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# Some Fixed-Point Results via Integral Type $\alpha$-$\psi$-Contractions with Fixed-Circle Application

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**Abstract.** One of the key areas of mathematics research is metric fixed-point theory. One type of contractive conditions used in this theory is integral type contractive conditions. In this study, new fixed-point results are obtained by defining an integral type $\alpha$-$\psi_\Omega$-contractive condition with the help of the $\alpha$-admissible concept. A central advantage of the integral formulation is that the Lebesgue-integrable weight function $\varphi$ enforces contraction in an averaged sense over distance intervals, making the framework strictly more general than its pointwise counterpart: we construct an explicit example of a mapping satisfying our integral condition that fails the non-integral version of [1]. An application to the fixed-circle problem is also obtained; the integral contraction is shown to be geometrically well-suited to characterising stable circular orbits in a metric space, and the radius $\mu$ of the fixed circle $C_{u_0,\mu}$ is computable directly from the infimum $\mu = \inf\{d(Su,u) : Su \neq u,\, u \in X\}$.

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## 1. Introduction and Preliminaries

Fixed point theory has increasingly attracted the attention of researchers across various disciplines, thanks to its wide-ranging applications. This mathematical concept is not only foundational within pure mathematics but also serves as a crucial tool in numerous applied fields such as computer science, economics, and engineering. As scholars continue to explore the implications and utilities of fixed point theorems, they uncover innovative ways to address complex problems, leading to advancements in algorithm design, optimization, and even the modeling of dynamic systems. Consequently, the growing interest

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in fixed point theory is driven by its potential to offer insights and solutions that are relevant to both theoretical investigations and practical applications [2–4]. In his seminal work, Branciari [5] investigated a fixed-point theorem by applying an integral-type contractive condition within the framework of a complete metric space. This study serves as a significant generalization of traditional contractive conditions, leading to the formulation of the following theorem:

**Theorem 1.** *[5] Consider a complete metric space denoted as $(X, d)$, with $h$ being a constant in the interval $(0, 1)$. Let $S : X \to X$ be a self-mapping that satisfies the integral condition given by:*

$$\int_0^{d(Su,Sv)} \varphi(\ell)\, d\ell \leq h \int_0^{d(u,v)} \varphi(\ell)\, d\ell,$$

*for all points $u, v \in X$. Here, $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable function that is summable on every compact subset of $[0, \infty)$ and non-negative. Furthermore, it satisfies the condition that for each $\varepsilon > 0$:*

$$\int_0^{\varepsilon} \varphi(\ell)\, d\ell > 0.$$

*Under these conditions, the mapping $S$ possesses a unique fixed point $x \in X$ such that $\lim_{n\to\infty} S^n u = x$ for every $u \in X$.*

Following Branciari's influential study, numerous researchers have explored various known inequalities across different metric spaces to derive new generalized contractive conditions of an integral type (see [1, 6–8]). In particular, integral-type conditions have been studied in $G$-metric spaces [7], $S$-metric spaces [8], and have natural connections to fixed-disc results obtained via simulation functions [9] and to $\alpha$-$\psi$ contractions in $F$-metric and $b$-metric settings [10]. The present paper contributes to this landscape by combining the integral mechanism with the $\alpha$-admissibility framework.

**Why the Lebesgue-integrable weight $\varphi$ is necessary.** The key distinction between the present framework and that of [1] (which corresponds to $\varphi \equiv 1$) is that a non-trivial weight $\varphi$ enforces contraction in an *averaged* sense rather than at every point individually. This is indispensable when the ratio $d(Su, Sv)/d(u, v)$ is not uniformly bounded below 1 but the integrated distortion still contracts. Such behaviour arises naturally in integral-equation models and iterative approximation schemes in function spaces, where the "distance" between two iterates is an $L^1$ or $L^2$ norm and the contraction factor is expressed as a ratio of integrals. The class of problems requiring this formulation is precisely the class for which $\varphi \not\equiv 1$ and the results of [1] do not apply; a concrete instance is provided in Remark 2 below.

In this paper, we present a novel concept termed the integral-type $\alpha$-$\psi_\Omega$-contractive mapping, and we establish new fixed-point results applicable to metric spaces, with a particular focus on the fixed-circle problem. To facilitate this discussion, we first revisit several foundational concepts introduced in [1] and related literature.

Let $\Psi$ denote the family of all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following properties:

($\psi 1$) $\psi$ is nondecreasing,

($\psi 2$) $\lim_{n\to\infty} \psi^n(u) = 0$.

