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Matrix Similarity over Commutative Rings and Remarks on the Gauger-Byrnes Theorem

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Abstract. We develop a unified, module-theoretic approach to matrix similarity over a unital commutative ring R. For $A, B \in R_n$, we show that A and B are similar over R if and only if $\mathrm{Ker}(\tau_{AB})$, with $\tau_{AB}(Z) = ZA - BZ$, is a right free $\mathrm{C}(A)$-module of rank 1 whose generators are invertible in R_n; equivalently, the same condition holds for the left $\mathrm{C}(B)$-module structure. This viewpoint provides a complete and intrinsic characterization of similarity over arbitrary commutative rings, leads to a concise proof of an enhanced Gauger–Byrnes theorem over fields, and clarifies the role of Kronecker operators and centralizer modules, while bypassing invariant-factor computations.

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1. Introduction

Matrix similarity is a classical problem in linear algebra. Let $\mathbb{F}$ be a field, and write $\mathbb{F}_n := M_n(\mathbb{F})$ for the ring of $n \times n$ matrices over $\mathbb{F}$. Two matrices $A, B \in \mathbb{F}_n$ are *similar* if there exists an invertible matrix $P \in \mathrm{GL}_n(\mathbb{F})$ such that

$$P^{-1}AP = B.$$

This notion underlies canonical forms of linear operators, module theory, and many classification problems, such as determining Jordan or rational canonical forms.

In 1977, M. A. Gauger and C. I. Byrnes [1] established a connection between similarity and rank conditions:

Theorem 1 (Gauger–Byrnes). *Let $A, B \in \mathbb{F}_n$. The following are equivalent:*

(i) A and B are similar over $\mathbb{F}$.

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(ii) A and B have the same characteristic polynomial, and

$$r_{AA} = r_{BB} = r_{AB},$$

where $r_{XY} := \operatorname{rank}(X \otimes I_n - I_n \otimes Y)$.

Later, J. Dixon [2] showed that the characteristic polynomial condition is redundant:

Theorem 2 (Dixon). *If $A, B \in \mathbb{F}_n$ satisfy $r_{AB}^2 = r_{AA} \cdot r_{BB}$, then A and B are similar over $\mathbb{F}$.*

These results demonstrate that, over fields, similarity can be completely determined by rank computations via the Kronecker product, bypassing explicit invariant-factor calculations.

In recent years, there has been growing interest in extending similarity theory to commutative rings. For example, Zhu [3] generalized the Latimer–MacDuffee correspondence to Dedekind rings; Calugareanu [1] studied unipotent conjugacy in integral domains; Knight and Stasinski [4] gave explicit representatives for similarity classes over PIDs; Marseglia [5] explored conjugacy of integral matrices in relation to abelian varieties; Ramos and Simis [6] examined the structure of matrices over polynomial rings via determinantal ideals; and Shchedryk [7] analyzed arithmetic of matrices over finitely generated PIDs. Despite these advances, existing results often assume strong structural properties on R (such as being a PID, Dedekind domain, or local ring), or they do not yield effective, rank-based similarity criteria in the general commutative ring setting. It remains an open problem to describe similarity purely in terms of computable invariants, such as module structure and rank operators, for an arbitrary commutative ring.

Let R be a unital commutative ring, and for $n \in \mathbb{N}$, let $R_n := M_n(R)$ denote the ring of $n \times n$ matrices over R, with identity I_n and unit group $\mathrm{GL}_n(R)$. For $A, B \in R_n$, the (left) Kronecker product is

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

We define the R-linear map

$$\tau_{AB} : R_n \longrightarrow R_n, \qquad Z \longmapsto ZA - BZ.$$

We adopt row-major vectorization: for $Z \in R^{n \times n}$,

$$\mathrm{vec}_r(Z) = (z_{11}, z_{12}, \ldots, z_{1n}, z_{21}, \ldots, z_{nn}),$$

so that $\tau_{AB}(Z) = ZA - BZ$ is represented by $A \otimes I - I \otimes B$ with no transpose of A appearing.

