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Cayley Graphs of Transformation Semigroups with Restriction on the Fixed Set Is Bijective

Natthawarun Pinin[1], Ekkachai Laysirikul[1,*], , Yanisa Chaiya[2,3]

[1] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand*
[2] *Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University (Rangsit Campus), Pathum Thani, 12120, Thailand*
[3] *Thammasat University Research Unit in Algebra and Its Applications, Thammasat University (Rangsit Campus), Pathum Thani 12120, Thailand*

Abstract. Let $PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\}$ be a subsemigroup of the full transformation semigroup $T(X)$, where the restriction of each transformation to Y is a permutation of Y. This structure can be viewed as an extension of the symmetric group to a transformation semigroup. In this paper, we study the Cayley digraphs $\mathrm{Cay}(PG_Y(X), A)$ generated by a nonempty set A of minimal idempotents. We describe basic structural properties of these digraphs, including initial and terminal vertices. We also examine their degrees, connectedness, and completeness. In addition, domination parameters of these digraphs are determined.

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1. Introduction

The notion of a Cayley graph for a finite group originates in 1878, when Arthur Cayley introduced it as a graphical method to reveal the internal structure of groups. The idea has since been generalized to other algebraic systems, in particular, to semigroups. Let S be a semigroup and $A \subseteq S$. We define the *Cayley digraph* $\mathrm{Cay}(S, A)$ *of* S *with respect to* A to be the directed graph whose vertices are the elements of S, and whose arcs are precisely the ordered pairs (x, xa) for $x \in S$ and $a \in A$. In this context, A is called a *connection set* of $\mathrm{Cay}(S, A)$. The Cayley digraphs of semigroups have been extensively investigated, see, for example [1–7].

*Corresponding Author.
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Email addresses: natthawarunp66@nu.ac.th (N. Pinin),
ekkachail@nu.ac.th (E. Laysirikul), yanisa@mathstat.sci.tu.ac.th (Y. Chaiya)

Among the various types of semigroups, transformation semigroups have drawn considerable interest. One reason for this is that transformation semigroups serve as a semigroup analogue of Cayley's classical theorem for groups, shedding light on their structural significance. Many authors have investigated structural and combinatorial aspects of Cayley graphs arising from transformation semigroups. For example, Tisklang and Panma [8] investigated the connectedness properties of Cayley graphs of finite transformation semigroups with restricted ranges. In another direction, Riyas and Geetha [9] analyzed Cayley graphs of full transformation semigroups generated by idempotents and established the presence of Hamiltonian cycles. Their later work [10] explored the Cayley graph of the symmetric inverse semigroup with respect to Green's $\mathcal{R}$-relation. Further developments include the work of Nupo and Pookpienlert [11], who studied conditions for connectedness and completeness of Cayley digraphs of transformation semigroups determined by fixed sets. More recently, Tisklang and Panma [12] described various characterizations of Cayley graphs of finite transformation semigroups with restricted ranges. Additionally, Riyas, Anusha, and Geetha [13] investigated Cayley graphs of full transformation semigroups with respect to Green's $\mathcal{L}$-class. In addition, Nupo and Pookpienlert [14] studied domination parameters of Cayley digraphs of the semigroups of transformations with fixed set. They provided explicit formulas and structural descriptions for several domination-related parameters, including the domination number and independent domination number, thereby extending the combinatorial analysis of Cayley digraphs of semigroups of transformation with fixed set. Recently, Nupo and Chaiya [15] analyzed Cayley digraphs of the full transformation semigroup with connection sets corresponding to Green's equivalence classes. They provided structural results and isomorphism conditions, showing how the algebraic partition of is reflected in the digraphs. This extends the study of Cayley digraphs of transformation semigroups to a broader class with detailed structural insights.

Let X be a nonempty set, and let $T(X)$ denote the full transformation semigroup on X. For a nonempty subset $Y \subseteq X$, the subsemigroup $PG_Y(X)$ of $T(X)$ is defined by

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\},$$

where $G(Y)$ denotes the permutation group on Y. Since $G(Y)$, consisting of all permutations on Y, is isomorphic to a subsemigroup of $PG_Y(X)$, it can be regarded as a substructure of $PG_Y(X)$. Consequently, properties of $PG_Y(X)$ may also provide insights into $G(Y)$. This semigroup was first introduced by Laysirikul [16] in 2016, who proved that $PG_Y(X)$ is regular and is an inverse semigroup if and only if $|X| \leq 2$ or $Y = X$. He further described the left regular, right regular, and completely regular elements of $PG_Y(X)$ and explored its relationship with other transformation semigroups. In 2021, Sommanee [17] studied Green's relations in $PG_Y(X)$, characterized its ideals, and established several isomorphism theorems. For finite sets, he also investigated the cardinalities of $PG_Y(X)$ and its subsets of idempotents, and computed their ranks. In 2022, Tantong and Sawatraksa [18] characterized the left regular, right regular, and completely regular elements of a variant of $PG_Y(X)$. Recently, in 2024, Chaiya [19] described all intra-regular and unit-regular elements of $PG_Y(X)$ and provided necessary and sufficient conditions for $PG_Y(X)$ to be intra-regular and unit-regular. The author also determined the number of these elements when X is finite, classified the maximal subsemigroups of $PG_Y(X)$, and proved that these maximal subsemigroups coincide with the maximal regular subsemigroups of $PG_Y(X)$ when $X \setminus Y$ is finite.

The purpose of this paper is to explore structural characteristics of Cayley digraphs arising from the semigroup $PG_Y(X)$ relative to their minimal idempotent.