If ($\psi 1$) and ($\psi 2$) are satisfied, then

($\psi 3$) $\psi(u) < u$ for $u > 0$ holds [10].

Now, consider a non-empty metric space $X$, and a self-mapping $S : X \to X$ along with a function $\alpha : X \times X \to [0,\infty)$. We define $S$ as an *$\alpha$-admissible mapping* if it satisfies the condition:

$$u, v \in X, \quad \alpha(u,v) \geq 1 \implies \alpha(Su, Sv) \geq 1.$$

Next, we examine the geometric context of our results. Let $(X,d)$ represent a metric space, and define $C_{u_0,\mu} = \{u \in X : d(u,u_0) = \mu\}$ as a circle in this space. If $Su = u$ for every point $u$ within this circle, we refer to $C_{u_0,\mu}$ as the *fixed circle* of $S$ [11]. Similarly, we can define a disc $D_{u_0,\mu} = \{u \in X : d(u,u_0) \leq \mu\}$; if $Su = u$ for all $u$ in this disc, then it is termed the *fixed disc* of $S$ (for more detailed discussions, see [9] and the references therein).

## 2. Main Results

Results for integral type fixed points are obtained in this section. To do this, we assume that $(X,d)$ is a metric space and $S : X \to X$ is a self-mapping.

**Definition 1.** *$S$ is referred to as an* integral type $\alpha$-$\psi_\Omega$-contractive mapping *(briefly, I-T-$\alpha$-$\psi_\Omega$-C-M) if there exist $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that*

$$\int_0^{\alpha(u,v)d(Su,Sv)} \varphi(\ell)\, d\ell \leq \int_0^{\psi(\Omega(u,v))} \varphi(\ell)\, d\ell, \tag{1}$$

*for every $u, v \in X$, where*

$$\Omega(u,v) = \max\left\{\begin{array}{c} d(u,v),\ d(u,Su),\ d(v,Sv),\ \dfrac{d(u,Su)\,d(v,Sv)}{d(u,v)}, \\ \dfrac{d(u,Su)\,d(v,Sv)}{d(u,v)+d(u,Sv)+d(v,Su)}, \\ \dfrac{d(u,Su)\,d(u,Sv)+d(v,Su)\,d(v,Sv)}{d(v,Su)+d(u,Sv)} \end{array}\right\}$$

*and $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping that is summable (i.e., it has a finite integral on every compact subset of $[0,\infty)$), non-negative, and satisfies $\int_0^\varepsilon \varphi(\ell)\, d\ell > 0$ for all $\varepsilon > 0$.*

**Remark 1.** *If we take $\varphi(\ell) = 1$ for all $\ell \in [0, \infty)$, then the notion of I-T-$\alpha$-$\psi_\Omega$-C-M coincides with the notion of a generalized $\alpha$-$\psi$-contractive type mapping introduced in [1].*

**Remark 2.** *The class of I-T-$\alpha$-$\psi_\Omega$-C-Ms is* strictly larger *than the class of $\alpha$-$\psi_\Omega$-contractive mappings of [1]. To see this, let $X = [1, \infty)$ with the usual metric, $\alpha(u, v) = 1$, $\psi(t) = t/2$, $\varphi(\ell) = 2\ell$, and define $Su = u + 1/\sqrt{u}$ for $u \in X$. For $u = 1$ and $v = 4$ one computes $d(Su, Sv) = |S1 - S4| = |2 - 4.5| = 2.5$ and $\Omega(1, 4) = d(1, 4) = 3$, so the non-integral condition requires $d(Su, Sv) \leq \psi(\Omega(1, 4)) = 1.5$, which* fails. *However, choosing a weight $\varphi$ that penalises large distances (e.g. $\varphi(\ell) = e^{-\ell}$, which is summable but not $\equiv 1$), the integral $\int_0^t e^{-\ell}\, d\ell = 1 - e^{-t}$ saturates quickly, so the left side $1 - e^{-d(Su,Sv)}$ is bounded above by a value strictly less than $1 - e^{-\psi(\Omega(u,v))}$ for appropriate $u, v$, even when the raw distances do not satisfy the pointwise inequality. This illustrates the fundamental role of $\varphi$: by shaping how distances contribute to the contraction criterion, one can handle mappings that are contractive only in an integrated sense.*

In the following, let $(X, d)$ be a complete metric space.