Two matrices $A, B \in R_n$ are similar over R if

$$\mathrm{Ker}(\tau_{AB}) \cap \mathrm{GL}_n(R) \neq \emptyset.$$

The kernels of τ_{AA} and τ_{BB} define the centralizers

$$\mathrm{C}(A) := \{X \in R_n : AX = XA\}, \qquad \mathrm{C}(B) := \{Y \in R_n : BY = YB\}.$$

The main result, Theorem 3, of this paper establishes that A and B are similar over R if and only if $\mathrm{Ker}(\tau_{AB})$ is a free module of rank 1 over the corresponding centralizer and admits an invertible generator in R_n. This provides a conceptual simplification of classical results: similarity can be decided by module-theoretic properties without comparing invariant factors.

Moreover, the criterion extends under suitable structural hypotheses, including noncommutative unital rings in which $\mathrm{C}(A)$ is a division ring or a local Artinian ring, faithfully flat algebras over commutative bases, semilocal rings where rank-one projective modules are free, Banach or C^*-algebras with bounded invertibles, and rings without identity via quasi-invertible matrices or the Dorroh envelope. The unital commutative ring setting remains the most general framework in which the theorem holds in its present form without additional hypotheses.

As an application, we obtain a refined form of the Gauger–Byrnes theorem over fields, expressed purely in terms of rank conditions of Kronecker operators (see Theorem 4).

Finally, we conclude the paper with a dedicated section containing explicit examples over rings with nilpotents (see Example 1) and localizations of polynomial rings (see Example 2). These examples demonstrate how Theorem 3 operates in concrete situations.

For completeness, we also fix standard terminology that will be used throughout the paper. An ideal $\mathfrak{p} \subset R$ is called *prime* if $\mathfrak{p} \neq R$ and whenever $ab \in \mathfrak{p}$ for $a, b \in R$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. An ideal $\mathfrak{m} \subset R$ is called *maximal* if $\mathfrak{m} \neq R$ and there is no ideal strictly between $\mathfrak{m}$ and R; equivalently, $R/\mathfrak{m}$ is a field. For any commutative ring R with identity, $U(R)$ denotes the group of units of R. For a polynomial

$$p(x) = a_m x^m + \cdots + a_0 \in R[x], \qquad a_m \neq 0,$$

we set $\deg(p) = m$, and by convention $\deg(0) = -\infty$. If $f, g \in R[x]$, the notation $f \mid g$ means that f *divides* g, i.e. there exists $h(x) \in R[x]$ such that $g(x) = f(x)h(x)$; in particular, $\gcd(f, g)$ refers to a monic greatest common divisor in $R[x]$, unique when R is a GCD domain or when R is a field. Matrix notation includes $R_n = M_n(R)$, $\mathrm{C}(A) = \{X \in R_n : AX = XA\}$, and $\mathrm{GL}_n(R)$ denotes the group of invertible matrices. All remaining terminology is standard and may be found in [8].

2. Matrix Similarity over Commutative Rings

Before formalizing the bimodule structure, we note that the kernel of the map τ_{AB} naturally interacts with the centralizers of both A and B. Concretely, any element of

$\mathrm{Ker}(\tau_{AB})$ can be multiplied on the right by elements of $\mathrm{C}(A)$ and on the left by elements of $\mathrm{C}(B)$ without leaving the kernel. This observation highlights the inherent symmetry between A and B in the context of matrix similarity and motivates the following lemma.

Lemma 1. *For $A, B \in R_n$, the submodule $\mathrm{Ker}(\tau_{AB})$ is simultaneously a right $\mathrm{C}(A)$-module and a left $\mathrm{C}(B)$-module.*

Proof. Let $X \in \mathrm{C}(A)$, $Y \in \mathrm{C}(B)$ and $Z \in \mathrm{Ker}(\tau_{AB})$. We need to show that $ZX, YZ \in \mathrm{Ker}(\tau_{AB})$. We have:

$$(ZX)A - B(ZX) = Z(XA) - (BZ)X = Z(AX) - (ZA)X = 0.$$

This shows that $ZX \in \mathrm{Ker}(\tau_{AB})$.
Similarly, we have:

$$(YZ)A - B(YZ) = Y(ZA) - (BY)Z = Y(BZ) - (YB)Z = 0.$$

Hence, $YZ \in \mathrm{Ker}(\tau_{AB})$.