2. Preliminaries

In this section, we introduce several basic definitions and notation that will be used throughout the paper. Unless specified otherwise, all sets considered here are finite. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. An ordered pair $(u, v) \in E(D)$ represents a directed edge from u to v. For convenience, we sometimes say that u is *adjacent* to v. The vertex u is regarded as the *initial* vertex of the arc, while v is its *terminal* vertex. Any arc of the form (u, u) is called a *loop*.

A digraph D is said to be *complete* if, for every two distinct vertices u and v, both arcs (u, v) and (v, u) belong to $E(D)$. Similarly, D is *semi-complete* if, for every pair of distinct vertices u and v, at least one of the arcs (u, v) or (v, u) belongs to $E(D)$. A sequence of distinct vertices $v_1, v_2, \ldots, v_n$ is called a *semidipath* if, for every index $i = 1, \ldots, n-1$, at least one of (v_i, v_{i+1}) or (v_{i+1}, v_i) lies in $E(D)$. If such a sequence connects vertices u and v, we say that D contains a $[u, v]$-semidipath. Moreover, if all arcs consistently go forward, i.e. $(v_i, v_{i+1}) \in E(D)$ for each i, then the sequence forms a *dipath* from u to v, referred to as a $[u, v]$-dipath.

A digraph D is *strongly connected* if every ordered pair of vertices u and v is linked by a dipath. It is *weakly connected* if every two vertices can be joined by a semidipath. The digraph D is *locally connected* if the existence of a dipath from u to v necessarily implies the existence of a dipath in the reverse direction. Furthermore, D is *unilaterally connected* if, for every pair of vertices u and v, at least one of the $[u, v]$-dipath or $[v, u]$-dipath exists.

For a vertex $v \in V(D)$, define the *out-neighborhood* and *in-neighborhood* respectively by

$$N^+(v) = \{u \in V(D) \backslash \{v\} : (v, u) \in E(D)\}, \qquad N^-(v) = \{u \in V(D) \backslash \{v\} : (u, v) \in E(D)\}.$$

Their cardinalities, denoted by $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$, are the *out-degree* and *in-degree* of v. A vertex with $d^+(v) = 0$ is called a *sink*, while one with $d^-(v) = 0$ is called a *source*. An *isolated* vertex is regarded as a vertex that is both a sink and a source.

A subset $I \subseteq V(D)$ is called an *independent set* of D if no two distinct vertices in I are adjacent; that is, no arc exists between any pair of distinct vertices in I. The *independence number* of D, denoted by $i(D)$, is the maximum cardinality of an independent set in D. Let $\emptyset \neq W \subseteq V(D)$. The set W is called a *dominating set* of D if, for every $v \in V(D) \setminus W$, there exists $u \in W$ such that $(u, v) \in E(D)$. In this case, we say that u *dominates* v, or that v is *dominated by* u. The *domination number* of D, denoted by $\gamma(D)$, is the minimum cardinality of a dominating set of D. A dominating set W is said to be *total* if, for each $v \in V(D)$, there exists $u \in W$ such that $(u, v) \in E(D)$. Moreover, a dominating set W is said to be *independent* if it is an independent set of D. The *total domination number* $\gamma_t(D)$ and the *independent domination number* $\gamma_i(D)$ are defined as the minimum cardinality of a dominating set having the corresponding property, analogous to the definition of $\gamma(D)$.

For additional background on digraph theory, we refer the reader to [20].

Let X be a nonempty set and let $T(X)$ be the full transformation semigroup on X. For a nonempty subset $Y \subseteq X$, the semigroup $PG_Y(X)$ is defined by

$$PG_Y(X) = \{\alpha \in T(X) : \alpha|_Y \in G(Y)\},$$

where $G(Y)$ is the permutation group on Y. For $\alpha \in PG_Y(X)$, the *image of an element* $x \in X$ *under* α is written as $x\alpha$. The set $X\alpha = \{x\alpha : x \in X\}$ is called the *range* of α. It is clear that Y is always contained in $X\alpha$. For each $x \in X\alpha$, define $x\alpha^{-1} = \{z \in X : z\alpha = x\}$. Moreover, the collection $\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}$ forms a partition of X. For $\alpha, \beta \in T(X)$, the composition $\alpha\beta$ is defined by $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in X$. Finally, we say that $\pi(\alpha)$ *refines* $\pi(\beta)$ if, for every $A \in \pi(\alpha)$, there exists $B \in \pi(\beta)$ such that $A \subseteq B$.

An element $\mu \in PG_Y(X)$ is called an *idempotent* if $\mu^2 = \mu$. It is well known that μ is idempotent if and only if $x\mu = x$ for all $x \in X\mu$. The set of all idempotents of $PG_Y(X)$ is denoted by $E(PG_Y(X))$. An idempotent μ is called *minimal* if, for each $\lambda \in E(PG_Y(X))$, the condition $\lambda \le \mu$ (with respect to the natural partial order on $E(PG_Y(X))$) implies $\lambda = \mu$. The set of all minimal idempotents of $PG_Y(X)$ is denoted by E_m. In [17, Proposition 5.1], it was shown that

$$E_m = \{\mu \in E(PG_Y(X)) : X\mu = Y\}.$$

Observe that for each $\mu \in E_m$, we have $\mu|_Y = \mathrm{id}_Y$, where id_Y denotes the identity function on Y. Consequently, if $\alpha \in PG_Y(X)$ satisfies $X\alpha = Y$, then $\alpha\mu = \alpha$ for all $\mu \in E_m$.

In this paper, we consider the Cayley digraph $\mathrm{Cay}(PG_Y(X), A)$, where A is an arbitrary nonempty subset of E_m. For convenience, we shall refer to this digraph as Γ throughout, under the assumption that $A \subseteq E_m$.