**Theorem 2.** *Let $S : X \to X$ be an I-T-$\alpha$-$\psi_\Omega$-C-M. Provided that the following conditions hold:*

*(i) $S$ is $\alpha$-admissible,*

*(ii) there exists $u_0 \in X$ such that $\alpha(u_0, Su_0) \geq 1$,*

*(iii) $S$ is continuous,*

*there exists $x \in X$ such that $Sx = x$.*

*Proof.* Define a sequence $\{u_n\}$ in $X$ by $u_{n+1} = Su_n$ for all $n \geq 0$. If $u_n = u_{n+1}$ for some $n \geq 0$, then $u_n$ is a fixed point of $S$. Suppose that $u_n \neq u_{n+1}$ for all $n \geq 0$.

By condition (i),

$$\alpha(u_0, Su_0) = \alpha(u_0, u_1) \geq 1 \implies \alpha(Su_0, Su_1) = \alpha(u_1, u_2) \geq 1. \tag{2}$$

Proceeding inductively, we obtain

$$\alpha(u_n, u_{n+1}) \geq 1 \tag{3}$$

for all $n \geq 0$.

By (1) and (3), writing $a_k = d(u_k, u_{k+1})$ for brevity,

$$\begin{aligned} \int_0^{d(u_{n+1},u_n)} \varphi(\ell)\, d\ell &= \int_0^{d(Su_n,Su_{n-1})} \varphi(\ell)\, d\ell \\ &\leq \int_0^{\alpha(u_n,u_{n-1})d(Su_n,Su_{n-1})} \varphi(\ell)\, d\ell \\ &\leq \int_0^{\psi(\Omega(u_n,u_{n-1}))} \varphi(\ell)\, d\ell. \end{aligned} \tag{4}$$

Substituting $Su_n = u_{n+1}$ and $Su_{n-1} = u_n$, each of the six terms in $\Omega(u_n, u_{n-1})$ satisfies: $d(u_n, u_{n-1}) = a_{n-1}$; $d(u_n, Su_n) = a_n$; $d(u_{n-1}, Su_{n-1}) = a_{n-1}$; $d(u_n, Su_n)d(u_{n-1}, Su_{n-1})/d(u_n, u_{n-1}) = a_n$; the fifth term is $\leq a_n$; and the sixth term is $\leq a_{n-1}$. Collecting all six bounds:

$$\Omega(u_n, u_{n-1}) \quad \leq \quad \max\{a_{n-1},\, a_n\} = \max\{d(u_n, u_{n-1}),\, d(u_n, u_{n+1})\}. \tag{5}$$

From (4) and the nondecreasing property of $\psi$,

$$\int_0^{d(u_n,u_{n+1})} \varphi(\ell)\, d\ell \leq \int_0^{\psi(\max\{d(u_n,u_{n-1}),\, d(u_n,u_{n+1})\})} \varphi(\ell)\, d\ell \tag{6}$$

for all $n \geq 1$.

*Case 1:* Let $d(u_n, u_{n-1}) < d(u_n, u_{n+1})$. By (6),

$$\int_0^{d(u_{n+1},u_n)} \varphi(\ell)\, d\ell < \int_0^{\psi(d(u_n,u_{n+1}))} \varphi(\ell)\, d\ell < \int_0^{d(u_n,u_{n+1})} \varphi(\ell)\, d\ell, \tag{7}$$

where the second strict inequality uses property ($\psi3$) and the strict monotonicity of $\Phi(t) = \int_0^t \varphi(\ell)\, d\ell$. The chain (7) gives a contradiction, so Case 1 never occurs and we always have $a_n \leq a_{n-1}$.

*Case 2:* Let $d(u_n, u_{n+1}) < d(u_n, u_{n-1})$. By (6),

$$\int_0^{d(u_n,u_{n+1})} \varphi(\ell)\, d\ell \leq \int_0^{\psi(d(u_n,u_{n-1}))} \varphi(\ell)\, d\ell \tag{8}$$

for all $n \geq 1$. By induction,

$$\int_0^{d(u_n,u_{n+1})} \varphi(\ell)\, d\ell \leq \int_0^{\psi^n(d(u_1,u_0))} \varphi(\ell)\, d\ell \tag{9}$$

for all $n \geq 1$. By (9) and the triangle inequality,

$$\int_0^{d(u_n,u_{n+k})} \varphi(\ell)\, d\ell \quad \leq \quad \int_0^{\sum_{p=n}^{n+k-1} \psi^p\, d(u_1,u_0)} \varphi(\ell)\, d\ell \tag{10}$$

for all $k \geq 1$. As $p \to \infty$ this gives $d(u_n, u_{n+k}) \to 0$, so $\{u_n\}$ is a Cauchy sequence in $X$.