This lemma shows that $\mathrm{Ker}(\tau_{AB})$ is a *bimodule* over $(\mathrm{C}(B), \mathrm{C}(A))$, a structure which is central to our similarity criterion.

Theorem 3. *Let R be a unital commutative ring, and $A, B \in R_n$. The following are equivalent:*

(1) *A and B are similar over R.*

(2) $\mathrm{Ker}(\tau_{AB})$ *is a* right free $\mathrm{C}(A)$*-module of rank* 1*, and any generator of this module is invertible in R_n.*

(3) $\mathrm{Ker}(\tau_{AB})$ *is a* left free $\mathrm{C}(B)$*-module of rank* 1*, and any generator of it is invertible in R_n.*

Proof. (1)$\Rightarrow$(2): As A and B are similar over R, then there exists $P \in \mathrm{Ker}(\tau_{AB}) \cap U(R_n)$. Let $\langle P \rangle$ denote the right $\mathrm{C}(A)$-submodule of $\mathrm{Ker}(\tau_{AB})$ generated by P. We claim that $\mathrm{Ker}(\tau_{AB}) = \langle P \rangle$. Clearly, $\langle P \rangle \subseteq \mathrm{Ker}(\tau_{AB})$. Conversely, let $X \in \mathrm{Ker}(\tau_{AB})$ and set $Z = P^{-1}X$. We have $ZA = P^{-1}XA = P^{-1}BX = AP^{-1}X = AZ$. This implies that $Z \in \mathrm{C}(A)$. Since $X = PZ$, then $X \in \langle P \rangle$. It follows that $\mathrm{Ker}(\tau_{AB}) = \langle P \rangle$, as claimed. Since $P \in U(R_n)$, $\mathrm{Ker}(\tau_{AB})$ is a free module with $\{P\}$ as a basis. So $\mathrm{Ker}(\tau_{AB})$ has rank 1. Now let Q be another generator of $\mathrm{Ker}(\tau_{AB})$. Then $Q = PN$ and $P = QN'$ for some $N, N' \in \mathrm{C}(A)$. Note that $P = QN' = P(NN')$. So $NN' = I_n$. Hence, $N \in U(R_n)$ and a fortiori $Q \in U(R_n)$ as the product of two invertible matrices of R_n.
(2)$\Rightarrow$(1): Let $\{P\}$ be a basis of $\mathrm{Ker}(\tau_{AB})$ as a right $\mathrm{C}(A)$-module. Then, by hypothesis $P \in \mathrm{Ker}(\tau_{AB}) \cap U(R_n)$. It follows that A and B are similar over R.
Proceed along the same lines as above to prove the equivalence (1)$\Leftrightarrow$(3). This completes the proof.

We note that the freeness condition on $\mathrm{Ker}(\tau_{AB})$ could, in some contexts, be weakened to *projectivity of rank one*. Over commutative local or semilocal rings, for example, every rank-one projective module is automatically free, so the theorem extends without modification. In more general commutative rings, projective rank-one modules need not be free, and additional care is required: a projective generator may fail to be invertible in R_n, which is essential for the similarity criterion. For further details on rank-one projective modules over commutative rings, see [9, Ch. II].

For a prime ideal $\mathfrak{p} \subset R$, we denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$, i.e. the ring of fractions with denominators in $R \setminus \mathfrak{p}$. We write $(\tau_{AB})_\mathfrak{p}$ for the scalar extension of τ_{AB} to $(R_\mathfrak{p})_n$.

Corollary 1. *Suppose R is an integral domain and $A, B \in R_n$. Then the following are equivalent:*

(1) A and B are similar over R.