We begin by characterizing those functions in $PG_Y(X)$ that form loops in Γ as follows:

Proposition 1. *Let* $\alpha \in PG_Y(X)$*. Then* $(\alpha, \alpha) \in E(\Gamma)$ *if and only if* $X\alpha = Y$*.*

Proof. Assume that $(\alpha, \alpha) \in E(\Gamma)$. Then $\alpha = \alpha\mu$ for some $\mu \in A$. Hence,

$$X\alpha = X\alpha\mu = (X\alpha)\mu \subseteq X\mu = Y \subseteq X\alpha,$$

and therefore $X\alpha = Y$.

Conversely, assume that $X\alpha = Y$. Then $\alpha = \alpha\mu$ for all $\mu \in A$ and hence $(\alpha, \alpha) \in E(\Gamma)$.

Proposition 1 completely identifies the loops in Γ. Hence, for the remainder of this paper, we regard Γ as a simple digraph.

3. Structure of the Cayley digraphs

In this section, we investigate the structural properties of the Cayley digraph Γ. We first focus on the characterization of its vertices, including initial and terminal vertices, as well as sinks and sources.

Theorem 1. *Let $\alpha, \beta \in V(\Gamma)$. If $(\alpha, \beta) \in E(\Gamma)$, then $\alpha|_Y = \beta|_Y$.*

Proof. Assume that $(\alpha, \beta) \in E(\Gamma)$. Then $\beta = \alpha\mu$ for some $\mu \in A$. Since $\mu|_Y = \mathrm{id}_Y$, for each $y \in Y$ we have $y\beta = y\alpha\mu = (y\alpha)\mu = y\alpha$. Hence, $\alpha|_Y = \beta|_Y$.

Theorem 2. *Let $\alpha \in PG_Y(X)$.*

1. *α is an initial vertex in Γ if and only if $X\alpha \neq Y$.*
2. *α is a terminal vertex if and only if the following statements hold:*
   (i) $X\alpha = Y$;
   (ii) $(X \setminus Y)\alpha \cap (X \setminus Y)\mu \neq \emptyset$ *for some* $\mu \in A$.

Proof. (1) Assume that α is an initial vertex. Then there exists $\beta \in PG_Y(X)$ such that $\beta \neq \alpha$ and $(\alpha, \beta) \in E(\Gamma)$. Hence, $\beta = \alpha\mu$ for some $\mu \in A$. If $X\alpha = Y$, then $\beta = \alpha\mu = \alpha$, which cannot hold as $\beta \neq \alpha$. Therefore, $X\alpha \neq Y$.

Conversely, assume that $X\alpha \neq Y$. For $\mu \in A$, we have $X\alpha\mu = (X\alpha)\mu \subseteq X\mu = Y \neq X\alpha$, so α and $\alpha\mu$ are distinct. Therefore, $(\alpha, \alpha\mu) \in E(\Gamma)$, which implies that α is an initial vertex.

(2) Assume that α is a terminal vertex. Then there exists $\beta \in PG_Y(X)$ such that $\beta \neq \alpha$ and $(\beta, \alpha) \in E(\Gamma)$. Hence, $\alpha = \beta\mu$ for some $\mu \in A$. Therefore, $X\alpha = X\beta\mu = Y \subseteq X\alpha$, and statement (i) holds. To show (ii), we again consider the condition $\beta \neq \alpha$. Under this condition, there exists $z \in X$ with $z\alpha \neq z\beta$. By Theorem 1, we have $\alpha|_Y = \beta|_Y$, which implies $z \in X \setminus Y$. If $z\beta \in Y$, then $z\alpha = z\beta\mu = z\beta$, which is a contradiction. Therefore, $z\beta \in X \setminus Y$. It follows that $z\alpha = (z\beta)\mu \in (X \setminus Y)\alpha \cap (X \setminus Y)\mu$, and hence statement (ii) holds.

Conversely, assume that conditions (i) and (ii) hold. Let μ be the function provided by condition (ii), and let $z \in (X \setminus Y)\alpha \cap (X \setminus Y)\mu$. Then $z = a\mu = b\alpha$ for some $a, b \in X \setminus Y$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a, & \text{if } x = b, \\ x\alpha, & \text{otherwise.} \end{cases}$$

We have $\beta \neq \alpha$ since $b\beta = a \neq z = b\alpha$, and $\beta|_Y = \alpha|_Y$ because $b \notin Y$. Moreover, $b\beta\mu = a\mu = z = b\alpha$, and $x\beta\mu = x\alpha\mu = x\alpha$ for all $x \in X \setminus \{b\}$. Thus $\beta\mu = \alpha$. Therefore, $(\beta, \alpha) \in E(\Gamma)$, and hence α is a terminal vertex.

Remark 1. *Theorem 2 shows that no vertex can be both initial and terminal.*

Recall that $\alpha \in V(\Gamma)$ is a *sink* if $d^+(\alpha) = |N^+(\alpha)| = 0$, and a *source* if $d^-(\alpha) = |N^-(\alpha)| = 0$. It is clear that α is a sink if and only if it is not an initial vertex, while α is a source if and only if it is not a terminal vertex. The following corollary, obtained directly from Theorem 2, provides characterizations of the sinks and sources in Γ.

Corollary 1. *Let $\alpha \in V(\Gamma)$.*

1. *α is a sink if and only if $X\alpha = Y$.*
2. *α is a source if and only if α satisfies one of the following conditions:*
   (i) $X\alpha \neq Y$;
   (ii) $X\alpha = Y$ and $(X \setminus Y)\alpha \cap (X \setminus Y)\mu = \emptyset$ for all $\mu \in A$.
3. *α is an isolated vertex if and only if $X\alpha = Y$ and $(X \setminus Y)\alpha \cap (X \setminus Y)\mu = \emptyset$ for all $\mu \in A$.*

Throughout the remainder of this work, $s^+(\Gamma)$, $s^-(\Gamma)$, and $s^0(\Gamma)$ denote the numbers of sinks, sources, and isolated vertices of Γ, respectively.

