{N}(\mathbf{s})+\mathbf{B}(\mathbf{s})\Big)ds \\
&= \Bigg(\frac{\sqrt{2}}{\sqrt{3}}\ tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big)+\frac{1}{\sqrt{3}}ln\big(\sqrt{1+s^2}+s\big)-\frac{1}{\sqrt{6}}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big)+a_1, \\
&\frac{1}{\sqrt{3}}ln\big(2+s^2\big)+\frac{1}{\sqrt{3}}sinh^{-1}(s)-\frac{2}{\sqrt{3}}\ tan^{-1}\big(\sqrt{1+s^2}\big)+a_2, \\
&\frac{s}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}\ tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big)+\frac{1}{\sqrt{3}}\sqrt{1+s^2}+\frac{1}{\sqrt{6}}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big)+a_3\Bigg), \\
\delta_2 &= \frac{1}{\sqrt{3}}\int \Big(\mathbf{T}(\mathbf{s})+\mathbf{D_2}(\mathbf{s})+\mathbf{D_1}(\mathbf{s})\Big)ds \\
&= \Bigg(\frac{\sqrt{2}}{\sqrt{3}}\ tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big)-\frac{1}{\sqrt{3}}ln\big(\sqrt{1+s^2}+s\big)+\frac{1}{\sqrt{6}}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big)+\frac{1}{\sqrt{3}}\sqrt{1+s^2}+a_4, \\
&\frac{1}{\sqrt{3}}ln\big(2+s^2\big)-\frac{1}{\sqrt{3}}sinh^{-1}(s)+\frac{2}{\sqrt{3}}\ tan^{-1}\big(\sqrt{1+s^2}\big)-\frac{1}{\sqrt{3}}\sqrt{1+s^2}+a_5, \\
&\frac{s}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}\ tan^{-1}\Big(\frac{s}{\sqrt{2}}\Big)+\frac{2}{\sqrt{3}}ln\big(\sqrt{1+s^2}+s\big)-\frac{1}{\sqrt{6}}ln\Big(\frac{\sqrt{2}\sqrt{1+s^2}+s}{\sqrt{2}\sqrt{1+s^2}-s}\Big)+a_6\Bigg),
\end{aligned}$$

where a_i, *for* $i=1,\ldots,6$ *are integral constants and the graphs of these curves are illustrated in Figure 4.*

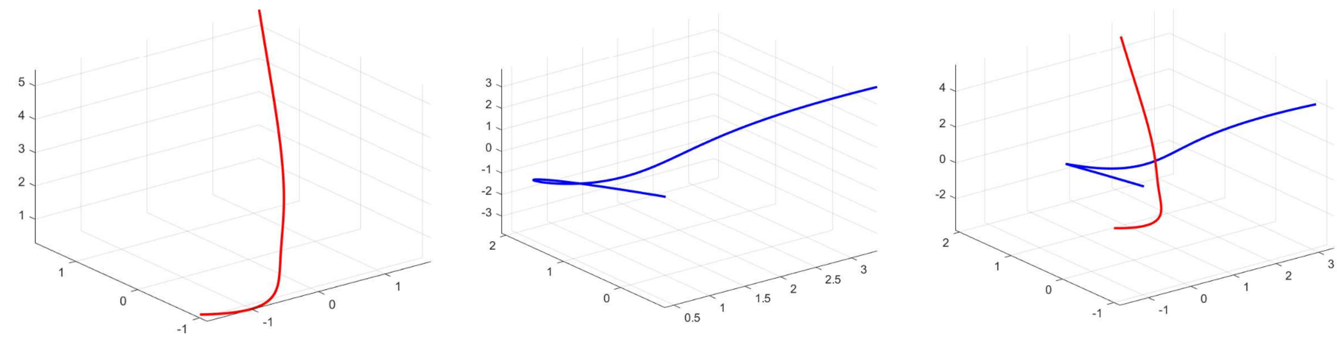

Figure 4: $\mathbf{TNB}$ compared to $\mathbf{TD_2D_1}-$ directional Smarandache Curves.

7. Conclusion

In this work, the geometric properties of directional Smarandache curves associated with the Flc-frame in the Euclidean 3−space were investigated. We designate four distinct categories of directional Smarandache curves: $\mathbf{TD_2}, \mathbf{TD_1}, \mathbf{D_2D_1}$, and $\mathbf{TD_2D_1}$ depending on specific combinations of the Flc frame vectors. For each category, we construct a comprehensive Frenet apparatus, which includes the corresponding Flc equipment, as well as tangent, normal, and binormal vectors, in addition to curvature and torsion. A comprehensive investigation of a third-order polynomial curve reveals the theoretical discoveries from which precise parameterizations for directional Smarandache curves are obtained. Graphical comparisons of directional Smarandache curves generated from the classical Frenet frame and the new Flc frame are presented. This work lays the groundwork for future research and advances our grasp of curve theory in differential geometry.

Acknowledgements

The authors express their sincere gratitude to the editor and anonymous reviewers for their constructive feedback and recommendations.

References

[1] Abbena, E., Solamon, S., Gray, A., *Modern differential geometry of curves and surfaces with Mathematica*, CRC Press.

[2] Ali, A. T. (2010). *Special Smarandache curves in the Euclidean space.* International Journal of Mathematical Combinatorics, 2, 30-36.

[3] Bektaş, Ö., Yüce, S. (2013). *Special Smarandache curves according to Darboux frame in Euclidean 3-Space.* Romanian Journal of Mathematics and Computer Science, 3(1), 48-59.

[4] Çetin, M., Tuncer, Y., and Karacan, M. K. (2014). *Smarandache curves according to Bishop frame in Euclidean 3-space.* Gen. Math. Notes, 20(2), 50-66.

[5] Chimienti, S. P., Bencze, M. (1998). *Smarandache anti-geometry.* Smarandache Notions Journal, 9(1-3), 48.

[6] Dede, M. (2019). *A new representation of tubular surfaces*, Houston J. Math., 45, 707-720.

[7] Baizeed, H., Elsharkawy, A., Cesarano, C., and Ramadan, A. A. (2025). *Smarandache curves for the integral curves with the quasi frame in Euclidean 3-space.* Azerbaijan Journal of Mathematics, 15(2), 219-239.

[8] Cesarano, C., Elsharkawy, A., Baizeed, H. (2025). *Construction and Analysis of Smarandache Curves for Integral Binormal Curves in Euclidean 3-Space.* Universal Journal of Mathematics and Applications, 8(3), 149-157.

[9] Elzawy, M., and Aloufi, A. (2024). *Some Special Smarandache-Ruled Surfaces According to quasi-Frame in Euclidean* $3-$*Space* E^3. Appl. Math, 18(4), 685-699.

[10] Güven, I. A. (2020). *Some integral curves with a new frame.* Open Mathematics, 18(1), 1332-1341.

[11] Lee, J. M. (2012). *Integral Curves and Flows.* In Introduction to Smooth Manifolds (pp. 205-248). New York, NY: Springer New York.

[12] Nurkan, S. K., and Güven, İ. (2022). *A new approach for Smarandache curves.* Turkish Journal of Mathematics and Computer Science, 14(1), 155-165.

[13] Turgut, M., and Yilmaz, S. (2008). *Smarandache curves in Minkowski space-time*, Inter national Journal of Mathematical Combinatorics, 3, 51-55.