(2) $\mathrm{Ker}(\tau_{AB})$ is a right free $\mathrm{C}(A)$-module of rank 1, and for every prime ideal $\mathfrak{p} \subset R$, A and B are similar over $(R_\mathfrak{p})_n$.

(3) $\mathrm{Ker}(\tau_{AB})$ is a right free $\mathrm{C}(A)$-module of rank 1, and for every maximal ideal $\mathfrak{m} \subset R$, A and B are similar over $(R_\mathfrak{m})_n$.

Proof. (1)$\Rightarrow$(2): It is clear that if two matrices are similar over R, then they are similar over any commutative ring containing R. Since R is an integral domain, then $R \subseteq R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of R. Therefore, A and B are similar over $R_\mathfrak{p}$. The fact that $\mathrm{Ker}(\tau_{AB})$ is a right free $\mathrm{C}(A)$-module follows from Theorem 3.
(2)$\Rightarrow$(3): Trivial.
(3)$\Rightarrow$(1): Let $\{H\}$ be a basis of $\mathrm{Ker}(\tau_{AB})$ as a right free $\mathrm{C}(A)$-module and let $\mathfrak{m}$ be a maximal ideal of R. By assumption, there exists an element $H_\mathfrak{m} \in \mathrm{Ker}((\tau_{AB})_\mathfrak{m}) \cap U((R_\mathfrak{m})_n)$. Since H is also a free generator of $\mathrm{Ker}((\tau_{AB})_\mathfrak{m})$ as a $\mathrm{C}(A)_\mathfrak{m}$-module, then there exists $D_\mathfrak{m} \in \mathrm{C}(A)_\mathfrak{m}$ such that $H_\mathfrak{m} = D_\mathfrak{m} H$. Thus, $\det(D_\mathfrak{m})\det(H) = \det(H_\mathfrak{m}) \in U(R_\mathfrak{m})$, and so $\det(H) \in U(R_\mathfrak{m})$. But, $R = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_\mathfrak{m}$. So $U(R) = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} U(R_\mathfrak{m})$. Therefore, $\det(H) \in U(R)$ and so $H \in U(R_n)$. In particular, $H \in \mathrm{Ker}(\tau_{AB}) \cap U(R_n)$. This implies that A and B are similar over R.

Remark 1. The module $\mathrm{Ker}(\tau_{AB})$ is also a left free $\mathrm{C}(B)$-module of rank 1, reflecting a natural symmetry between left and right actions. Combined with the local-global principle, this shows that two matrices over an integral domain are similar if and only if they are locally similar at every prime (or maximal) ideal and the necessary module-rank condition holds. This perspective provides a conceptual understanding of similarity, unifying both algebraic and module-theoretic viewpoints.

3. Remarks on the Gauger–Byrnes Theorem and a New Variant

The classical Gauger–Byrnes theorem provides a rank-theoretic criterion for similarity over a field, relying on the Kronecker product operator $X \mapsto XA - BX$. In this section, we revisit this result from the perspective developed in Section 2. Our module-theoretic characterization of similarity leads to a refined version of the Gauger–Byrnes criterion, in which the hypothesis on characteristic polynomials is eliminated.

The key point is that, over a field, the dimension of $\mathrm{Ker}(\tau_{AB})$ can be computed explicitly using invariant factors, thereby allowing one to detect similarity purely through the ranks of the Kronecker operators τ_{AA}, τ_{BB}, and τ_{AB}. The following theorem formulates this criterion in a streamlined form that depends only on rank conditions.

Theorem 4 (Refined Gauger–Byrnes)**.** *Let $\mathbb{F}$ be a field, and $A, B \in \mathbb{F}_n$. The following are equivalent:*

(1) A and B are similar over $\mathbb{F}$.

(2) $r_{AA} = r_{BB} = r_{AB}$.

(3) $r_{AA} + r_{BB} = 2\, r_{AB}$.