Theorem 3. *Let X be a finite set with $|X| = n, |Y| = m$, and $|\bigcup_{\mu \in A}(X \setminus Y)\mu| = k$.*

1. $s^+(\Gamma) = m!\, m^{n-m}$.
2. $s^-(\Gamma) = m!(n^{n-m} - m^{n-m} + (m-k)^{n-m})$.
3. $s^0(\Gamma) = m! \cdot (m-k)^{n-m}$.

Proof. It suffices to count the functions in $PG_Y(X)$ that satisfy conditions (1), (2), and (3) of Corollary 1, respectively.

(1) To count the sinks in Γ, we consider the functions $\alpha \in PG_Y(X)$ whose image is exactly Y. There are $m!$ ways to permute the elements of Y, and for each function, there are m^{n-m} ways to assign the elements of $X \setminus Y$ to elements of Y. Hence, the total number of sinks is

$$s^+(\Gamma) = m!\, m^{n-m}.$$

(2) We count the sources α in two types.

Type 1: Functions α whose image is not Y. The number of such sources equals

$$|PG_Y(X)| - s^+(\Gamma) = m!\, n^{n-m} - m!\, m^{n-m}.$$

Type 2: Functions α with $X\alpha = Y$ that satisfy $\left(\bigcup_{\mu \in A}(X \setminus Y)\mu\right) \cap (X \setminus Y)\alpha = \emptyset$. Note that $\bigcup_{\mu \in A}(X \setminus Y)\mu \subseteq Y$. There are $(m-k)^{n-m}$ functions from $X \setminus Y$ into $Y \setminus \bigcup_{\mu \in A}(X \setminus Y)\mu$, so the number of sources of the second type is

$$m!\,(m-k)^{n-m}.$$

Adding the two types together, we obtain the total number of sources:

$$s^{-}(\Gamma) = m!\big(n^{n-m} - m^{n-m} + (m-k)^{n-m}\big).$$

(3) Observe that the isolated vertices in Γ correspond precisely to the sources of type 2. Hence, their number is

$$s^{0}(\Gamma) = m!\,(m-k)^{n-m}.$$

Observe that the sinks of Γ consist of terminal vertices together with isolated vertices, whereas the sources consist of initial vertices together with isolated vertices. Since isolated vertices have both in-degree and out-degree equal to zero, we next restrict our attention to the study of the out-degrees of the initial vertices and the in-degrees of the terminal vertices in Γ.

Let α be an initial vertex of Γ. Then $X\alpha \neq Y$ and $N^{+}(\alpha) = \{\alpha\mu : \mu \in A\}$. The out-degree of α is given by $d^{+}(\alpha) = |N^{+}(\alpha)|$. In particular, we determine $d^{+}(\alpha)$ in the cases where $A = E_m$ and where A is a singleton subset of E_m, as follows:

Theorem 4. *Let α be an initial vertex of Γ.*

1. *If $A = \{\mu\}$, then $d^{+}(\alpha) = 1$.*
2. *If $A = E_m$, then $d^{+}(\alpha) = m^{k-m}$, where $k = |X\alpha|$ and $m = |Y|$.*

Proof. By the above observation, we have $d^{+}(\alpha) = \big|N^{+}(\alpha)\big| = \big|\{\alpha\mu : \mu \in A\}\big|$. Clearly, (1) holds. To prove (2), assume that $A = E_m$. Since $\mu|_Y = \mathrm{id}_Y$ for all $\mu \in E_m$, the number of distinct elements of the form $\alpha\mu$ is equal to the number of all mappings from $X\alpha \setminus Y$ into Y. Hence, $d^{+}(\alpha) = m^{k-m}$, as required.

Let α be a terminal vertex of Γ. Then

$$N^{-}(\alpha) = \{\beta \in PG_Y(X) \setminus \{\alpha\} : \beta\mu = \alpha \text{ for some } \mu \in A\}.$$

The in-degree of α is given by $d^{-}(\alpha) = |N^{-}(\alpha)|$. As before, we determine $d^{-}(\alpha)$ in the cases where $A = E_m$ and where A is a singleton subset of E_m, as follows:

Theorem 5. *Let $A = \{\mu\}$, and let α be a terminal vertex of Γ. Then $\beta \in N^{-}(\alpha)$ if and only if β satisfies the following conditions:*

1. *$X\beta \neq Y$;*
2. *$\beta|_Y = \alpha|_Y$;*
3. *$(z\alpha^{-1} \setminus Y)\beta \subseteq z\mu^{-1}$ for all $z \in (X \setminus Y)\alpha$.*

Proof. Assume that $\beta \in N^{-}(\alpha)$. Then $\beta \neq \alpha$ and $(\beta, \alpha) \in E(\Gamma)$, that is, $\alpha = \beta\mu$. Condition (1) follows from Theorem 2, and condition (2) follows from Theorem 1. To verify (3), let $z \in (X \setminus Y)\alpha$ and let $x \in (z\alpha^{-1} \setminus Y)\beta$. Then $x = w\beta$ for some $w \in z\alpha^{-1} \setminus Y$.

This implies that $w\alpha = z$, and hence $x\mu = (w\beta)\mu = w\alpha = z$. Therefore, $x \in z\mu^{-1}$, and consequently, $(z\alpha^{-1} \setminus Y)\beta \subseteq z\mu^{-1}$.

Conversely, assume that β satisfies all three conditions. Then $\beta \neq \alpha$. To show that $\alpha = \beta\mu$, let $x \in X$.

Case 1: $x \in Y$. By condition (2), $x\beta = x\alpha \in Y$. Since $\mu \in E_m$, we have $\mu|_Y = \mathrm{id}_Y$, and hence $x\beta\mu = x\beta = x\alpha$.