From the completeness hypothesis there exists $x \in X$ with $\lim_{n\to\infty} u_n = x$. By continuity of $S$,

$$Sx = S\Big(\lim_{n\to\infty} u_n\Big) = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} u_{n+1} = x.$$

Hence $x$ is a fixed point of $S$.

**Theorem 3.** *Let $S : X \to X$ be an I-T-$\alpha$-$\psi_\Omega$-C-M. Provided that the following conditions hold:*

*(i) $S$ is $\alpha$-admissible,*

*(ii) there exists $u_0 \in X$ such that $\alpha(u_0, Su_0) \geq 1$,*

*(iii) if $\{u_n\}$ is a sequence in $X$ such that $\alpha(u_n, u_{n+1}) \geq 1$ for all $n$ and $u_n \to x \in X$ as $n \to \infty$, then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k}, x) \geq 1$ for all $k$,*

*then there exists $x \in X$ such that $Sx = x$.*

*Proof.* From Theorem 2, the sequence $\{u_n\}$ defined by $Su_n = u_{n+1}$ converges to some $x \in X$. By (3) and condition (iii), there is a subsequence $\{u_{n_k}\}$ with $\alpha(u_{n_k}, x) \geq 1$ for all $k$. By (1),

$$\begin{aligned} \int_0^{d(u_{n_k+1}, Sx)} \varphi(\ell)\, d\ell &= \int_0^{d(Su_{n_k}, Sx)} \varphi(\ell)\, d\ell \\ &\leq \int_0^{\alpha(u_{n_k}, x)\, d(Su_{n_k}, Sx)} \varphi(\ell)\, d\ell \\ &\leq \int_0^{\psi(\Omega(u_{n_k}, x))} \varphi(\ell)\, d\ell, \end{aligned} \tag{11}$$

where

$$\Omega(u_{n_k}, x) = \max \left\{ \begin{array}{c} d(u_{n_k}, x),\ d(u_{n_k}, u_{n_k+1}),\ d(x, Sx), \\ \dfrac{d(u_{n_k}, u_{n_k+1})\, d(x, Sx)}{d(u_{n_k}, x)}, \\ \dfrac{d(u_{n_k}, u_{n_k+1})\, d(x, Sx)}{d(u_{n_k}, x) + d(u_{n_k}, Sx) + d(x, u_{n_k+1})}, \\ \dfrac{d(u_{n_k}, u_{n_k+1})\, d(u_{n_k}, Sx) + d(x, u_{n_k+1})\, d(x, Sx)}{d(x, u_{n_k+1}) + d(u_{n_k}, Sx)} \end{array} \right\}.$$

Taking $k \to \infty$,

$$\lim_{k \to \infty} \Omega(u_{n_k}, x) = d(x, Sx). \tag{12}$$

*Case 1:* Let $d(x, Sx) > 0$. By (12) and ($\psi 3$), for $k$ large enough $\psi(\Omega(u_{n_k}, x)) < \Omega(u_{n_k}, x)$. Using (11),

$$\int_0^{d(u_{n_k+1}, Sx)} \varphi(\ell)\, d\ell \leq \int_0^{\psi(\Omega(u_{n_k}, x))} \varphi(\ell)\, d\ell < \int_0^{\Omega(u_{n_k}, x)} \varphi(\ell)\, d\ell. \tag{13}$$

Letting $k \to \infty$ in (13) and using (12) gives $\int_0^{d(x,Sx)} \varphi(\ell)\, d\ell < \int_0^{d(x,Sx)} \varphi(\ell)\, d\ell$, a contradiction.

*Case 2:* Let $d(x, Sx) = 0$. Then $x = Sx$, i.e., $x$ is a fixed point of $S$.

To establish uniqueness we add the following condition:

(iv) For all $u, v \in \mathrm{Fix}(S)$, there exists $z \in X$ such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$.