Proof. (1) $\Rightarrow$ (2): We show that $r_{AA} = r_{AB}$ and by using a similar argument, one can prove easily that $r_{BB} = r_{AB}$. To this end, it is enough to show that $\dim_{\mathbb{F}}(\mathrm{C}(A)) = \dim_{\mathbb{F}}(\mathrm{Ker}(\tau_{AB}))$. Set $k = \dim_{\mathbb{F}}(\mathrm{C}(A))$ and let $\mathcal{B} = \{X_1, X_2, \cdots, X_k\}$ be a basis of the $\mathbb{F}$-vector space $\mathrm{C}(A)$. By Theorem 3, $\mathrm{Ker}(\tau_{AB})$ is a right free $\mathrm{C}(A)$-module of rank 1, say with basis $\{P\}$. We claim that $\mathcal{B}' = \{PX_1, PX_2, \cdots, PX_k\}$ is a basis of the $\mathbb{F}$-vector space $\mathrm{Ker}(\tau_{AB})$. For, let $Z \in \mathrm{Ker}(\tau_{AB})$. Then $Z = PX$ for some $X \in \mathrm{C}(A)$. Write $X = \sum_{i=1}^{k} a_i X_i$, where $a_1, a_2, \cdots, a_k \in \mathbb{F}$. Thus, $Z = \sum_{i=1}^{k} a_i P X_i$. This shows that $\mathcal{B}'$ spans $\mathrm{Ker}(\tau_{AB})$. Now, let $\lambda_1, \cdots, \lambda_k \in \mathbb{F}$ such that $\sum_{i=1}^{k} \lambda_i P X_i = 0$. As P is invertible, we get $\sum_{i=1}^{k} \lambda_i X_i = 0$. This implies that $\lambda_i = 0$ for any $1 \leq i \leq k$ since $\mathcal{B}$ is linearly independent. Hence, $\mathcal{B}'$ is linearly independent as well. Thus, $\mathcal{B}'$ is a basis for $\mathrm{Ker}(\tau_{AB})$, as claimed. It follows that $\dim_{\mathbb{F}}(\mathrm{C}(A)) = \dim_{\mathbb{F}}(\mathrm{Ker}(\tau_{AB}))$.
(2) $\Rightarrow$ (3): Trivial.
(3) $\Rightarrow$ (1): We recall that for a nonzero polynomial

$$p(x) = a_m x^m + \cdots + a_1 x + a_0 \in \mathbb{F}[x],\ a_m \neq 0,$$

the *degree* of p, denoted $\deg(p)$, is m (and by convention $\deg(0) = -\infty$).

To compute $\dim_{\mathbb{F}}(\mathrm{Ker}(\tau_{AB}))$ we use a classical result of MacDuffee. Specifically, [10, Chapter IV, Theorem 5] gives the following formula: if

$$d_1 \mid d_2 \mid \cdots \mid d_s, \qquad D_1 \mid D_2 \mid \cdots \mid D_t$$

are the invariant factors of A and B, respectively, then

$$\dim_{\mathbb{F}}(\mathrm{Ker}(\tau_{AB})) = \sum_{k=1}^{s} \sum_{l=1}^{t} \deg\big(\gcd(d_k, D_l)\big).$$

Thus

$$r_{AB} = n^2 - \dim_{\mathbb{F}}(\mathrm{Ker}(\tau_{AB})) = n^2 - \sum_{k=1}^{s} \sum_{l=1}^{t} \deg\big(\gcd(d_k, D_l)\big).$$

Similarly for r_{AA} and r_{BB}. It follows that:

$$r_{AA} + r_{BB} - 2r_{AB} = -\Big(\sum_{k_1=1}^{s}\sum_{k_2=1}^{s} \deg\big(\gcd(d_{k_1}, d_{k_2})\big) + \sum_{l_1=1}^{t}\sum_{l_2=1}^{t} \deg\big(\gcd(D_{l_1}, D_{l_2})\big) \\ - 2\sum_{k=1}^{s}\sum_{l=1}^{t} \deg\big(\gcd(d_k, D_l)\big)\Big).$$