Case 2: $x \in X \setminus Y$. Let $z = x\alpha$. Then $z \in (X \setminus Y)\alpha$ and $x \in z\alpha^{-1} \setminus Y$. By condition (3), $x\beta \in z\mu^{-1}$, and therefore $x\beta\mu = z = x\alpha$.

Thus $\beta\mu = \alpha$, and hence $\beta \in N^-(\alpha)$.

Theorem 6. *Let $A = \{\mu\}$, and let α be a terminal vertex of Γ. Then*

$$d^-(\alpha) = \left(\prod_{z \in (X\setminus Y)\alpha} |z\mu^{-1}|^{|z\alpha^{-1}\setminus Y|} \right) - 1.$$

Proof. By Theorem 5, we have

$$N^-(\alpha) \cup \{\alpha\} = \left\{ \beta \in PG_Y(X) : \beta|_Y = \alpha|_Y \text{ and } (z\alpha^{-1}\setminus Y)\beta \subseteq z\mu^{-1} \text{ for all } z \in (X\setminus Y)\alpha \right\}.$$

Let $\beta \in N^-(\alpha) \cup \{\alpha\}$. Then $x\beta = x\alpha$ for all $x \in Y$. For each $x \in X \setminus Y$, set $z = x\alpha$. Then $z \in (X \setminus Y)\alpha$ and $x \in z\alpha^{-1} \setminus Y$. Hence, $x\beta \in z\mu^{-1} = x\alpha\mu^{-1}$, so there are $|(x\alpha)\mu^{-1}|$ possible choices for $x\beta$. It is easy to see that the collection of sets $\{\, z\alpha^{-1}\setminus Y : z \in (X\setminus Y)\alpha \,\}$ forms a partition of $X \setminus Y$. Therefore, the number of possible choices for $\beta \in N^-(\alpha) \cup \{\alpha\}$ is

$$\begin{aligned} \prod_{x \in X\setminus Y} |x\alpha\mu^{-1}| &= \prod_{z \in (X\setminus Y)\alpha} \left(\prod_{x \in z\alpha^{-1}\setminus Y} |x\alpha\mu^{-1}| \right) \\ &= \prod_{z \in (X\setminus Y)\alpha} \left(\prod_{x \in z\alpha^{-1}\setminus Y} |z\mu^{-1}| \right) \\ &= \prod_{z \in (X\setminus Y)\alpha} |z\mu^{-1}|^{|z\alpha^{-1}\setminus Y|}. \end{aligned}$$

Since α itself is included in this count, it follows that

$$d^-(\alpha) = \left(\prod_{z \in (X\setminus Y)\alpha} |z\mu^{-1}|^{|z\alpha^{-1}\setminus Y|} \right) - 1,$$

as required.

Next, we investigate the in-neighborhoods of terminal vertices in the case $A = E_m$. Accordingly, for a subset $Z \subseteq X$ and $\alpha \in PG_Y(X)$, we define $\pi_Z(\alpha) = \{P \in \pi(\alpha) : P \cap Z \neq \emptyset\}$.

Theorem 7. *Let $A = E_m$ and let α be a terminal vertex of Γ. Then $\beta \in N^-(\alpha)$ if and only if $\beta|_Y = \alpha|_Y$ and there exists a subset $B \subseteq X \setminus Y$ such that the following conditions hold:*

1. $\beta|_B = \alpha|_B$*;*
2. $\pi_{X\setminus(Y\cup B)}(\beta)$ *refines* $\pi_{X\setminus(Y\cup B)}(\alpha)$*;*
3. $\big(X \setminus (Y \cup B)\big)\beta \subseteq X \setminus Y$.

Proof. Assume that $\beta \in N^-(\alpha)$. Then $\beta \neq \alpha$ and $(\beta, \alpha) \in E(\Gamma)$, that is, $\alpha = \beta\mu$ for some $\mu \in A$. By Theorem 1, we have $\beta|_Y = \alpha|_Y$. Define

$$B = \{x \in X \setminus Y : x\alpha = x\beta\}.$$

Since $\beta \neq \alpha$, it follows that $B \subsetneq X \setminus Y$. Moreover, B clearly satisfies condition (1). Since $\beta\mu = \alpha$, we have that $\pi(\beta)$ refines $\pi(\alpha)$. Consequently, $\pi_{X\setminus(Y\cup B)}(\beta)$ refines $\pi_{X\setminus(Y\cup B)}(\alpha)$, and condition (2) holds. It is clear that $x\beta \neq x\alpha$ for all $x \in X \setminus (Y \cup B)$. Since $\mu|_Y = \mathrm{id}_Y$, we obtain $\big(X \setminus (Y \cup B)\big)\beta \subseteq X \setminus Y$, and hence condition (3) is satisfied.

Conversely, assume that $\beta|_Y = \alpha|_Y$, and let $B \subseteq X \setminus Y$ be a subset satisfying conditions (1)–(3). To show that $\beta \in N^-(\alpha)$, it suffices to construct $\mu \in E_m$ such that $\alpha = \beta\mu$. For each $x \in \big(X \setminus (Y \cup B)\big)\beta$, we have $x = x'\beta$ for some $x' \in X \setminus (Y \cup B)$. Fix an element $y \in Y$, and define a map $\mu : X \to X$ by

$$x\mu = \begin{cases} x, & \text{if } x \in (Y \cup B)\beta, \\ x'\alpha, & \text{if } x \in \big(X \setminus (Y \cup B)\big)\beta, \\ y, & \text{otherwise.} \end{cases}$$

By condition (2), the mapping μ is well defined. It follows from Theorem 2(2) that $X\mu = Y$. Furthermore, condition (3) implies that $\mu|_Y = \mathrm{id}_Y$. Hence, $\mu \in E_m$. To show that $\beta\mu = \alpha$, let $x \in X$. If $x \in Y \cup B$, then $x\beta\mu = (x\beta)\mu = x\beta = x\alpha$. If $x \in X \setminus (Y \cup B)$, then $x\beta\mu = (x\beta)\mu = x\alpha$. In both cases, we obtain $x\beta\mu = x\alpha$ for all $x \in X$. Therefore, $\beta\mu = \alpha$, and hence $\beta \in N^-(\alpha)$.