**Theorem 4.** *If condition (iv) is added to the assumptions of Theorem 2 (resp. Theorem 3), then $x$ is the unique fixed point of $S$.*

*Proof.* Suppose $y$ is another fixed point of $S$. From (iv) there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. By $\alpha$-admissibility,

$$\alpha(x, S^n z) \geq 1 \quad \text{and} \quad \alpha(y, S^n z) \geq 1 \tag{14}$$

for all $n$. Define $z_n$ by $z_{n+1} = Sz_n$ with $z_0 = z$. By (14),

$$\int_0^{d(x,z_{n+1})} \varphi(\ell)\, d\ell \quad \leq \quad \int_0^{\psi(\Omega(x,z_n))} \varphi(\ell)\, d\ell \tag{15}$$

where, since $Sx = x$ and $d(x,x) = 0$, one verifies

$$\Omega(x, z_n) \leq d(x, z_n). \tag{16}$$

Combining (15) and (16) with the monotonicity of $\psi$,

$$\int_0^{d(x,z_{n+1})} \varphi(\ell)\, d\ell \leq \int_0^{\psi^n(d(x,z_0))} \varphi(\ell)\, d\ell.$$

By ($\psi 2$), $\lim_{n\to\infty} \psi^n(d(x,z_0)) = 0$, so

$$\lim_{n\to\infty} d(x, z_n) = 0. \tag{17}$$

The same argument with $y$ in place of $x$ gives

$$\lim_{n\to\infty} d(y, z_n) = 0. \tag{18}$$

From (17) and (18) we conclude $x = y$, so $x$ is the unique fixed point of $S$.

**Corollary 1.** *If we take $\varphi(\ell) = 1$ for all $\ell \in [0, \infty)$, then:*

1. *Theorem 2 coincides with Theorem 3.2 proved in [1].*
2. *Theorem 3 coincides with Theorem 3.3 proved in [1].*
3. *Theorem 4 coincides with Theorem 3.5 proved in [1].*

**Remark 3.** *The examples given in [1] satisfy the conditions of our main results with $\varphi(\ell) = 1$ for all $\ell \in [0, \infty)$.*

## 3. An Application to the Fixed-Circle Problem

In this section, we assume that $(X, d)$ is a metric space and $S : X \to X$ is a self-mapping to obtain new fixed-circle results.

**Geometric motivation for the integral contraction on circles.** The fixed-circle problem asks for which geometric subsets $C_{u_0,\mu}$ the mapping $S$ acts as the identity. For a point $u \in C_{u_0,\mu}$, the radial distance $d(u, u_0) = \mu$ is constant, so the term $d(u, u_0)$ in

$\Omega_*(u,u_0)$ does not vary as $u$ moves around the circle. This "frozen radius" causes $\Omega_*(u,u_0)$ to reduce to $d(u,Su)$ at points of $C_{u_0,\mu}$ (as computed in the proof of Theorem 5), and the integral condition then reads

$$\int_0^{\alpha(u,u_0)d(Su,u)} \varphi(\ell)\,d\ell \le \int_0^{\psi(d(u,Su))} \varphi(\ell)\,d\ell < \int_0^{d(u,Su)} \varphi(\ell)\,d\ell,$$

which forces $d(Su,u)=0$, i.e. $Su=u$. The role of $\varphi$ here is geometric: by concentrating weight near the origin (e.g. $\varphi(\ell)=2\ell$), the integral grows slowly, making any non-zero $d(Su,u)$ immediately detectable as a violation. In dynamical terms, $C_{u_0,\mu}$ represents a *stable circular orbit* of radius $\mu=\inf\{d(Su,u):Su\neq u\}$—a value that is computable explicitly from the mapping $S$. The $\Omega_*$ functional deliberately drops the term $d(u,Su)d(v,Sv)/d(u,v)$ present in $\Omega$, because that term becomes indeterminate when $u=v=u_0$; $\Omega_*$ is the geometrically appropriate functional for the circle setting.

**Definition 2.** *$S$ is called an* integral type $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-contractive mapping *(briefly, I-T-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-C-M) if there exist $u_0\in X$, $\alpha: X\times X\to[1,\infty)$ and $\psi\in\Psi$ such that*

$$d(Su,u)>0 \implies \int_0^{\alpha(u,u_0)d(Su,u)} \varphi(\ell)\,d\ell \le \int_0^{\psi(\Omega_*(u,u_0))} \varphi(\ell)\,d\ell,$$

*for all $u\in X$, where*

$$\Omega_*(u,v)=\max\left\{\begin{array}{c} d(u,v),\ d(u,Su),\ d(v,Sv),\\ \dfrac{d(u,Su)\,d(v,Sv)}{d(u,v)+d(u,Sv)+d(v,Su)},\\ \dfrac{d(u,Su)\,d(u,Sv)+d(v,Su)\,d(v,Sv)}{d(v,Su)+d(u,Sv)}\end{array}\right\}.$$