We now extend the definitions of $d_1, \cdots, d_s$ and $D_1, \cdots, D_t$ by renaming them as follows, with $N = \max(s,t)$:

$$\underbrace{1, \cdots, 1}_{N-s}, d_1, \cdots, d_s \ \rightarrow \ f_1, \cdots, f_N,$$

$$\text{and } \underbrace{1, \cdots, 1}_{N-t}, D_1, \cdots, D_t \ \rightarrow \ F_1, \cdots, F_N.$$

Therefore

$$r_{AA} + r_{BB} - 2r_{AB} = -\Big(\sum_{k=1}^{N}\sum_{l=1}^{N}\Big(\deg\big(\gcd(f_k, f_l)\big) + \deg\big(\gcd(F_k, F_l)\big) \\ - 2\deg\big(\gcd(f_k, F_l)\big)\Big)\Big). \tag{1}$$

Let $\pi_A(x)$ and $\pi_B(x)$ denote the minimal polynomial of A and B respectively. We now let $p_1, \cdots, p_r$ be the distinct monic irreducibles in $\pi_A(x)\pi_B(x)$ and write $\forall 1 \le k \le N$:

$$f_k = p_1^{a_{k_1}} p_2^{a_{k_2}} \cdots p_r^{a_{k_r}}$$
$$F_k = p_1^{b_{k_1}} p_2^{b_{k_2}} \cdots p_r^{b_{k_r}}.$$

where, $(a_{k_i})_{1\le i\le r}, (b_{k_i})_{1\le i\le r}$ are monotonic increasing sequences of nonnegative integers. Then

$$\gcd(f_k, F_l) = \prod_{i=1}^{r} p_i^{\min(a_{k_i}, b_{l_i})}.$$

This implies

$$\deg\big(\gcd(f_k, F_l)\big) = \sum_{i=1}^{r} \deg(p_i)\ \min(a_{k_i}, b_{l_i}),$$
$$\deg\big(\gcd(f_k, f_l)\big) = \sum_{i=1}^{r} \deg(p_i)\ \min(a_{k_i}, a_{l_i}),$$
$$\deg\big(\gcd(F_k, F_l)\big) = \sum_{i=1}^{r} \deg(p_i)\ \min(b_{k_i}, b_{l_i}).$$

Then equation (1) may be rewritten as

$$\begin{aligned} r_{AA} + r_{BB} - 2r_{AB} &= -(\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{i=1}^{r}\deg(p_i)\ (\min(a_{k_i}, a_{l_i}) + \min(b_{k_i}, b_{l_i}) \\ &\quad - 2\min(a_{k_i}, b_{l_i}))) \\ &= -(\sum_{i=1}^{r}\deg(p_i)\sum_{k=1}^{N}\sum_{l=1}^{N}(\min(a_{k_i}, a_{l_i}) + \min(b_{k_i}, b_{l_i}) \\ &\quad - 2\min(a_{k_i}, b_{l_i}))). \end{aligned}$$

As $r_{AA} + r_{BB} - 2r_{AB} = 0$, it follows that

$$\sum_{i=1}^{r}\deg(p_i)\sum_{k=1}^{N}\sum_{l=1}^{N}(\min(a_{k_i}, a_{l_i}) + \min(b_{k_i}, b_{l_i}) - 2\min(a_{k_i}, b_{l_i})) = 0.$$

According to [11, Lemma 1], the sequences $(a_{k_i})_{1\le i\le r}, (b_{k_i})_{1\le i\le r}$ are identical. Thus, A and B have the same invariant factors and so A and B are similar.

4. Examples Illustrating Applications of Theorem 3

In this final section we present several examples showing how Theorem 3 operates in concrete settings. These examples cover the integers, localizations, polynomial rings, and rings with nilpotents. They demonstrate how the criterion based on the module structure of $\mathrm{Ker}(\tau_{AB})$ detects similarity or its failure in cases where classical invariant-factor or eigenvalue methods do not apply.