A key combinatorial concept required for our analysis of terminal in-degrees is the Stirling number of the second kind. For positive integers n and k, $S(n, k)$ represents the number of ways to divide a set of n elements into exactly k nonempty subsets. An explicit representation of $S(n, k)$ is given by

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n.$$

We also use the notation $P(n, k)$ to denote the number of permutations of k distinct elements chosen from an n-element set, given by

$$P(n, k) = \frac{n!}{(n - k)!}.$$

Throughout, for any $m \in \mathbb{N}$, X_m denotes the set of the first m positive integers, that is, $X_m = \{1, 2, \ldots, m\}$.

Theorem 8. *Let $A = E_m$ and let α be a terminal vertex of Γ. For each subset B with $B \subsetneq X \setminus Y$, write $\pi_{X\setminus(Y\cup B)}(\alpha) = \{P_1, \ldots, P_t\}$, and define $n_i = |P_i \setminus (Y \cup B)|$ for $i = 1, \ldots, t$, and $K(B) = X_{n_1} \times X_{n_2} \times \cdots \times X_{n_t}$. Then*

$$d^-(\alpha) = \sum_{B \subsetneq X\setminus Y} \sum_{(k_1,\ldots,k_t)\in K(B)} \left(\prod_{i=1}^{t} S(n_i, k_i)\right) P(m, k_1 + k_2 + \cdots + k_t)\,,$$

where $m = |X \setminus Y|$.

Proof. For each subset B with $B \subsetneq X \setminus Y$, Theorem 7 characterizes the in-neighbors β of α as those mappings satisfying $\beta|_{Y\cup B} = \alpha|_{Y\cup B}$, $\pi_{X\setminus(Y\cup B)}(\beta)$ refines $\pi_{X\setminus(Y\cup B)}(\alpha)$, and $\big(X \setminus (Y \cup B)\big)\beta \subseteq X \setminus Y$. To count such mappings β, observe that the condition $\beta|_{Y\cup B} = \alpha|_{Y\cup B}$ fixes the values of β on $Y \cup B$. Hence, it suffices to consider the images under β of elements in $X \setminus (Y \cup B)$.

Let $\pi_{X\setminus(Y\cup B)}(\alpha) = \{P_1, \ldots, P_t\}$, where $n_i = |P_i \setminus (Y \cup B)|$ for $i = 1, \ldots, t$. Since $\pi_{X\setminus(Y\cup B)}(\beta)$ must refine $\pi_{X\setminus(Y\cup B)}(\alpha)$, the set $P_i \setminus (Y \cup B)$ can be partitioned into k_i nonempty blocks in $S(n_i, k_i)$ ways. Thus, for fixed $(k_1, \ldots, k_t)$, there are

$$\prod_{i=1}^{t} S(n_i, k_i)$$

ways to form such refinements. Moreover, since $\big(X \setminus (Y \cup B)\big)\beta \subseteq X \setminus Y$, the images of all these blocks can be assigned injectively to elements of $X \setminus Y$ in

$$P\big(m, k_1 + k_2 + \cdots + k_t\big)$$

ways, where $m = |X \setminus Y|$. Therefore, the total number of in-neighbors of α corresponding to a fixed subset B is

$$\sum_{(k_1,\ldots,k_t)\in K(B)} \left(\prod_{i=1}^{t} S(n_i, k_i)\right) P(m, k_1 + k_2 + \cdots + k_t)\,.$$

Since B ranges over all proper subsets of $X \setminus Y$, summing over all such B yields

$$d^-(\alpha) = \sum_{B \subsetneq X\setminus Y} \sum_{(k_1,\ldots,k_t)\in K(B)} \left(\prod_{i=1}^{t} S(n_i, k_i)\right) P(m, k_1 + k_2 + \cdots + k_t)\,.$$

This completes the proof.

To further understand the structure of Γ, we next investigate its connectedness. This leads us to examine the semidipaths that occur in the graph.

Lemma 1. *Let $\alpha, \beta \in V(\Gamma)$ be distinct. If $[\alpha, \beta]$-semidipath exists in Γ, then $\alpha|_Y = \beta|_Y$.*

Proof. Assume that there exists a $[\alpha, \beta]$-semidipath in Γ. If $(\alpha, \beta) \in E(\Gamma)$, then, by Theorem 1, we obtain that $\alpha|_Y = \beta|_Y$. If $(\alpha, \beta) \notin E(\Gamma)$, then there exist $\gamma_1, \ldots, \gamma_n \in PG_Y(X)$ such that $\alpha, \gamma_1, \ldots, \gamma_n, \beta$ form a semidipath. Hence, $(\alpha, \gamma_1) \in E(\Gamma)$ or $(\gamma_1, \alpha) \in E(\Gamma)$. In particular, both situations lead, by Theorem 1, to $\alpha|_Y = \gamma_1|_Y$. By the same reason, we conclude that $\gamma_1|_Y = \gamma_2|_Y, \gamma_2|_Y = \gamma_3|_Y, \ldots, \gamma_{n-1}|_Y = \gamma_n|_Y$, and $\gamma_n|_Y = \beta|_Y$. This completes the proof.