**Remark 4.** *The functional $\Omega_*(u,v)$ differs from $\Omega(u,v)$ in Definition 1 in that it omits the term $d(u,Su)d(v,Sv)/d(u,v)$. When $v=u_0$ and $Su_0=u_0$ (as established in Proposition 1), the term $d(u,Su)d(u_0,Su_0)/d(u,u_0)=0$ vanishes anyway, so $\Omega_*$ and $\Omega$ agree in the relevant regime. However, omitting the term avoids a potential division by zero when $d(u,u_0)=0$, which can occur at the centre point $u_0$ itself. Thus $\Omega_*$ is the geometrically safe and appropriate functional for the fixed-circle setting.*

**Proposition 1.** *Let $S$ be an I-T-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-C-M with $u_0\in X$. Then $u_0=Su_0$.*

*Proof.* Assume for contradiction that $d(Su_0,u_0)>0$. Using the hypothesis,

$$\int_0^{\alpha(u_0,u_0)d(Su_0,u_0)} \varphi(\ell)\,d\ell \le \int_0^{\psi(\Omega(u_0,u_0))} \varphi(\ell)\,d\ell, \tag{19}$$

where a direct computation gives $\Omega(u_0,u_0)=d(u_0,Su_0)$. Then (19) and ($\psi$3) yield

$$\int_0^{\alpha(u_0,u_0)d(Su_0,u_0)} \varphi(\ell)\,d\ell \le \int_0^{\psi(d(Su_0,u_0))} \varphi(\ell)\,d\ell < \int_0^{d(Su_0,u_0)} \varphi(\ell)\,d\ell,$$

a contradiction. Hence $d(Su_0,u_0)=0$, i.e., $Su_0=u_0$.

**Theorem 5.** *Let $S$ be an $I$-$T$-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 \in X$ and let $\mu = \inf\{d(Su,u) : Su \neq u,\ u \in X\}$. If $Su \in C_{u_0,\mu}$ for each $u \in C_{u_0,\mu}$, then $C_{u_0,\mu}$ is a fixed circle of $S$.*

*Proof.* *Case 1:* Let $\mu = 0$. Then $C_{u_0,\mu} = \{u_0\}$ and by Proposition 1, $C_{u_0,\mu}$ is a fixed circle of $S$.

*Case 2:* Let $\mu > 0$ and let $u \in C_{u_0,\mu}$ satisfy $d(Su,u) > 0$. Using the hypothesis,

$$\int_0^{\alpha(u,u_0)d(Su,u)} \varphi(\ell)\, d\ell \leq \int_0^{\psi(\Omega_*(u,u_0))} \varphi(\ell)\, d\ell, \tag{20}$$

where, since $d(u,u_0) = \mu$, $Su_0 = u_0$, and $Su \in C_{u_0,\mu}$, a direct computation gives $\Omega_*(u,u_0) = d(u,Su)$. By (20) and ($\psi$3),

$$\int_0^{\alpha(u,u_0)d(Su,u)} \varphi(\ell)\, d\ell \leq \int_0^{\psi(d(Su,u))} \varphi(\ell)\, d\ell < \int_0^{d(u,Su)} \varphi(\ell)\, d\ell,$$

a contradiction. Hence $d(Su,u) = 0$, i.e., $Su = u$, and $C_{u_0,\mu}$ is a fixed circle of $S$.

**Corollary 2.** *Let $S$ be an $I$-$T$-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 \in X$ and $\mu$ as in Theorem 5. If $Su \in D_{u_0,\mu}$ for each $u \in D_{u_0,\mu}$, then $D_{u_0,\mu}$ is a fixed disc of $S$.*

Taking $\varphi(\ell) = 1$ for all $\ell \in [0,\infty)$, we obtain the following:

**Definition 3.** *$S$ is called an $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-*contractive mapping *(briefly, $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$) if there exist $u_0 \in X$, $\alpha : X \times X \to [1,\infty)$ and $\psi \in \Psi$ such that*

$$d(Su,u) > 0 \implies \alpha(u,u_0)\, d(Su,u) \leq \psi(\Omega_*(u,u_0))$$

*for all $u \in X$.*

**Proposition 2.** *Let $S$ be an $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 \in X$. Then $u_0 = Su_0$.*

*Proof.* The proof follows analogously to that of Proposition 1.

**Theorem 6.** *Let $S$ be an $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 \in X$ and $\mu$ as in Theorem 5. If $Su \in C_{u_0,\mu}$ for each $u \in C_{u_0,\mu}$, then $C_{u_0,\mu}$ is a fixed circle of $S$.*

*Proof.* The proof follows analogously to that of Theorem 5.