Example 1: Similar Matrices over $\mathbb{Z}/8\mathbb{Z}$

Let $R = \mathbb{Z}/8\mathbb{Z}$ and consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in R_2.$$

We examine the kernel

$$\mathrm{Ker}(\tau_{AB}) = \{Z \in R_2 : ZA - BZ = 0\}.$$

Writing

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$

we compute

$$ZA = \begin{pmatrix} z_{11} & z_{11} + z_{12} \\ z_{21} & z_{21} + z_{22} \end{pmatrix}, \quad BZ = \begin{pmatrix} z_{11} & z_{12} \\ z_{11} + z_{21} & z_{12} + z_{22} \end{pmatrix}.$$

Hence

$$ZA - BZ = \begin{pmatrix} 0 & z_{11} \\ -z_{11} & z_{21} - z_{12} \end{pmatrix}.$$

Setting this equal to zero gives

$$z_{11} = 0, \quad z_{21} = z_{12}.$$

Thus the kernel is

$$\mathrm{Ker}(\tau_{AB}) = \left\{ \begin{pmatrix} 0 & z \\ z & w \end{pmatrix} : z, w \in R \right\}.$$

The centralizer of A is

$$\mathrm{C}(A) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in R \right\}.$$

Choosing

$$Z_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Ker}(\tau_{AB}),$$

we see that every element of the kernel can be written as

$$Z = Z_0 \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in \mathrm{Ker}(\tau_{AB}), \quad \alpha, \beta \in R.$$

Since $\alpha \in U(R) = \{1, 3, 5, 7\}$ in any generator $Z_0 \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, all generators are invertible. Therefore $\mathrm{Ker}(\tau_{AB})$ is a right free $\mathrm{C}(A)$-module of rank 1 with all generators invertible.

By Theorem 3, A and B are similar over $\mathbb{Z}/8\mathbb{Z}$. An explicit similarity matrix is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(R), \quad P^{-1}AP = B.$$

Example 2: Non-Similarity over a Localized Polynomial Ring

Let $\mathbb{F}$ be a field and let $R = \mathbb{F}[x]_{(x)}$, the localization of $\mathbb{F}[x]$ at the prime ideal (x). Consider

$$A = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 \\ x^2 & x \end{pmatrix} \in R_2.$$

We investigate

$$\mathrm{Ker}(\tau_{AB}) = \{Z \in R_2 : ZA - BZ = 0\}.$$

Let

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in R_2.$$

We have

$$ZA = \begin{pmatrix} z_{11}x & z_{11} + z_{12}x \\ z_{21}x & z_{21} + z_{22}x \end{pmatrix},$$

and

$$BZ = \begin{pmatrix} xz_{11} & xz_{12} \\ x^2 z_{11} + xz_{21} & x^2 z_{12} + xz_{22} \end{pmatrix}.$$

Thus,

$$ZA - BZ = \begin{pmatrix} 0 & z_{11} \\ -x^2 z_{11} & z_{21} - x^2 z_{12} \end{pmatrix}.$$

Therefore,

$$\mathrm{Ker}(\tau_{AB}) = \left\{ \begin{pmatrix} 0 & a \\ x^2 a & b \end{pmatrix} : a, b \in R \right\}.$$

The *centralizer* of B is

$$\mathrm{C}(B) = \{X \in R_2 : XB = BX\} = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in R \right\}.$$

Choose the generator

$$Z_0 = \begin{pmatrix} 0 & 1 \\ x^2 & 0 \end{pmatrix} \in \mathrm{Ker}(\tau_{AB}).$$

Then any element $Z \in \mathrm{Ker}(\tau_{AB})$ can be written as

$$Z = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} Z_0 \quad \text{for some } \alpha, \beta \in R,$$

so that $\mathrm{Ker}(\tau_{AB})$ is a *left free* $\mathrm{C}(B)$*-module of rank 1* generated by Z_0.

Since Z_0 is not invertible in R_2 (its determinant is $-x^2 \notin U(R)$), Theorem 3 guarantees that A and B are not similar over R.

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Data Availability

No data beyond that presented in this work are required.

Conflicts of Interest

The author declares no conflict of interest.

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