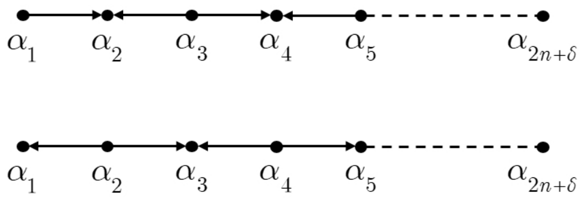

Figure 1: $[\alpha_1, \alpha_{2n+\delta}]$-semidipath

Lemma 2. *Let $P := \alpha_1, \alpha_2, \ldots, \alpha_{2n+\delta}$ be the semidipath of Γ where $\delta = 0, 1$. Then P is either one of the following forms:*

(i) $(\alpha_{2i\pm1}, \alpha_{2i}) \in E(\Gamma)$ *for all* $i = 1, \ldots, n$;

(ii) $(\alpha_{2i}, \alpha_{2i\pm1}) \in E(\Gamma)$ *for all* $i = 1, \ldots, n$.

Proof. From the definition of P, we obtain that either $(\alpha_1, \alpha_2) \in E(\Gamma)$ or $(\alpha_2, \alpha_1) \in E(\Gamma)$.

Case 1: $(\alpha_1, \alpha_2) \in E(\Gamma)$. Then α_2 is terminal. By Theorem 2, we obtain that $X\alpha_2 = Y$, and therefore it cannot be initial. This implies that (α_2, α_3) cannot occur in $E(\Gamma)$, and thus $(\alpha_3, \alpha_2) \in E(\Gamma)$. Then α_3 is initial. By Theorem 2, we obtain that $X\alpha_3 \neq Y$, and thus it cannot be terminal. Consequently, (α_4, α_3) cannot occur in $E(\Gamma)$, and therefore $(\alpha_3, \alpha_4) \in E(\Gamma)$. By repeating the same reasoning, we conclude that P is of the form (i).

Case 2: $(\alpha_2, \alpha_1) \in E(\Gamma)$. In this case, by following the same reasoning as in Case 1 and observing first that α_2 is an initial vertex and therefore cannot be terminal, we conclude that P is of the form (ii).

Theorem 9. Γ *is weakly connected if and only if* $|Y| = 1$.

Proof. Assume that $|Y| \geq 2$. Then there exist distinct elements $a, b \in Y$. Define $\alpha : X \to X$ and $\beta : X \to X$ as follows:

$$x\alpha = \begin{cases} x, & \text{if } x \in Y, \\ a, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad x\beta = \begin{cases} b, & \text{if } x = a, \\ x, & \text{if } x \in Y \setminus \{a, b\}, \\ a, & \text{otherwise.} \end{cases}$$

Clearly, $\alpha, \beta \in PG_Y(X)$ and $\alpha|_Y \neq \beta|_Y$. By Lemma 1, we obtain that Γ contains neither a $[\alpha, \beta]$-semidipath nor a $[\beta, \alpha]$-semidipath. Therefore, Γ is not weakly connected.

Conversely, assume that $|Y| = 1$. Let $Y = \{y\}$. In this case, we have $E_m = \{\mu\} = A$, where μ is the constant function with $X\mu = \{y\}$. For any $\alpha \in PG_Y(X)$ such that $\alpha \neq \mu$, we have $\alpha\mu = \mu$; that is, $(\alpha, \mu) \in E(\Gamma)$, and this provides a semidipath joining α and μ. Now consider any distinct $\alpha, \beta \in PG_Y(X)$ such that $\alpha \neq \mu \neq \beta$. Then α, μ, β form a semidipath joining α and β. Therefore, Γ is weakly connected.

Remark 2. *In the case where* $|Y| = 1$, Γ *is an in-star digraph whose center is the unique constant function.*

Theorem 10. Γ *is strongly connected if and only if* $|X| = 1$.

Proof. Assume that Γ is strongly connected. Then Γ is weakly connected, and thus $|Y| = 1$ by Theorem 9. By Remark 2, Γ is an in-star digraph whose center is the unique constant function, denoted by μ. If $X \neq Y$, then there exists $\alpha \in PG_Y(X)$ such that $\alpha \neq \mu$, and consequently there is no $[\mu, \alpha]$-dipath. Hence $X = Y$, and therefore $|X| = 1$.

Conversely, assume that $|X| = 1$. In this case, Γ is a trivial graph and is obviously strongly connected.

Theorem 11. Γ *is locally connected if and only if* $X = Y$.

Proof. Assume that Γ is locally connected and $X \neq Y$. Consider $\mu \in A$. Since $\mu = \mathrm{id}_X\mu$, we obtain that $(\mathrm{id}_X, \mu) \in E(\Gamma)$, and this forms the $[\mathrm{id}_X, \mu]$-dipath. By the local connectedness of Γ, the $[\mu, \mathrm{id}_X]$-dipath also exists in Γ. This implies that id_X is a terminal vertex. By Theorem 2(2), we obtain that $X = X\mathrm{id}_X = Y$, a contradiction.

Conversely, assume that $X = Y$. Then $E_m = \{\mathrm{id}_X\}$, which implies that there are no non-loop edges. Thus, Γ is an edgeless graph (i.e., a null graph) and is clearly locally connected.

Theorem 12. Γ *is unilaterally connected if and only if* $|Y| = 1$ *and* $|X \setminus Y| \leq 1$.

Proof. Assume that $|Y| \neq 1$ or $|X \setminus Y| \geq 2$. Consider first the case $|Y| \neq 1$. By Theorem 9, Γ is not weakly connected and hence it is not unilaterally connected. For the case $|X \setminus Y| \geq 2$, let $a, b \in X \setminus Y$ be distinct. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = a, \\ x, & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in PG_Y(X)$ and $\alpha \neq \mathrm{id}_X$. However, $X\alpha \neq Y \neq X\mathrm{id}_X$, which means that neither of them is a terminal vertex in Γ. Thus, neither the $[\alpha, \mathrm{id}_X]$-dipath nor the $[\mathrm{id}_X, \alpha]$-dipath exists in Γ. Therefore, in this case, Γ is also not unilaterally connected.