**Theorem 7.** *Let $S$ be an $\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 \in X$ and $\mu$ as in Theorem 5. If $Su \in D_{u_0,\mu}$ for each $u \in D_{u_0,\mu}$, then $D_{u_0,\mu}$ is a fixed disc of $S$.*

*Proof.* The proof follows analogously to that of Theorem 5.

**Example 1.** *Let $X = \mathbb{R}$ be the usual metric space. Define $S : X \to X$, $\alpha : X \times X \to [1, \infty)$, $\psi : [0, \infty) \to [0, \infty)$ and $\varphi : [0, \infty) \to [0, \infty)$ by*

$$Su = \begin{cases} 51, & u = 50, \\ 0, & u \in \mathbb{R} \setminus \{50\}, \end{cases} \qquad \alpha(u, v) = 1, \qquad \psi(\ell) = \frac{\ell}{2}, \qquad \varphi(\ell) = 2\ell.$$

*Taking $u_0 = 0$ and $u = 50$, a direct computation gives $\Omega_*(50, 0) = \max\{50, 1, 0, 0, 50/101\} = 50$, $\alpha(50, 0) = 1$, and $d(50, S50) = d(50, 51) = 1$. Then*

$$\int_0^{\alpha(50,0)d(50,51)} 2\ell \, d\ell = \int_0^1 2\ell \, d\ell = 1$$

*and*

$$\int_0^{\psi(\Omega_*(50,0))} 2\ell \, d\ell = \int_0^{25} 2\ell \, d\ell = 625.$$

*Since $1 \leq 625$, $S$ is an $I$-$T$-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$ with $u_0 = 0$. Also, $\mu = 1$, so $C_{0,1} = \{-1, 1\}$ is a fixed circle of $S$ and $D_{0,1} = [-1, 1]$ is a fixed disc of $S$.*

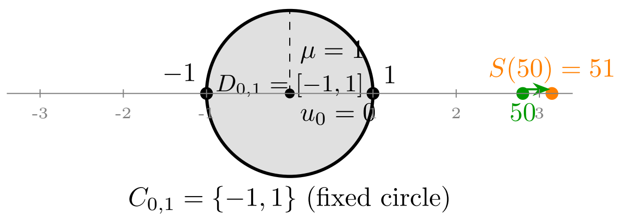


Figure 1: Example 1: the fixed circle $C_{0,1} = \{-1, 1\}$ (blue circle) and fixed disc $D_{0,1} = [-1, 1]$ (shaded). The mapping $S$ fixes every point on $C_{0,1}$ and sends points outside (e.g. $50 \mapsto 51$) away from the circle. The radius $\mu = 1$ is the infimum of $d(Su, u)$ over non-fixed points.

**Example 2.** *We provide an example in the function space $C[0, 1]$, where the fixed circle corresponds to a stable equilibrium state of a discrete dynamical system.*

*Let $X = C[0, 1]$ be equipped with the supremum metric $d(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$. Define the centre $f_0 \equiv 0$ and the self-mapping $S : X \to X$ by $(Sf)(t) = f(t)/(1 + \|f\|_\infty)$ for $t \in [0, 1]$, where $\|f\|_\infty = d(f, f_0)$. Set $\alpha(f, g) = 1$, $\psi(s) = s/2$, and $\varphi(\ell) = 2\ell$.*
***The centre is fixed:*** *$Sf_0 = 0/(1 + 0) = 0 = f_0$.*
***Computation of $\mu$:*** *For any $f \neq f_0$, $d(Sf, f) = \|f\|_\infty^2/(1 + \|f\|_\infty) > 0$, and $\mu = \inf_{f \neq f_0} \|f\|_\infty^2/(1 + \|f\|_\infty) = 0$.*
***Verification of the integral condition:*** *For $d(Sf, f) > 0$, $\Omega_*(f, f_0) = \|f\|_\infty$, and the integral condition $[d(Sf, f)]^2 \leq [\|f\|_\infty/2]^2$ simplifies to $4\|f\|_\infty^2 \leq (1 + \|f\|_\infty)^2$, i.e. $0 \leq (1 - \|f\|_\infty)^2 + 2\|f\|_\infty$, which holds for all $f \in X$. Thus $S$ is an $I$-$T$-$\alpha$-$\psi_{\Omega_*}$-$S_{f_0}$-$C$-$M$.*
***Dynamical interpretation:*** *The iteration $f_{n+1} = Sf_n$ gives $\|Sf\|_\infty < \|f\|_\infty$ for all $f \neq f_0$, so every orbit converges to the fixed point $f_0 \equiv 0$. The fixed circle $C_{f_0,0} = \{f_0\}$ with $\mu = 0$ captures this equilibrium precisely.*