Conversely, assume that $|Y| = 1$ and $|X \setminus Y| \leq 1$. Then $|PG_Y(X)| \leq 2$ and $|E_m| = 1$. In this situation, Γ is either a trivial graph or a dipath P_2, and both of these are unilaterally connected.

Corollary 2. Γ *is semi-complete if and only if* $|Y| = 1$ *and* $|X \setminus Y| \leq 1$.

Proof. Assume that Γ is semi-complete. Then Γ is unilaterally connected, and by Theorem 12, we obtain that $|Y| = 1$ and $|X \setminus Y| \leq 1$.

Conversely, in this case, Γ is either a trivial graph or a dipath P_2, and both of these are semi-complete.

Corollary 3. Γ *is complete if and only if* $|X| = 1$.

Proof. Assume that Γ is complete. Then it is strongly connected, and hence, by Theorem 10, we obtain that $|X| = 1$.

Conversely, if $|X| = 1$, then Γ is a trivial graph and is clearly complete.

4. Domination parameters of the Cayley digraphs

In this section, we determine the domination parameters of Γ. Throughout, let X be a finite set with n elements, and let $Y \subseteq X$ be nonempty with $|Y| = m$. Moreover, denote $k = \left|\bigcup_{\mu \in A} (X \setminus Y)\mu\right|$. Let $S^-(\Gamma)$ be the set of all sources in Γ. Then, by Theorem 3, we have $|S^-(\Gamma)| = s^-(\Gamma)$.

Lemma 3. $S^-(\Gamma)$ *is the smallest dominating set of* Γ.

Proof. We begin by showing that $S^-(\Gamma)$ is a dominating set of Γ. Let $\alpha \in V(\Gamma) \setminus S^-(\Gamma)$. This means that α is not a source and so there exists $\beta \in V(\Gamma)$ such that $(\beta, \alpha) \in E(\Gamma)$. This means that β is an initial vertex, and by Theorem 2(1), we obtain that $X\beta \neq Y$. By Corollary 1(2), we obtain that β is a source and this implies that $\beta \in S^-(\Gamma)$ and β dominates α. Therefore, $S^-(\Gamma)$ is a dominating set of Γ.

To verify that $S^-(\Gamma)$ is the smallest dominating set of Γ, let W be any dominating set of Γ and $\alpha \in S^-(\Gamma)$. If $\alpha \notin W$, then there exists $\beta \in W$ such that $(\beta, \alpha) \in E(\Gamma)$. However, this contradicts that α is a source. Therefore $\alpha \in W$ and so $S^-(\Gamma) \subseteq W$. Therefore, $S^-(\Gamma)$ is the smallest dominating set of Γ.

Combining Lemma 3 and Theorem 3(2) we obtain the following:

Theorem 13. $\gamma(\Gamma) = s^-(\Gamma) = m!\,(n^{n-m} - m^{n-m} + (m-k)^{n-m})$.

Theorem 14. Γ *has no total dominating set.*

Proof. Assume that Γ has a total dominating set, say W. Then W is in particular, a dominating set of Γ; hence, by Lemma 3, we have $S^-(\Gamma) \subseteq W$. However, every vertex in $S^-(\Gamma)$ is a source and therefore is not dominated by any other vertex in W. This contradicts the assumption that W is a total dominating set. Thus, Γ has no total dominating set.

Theorem 15. $\gamma_i(\Gamma) = \gamma(\Gamma)$.

Proof. To show that $S^-(\Gamma)$ is an independent set of Γ, let $\alpha, \beta \in S^-(\Gamma)$ be distinct. Since both α and β are sources, neither is a terminal vertex; hence, no arc exists between α and β. Therefore, $S^-(\Gamma)$ is an independent set. By Lemma 3, $S^-(\Gamma)$ is the smallest independent dominating set of Γ. Consequently, $\gamma_i(\Gamma) = s^-(\Gamma) = \gamma(\Gamma)$.

The proof of Theorem 15 shows that $S^-(\Gamma)$ is an independent set of Γ. Therefore, we obtain the lower bound of the independence number of Γ as follows:

Corollary 4. $i(\Gamma) \geq \gamma(\Gamma)$.

5. Conclusion

In this paper, we have investigated the structural and combinatorial properties of the Cayley digraph $\Gamma = \mathrm{Cay}(PG_Y(X), A)$, where $PG_Y(X)$ is the transformation semigroup with restriction on the fixed set being bijective and A is a nonempty subset of minimal idempotents. We successfully characterized the basic components of the digraph, providing conditions for elements to be loops, initial vertices, and terminal vertices. Furthermore, we classified the sinks, sources, and isolated vertices of Γ and derived explicit formulas for their cardinalities. Our study extended to the calculation of vertex degrees. We calculated the out-degrees of initial vertices and applied Stirling numbers of the second kind to formulate the in-degrees of terminal vertices. Regarding the connectivity of Γ, we found that the digraph is strongly connected and complete if and only if $|X| = 1$. We also proved that Γ is weakly connected if and only if $|Y| = 1$, and locally connected if and only if $X = Y$. Additionally, characterizations for unilaterally connectedness and semi-completeness were provided. Finally, we analyzed the domination parameters of Γ. We demonstrated that the set of sources, $S^-(\Gamma)$, constitutes the smallest dominating set. Consequently, the domination number $\gamma(\Gamma)$ and the independent domination number $\gamma_i(\Gamma)$ are equal to the number of sources, while the digraph admits no total dominating set. These results provide a comprehensive understanding of the algebraic and graph-theoretic structure of Cayley digraphs associated with the semigroup $PG_Y(X)$.

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