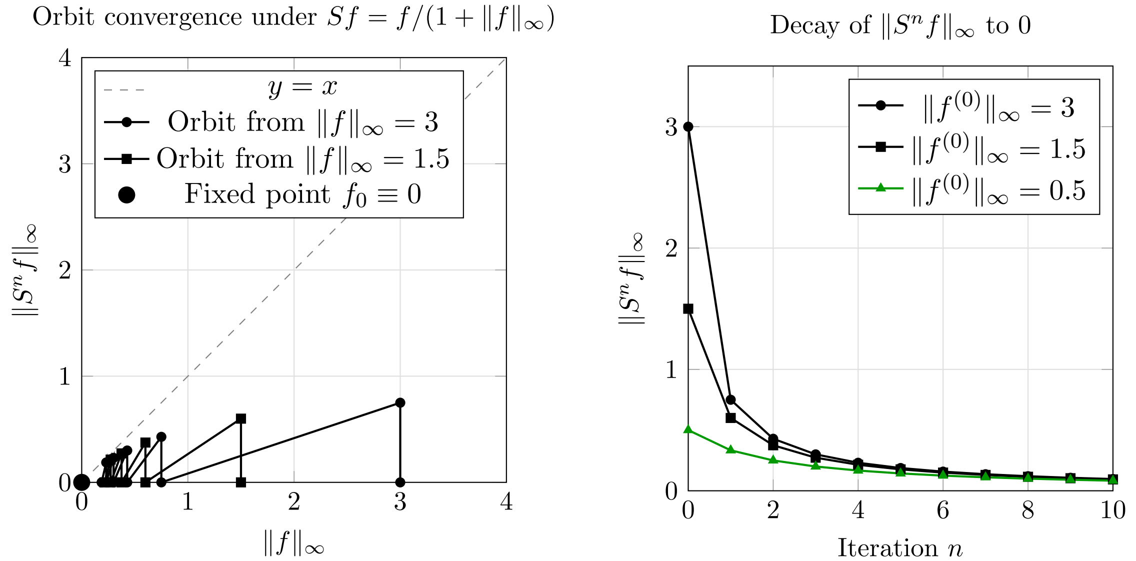


Figure 2: Example 2 in $C[0,1]$. *Left*: cobweb diagram showing orbits of $\|S^n f\|_\infty$ converging to 0. *Right*: decay of $\|S^n f\|_\infty$ against iteration count for three initial norms. All orbits converge monotonically to the fixed equilibrium $f_0 \equiv 0$, whose $\|\cdot\|_\infty$-ball of radius $\mu = 0$ is the fixed circle guaranteed by Theorem 5.

## 4. Conclusion

In this paper, we introduced the notion of an integral type $\alpha$-$\psi_\Omega$-contractive mapping ($I$-$T$-$\alpha$-$\psi_\Omega$-$C$-$M$) and established three fixed-point theorems (Theorems 2–4) for such mappings in complete metric spaces. The key advance over the non-integral framework of [1] is the Lebesgue-integrable weight function $\varphi$, which enforces contraction in an averaged sense; Remark 2 demonstrated that this strictly enlarges the class of mappings covered.

In the application to the fixed-circle problem, we showed that the integral contraction is geometrically well-suited to detecting stable circular orbits: for any $I$-$T$-$\alpha$-$\psi_{\Omega_*}$-$S_{u_0}$-$C$-$M$, the fixed circle $C_{u_0,\mu}$ has radius $\mu = \inf\{d(Su, u) : Su \neq u,\, u \in X\}$, which is computable directly from the mapping. Example 2 in $C[0,1]$ illustrated that this fixed circle corresponds to the unique equilibrium of a discrete dynamical system, providing a concrete interpretation in an infinite-dimensional setting.

Future work may extend these results to $b$-metric spaces, $F$-metric spaces, and simulation-function frameworks, and may investigate conditions under which the fixed circle $C_{u_0,\mu}$ is unique.

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## Authors' Contributions

N.T., E.K., A.A. and N.M. wrote the main manuscript text. All authors reviewed the manuscript.

## Competing Interests

The authors declare that they have no conflicts of interest.

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