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A Revisit of SRCFs Through the Farey Graph

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Abstract. We introduce the notion of "well directed paths from infinity" in the Farey graph and show that the vertices of such a path, in the infinite case, have almost strictly increasing denominators, that is, the subsequence of denominators corresponding to the indices where the sequence fails to strictly increase, is itself a strictly increasing sequence. We establish a one-to-one correspondence between semi-regular continued fractions and well directed paths in the Farey Graph and utilize this correspondence to describe various kinds of semi-regular continued fraction expansions of real numbers.

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Key Words and Phrases: Semi-regular continued fraction, convergent, farey graph, well directed path from ∞.

1. Introduction

The notion of continued fraction is very old; Euler used continued fractions to study Euler's number e in the 18th century. The study of semi-regular continued fractions goes back to Lagrange in the 18th century ([1]). Indeed, continued fractions have been studied in various contexts by several researchers from time to time. In this article, we study semi-regular continued fractions by means of the Farey graph. The connection of regular (also called simple) continued fractions to the Farey graph is classically known. There are instances of studying semi-regular continued fractions by means of the Farey graph (e.g, Ford [2]). Our efforts in this regard may be considered to be a step in this direction.

In the 20th century, several interesting articles on various kinds of semi-regular continued fractions appeared (e.g., [3], [4]). Our approach to semi-regular continued fractions through certain paths from infinity helps us to get new results besides deriving certain known ones, just in one go. This is evidence of its importance.

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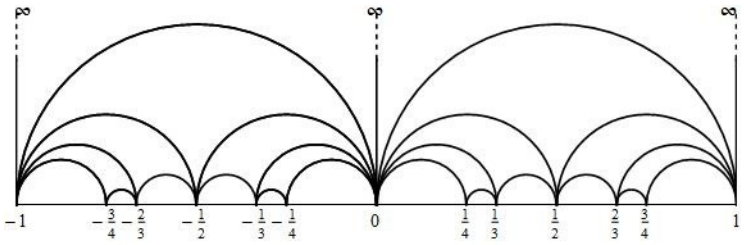

Figure 1: A part of $\mathcal{F}$ on $[-1, 1]$, where the multiple labels marked ∞ represent the same boundary point.

The modular group $\mathrm{PSL}(2, \mathbb{Z})$ acts on the extended set of rationals $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{x}{y} = \frac{ax + by}{cx + dy},$$

where $x, y \in \mathbb{Z}$, $(x, y) = 1$ with the usual convention that ∞ is represented by $\frac{x}{0}$ $(x \neq 0)$. The graph $\mathcal{F}$ on the vertex set $\hat{\mathbb{Q}}$, where the adjacency of p/q, r/s is given by the condition $ps - qr = \pm 1$, is called the Farey graph ([5]). If $x, y \in \hat{\mathbb{Q}}$ are adjacent, we write $x \sim y$.

The edges of $\mathcal{F}$ are represented as hyperbolic geodesics in the upper half plane $\mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$, that is, as Euclidean semi-circles or half lines perpendicular to the real line. In Figure 1, a few edges are displayed for certain vertices lying in $[-1, 1]$. The multiple vertical lines labeled "∞" represent the same point in the boundary of the upper half-plane, following standard conventions for hyperbolic tessellations. The fact that no two edges of $\mathcal{F}$ cross in $\mathcal{U}$ plays a key role in this article.

Next, we mention a few definitions and basic results of semi-regular continued fractions. For more details, see [6–8].

A pair of finite or infinite sequences $\{\epsilon_i\}_{i\geq 1}$ and $\{a_i\}_{i\geq 0}$ with $a_0 \in \mathbb{Z}$, for $i \geq 1$, $a_i \in \mathbb{N}$, and $\epsilon_i = \pm 1$, is called a *semi-regular continued fraction* (abbreviated as *SRCF*) if $\epsilon_{n+1} + a_n \geq 1$. A finite SRCF is expressed as

$$a_0 + \frac{\epsilon_1}{a_1+} \; \frac{\epsilon_2}{a_2+} \; \frac{\epsilon_3}{a_3+} \; \cdots \frac{\epsilon_n}{a_n}$$

whereas an infinite one is expressed as

$$a_0 + \frac{\epsilon_1}{a_1+} \; \frac{\epsilon_2}{a_2+} \; \frac{\epsilon_3}{a_3+} \; \cdots \frac{\epsilon_n}{a_n+} \cdots .$$

The integer a_i is called the i-th *partial denominator*, and ϵ_i is called the i-th *partial numerator*. For $k \geq 0$, the value of the expression

$$\frac{p_0}{q_0} = a_0, \; \frac{p_k}{q_k} = a_0 + \frac{\epsilon_1}{a_1+} \; \frac{\epsilon_2}{a_2+} \; \frac{\epsilon_3}{a_3+} \; \cdots \frac{\epsilon_k}{a_k} \; (\text{ for } k \geq 1),$$

where $(p_k, q_k) = 1$, is called the *k-th convergent*, and the sequence $\{p_k/q_k\}_{k\geq 0}$ is called the *sequence of convergents* of the continued fraction. Obviously, the sequence of convergents of a finite SRCF is a finite sequence. The following SRCF is called its *tail at the i-th stage*

$$a_i + \; \frac{\epsilon_{i+1}}{a_{i+1}+} \cdots .$$

If $\epsilon_i = 1$ for each i, the SRCF is called *regular* or *simple*, and if $\epsilon_i = -1$ for each i, it is called *backward*. If each a_i is even, then it is called an *even continued fraction*. The *odd* continued fractions are also defined similarly (see [3, 9]).

In [10], paths from infinity in $\mathcal{F}_{1,2}$ are used to define $\mathcal{F}_{1,2}$-continued fractions. In [11], $\mathcal{F}_{1,3}$-continued fractions are defined; in fact, a one to one correspondence between "well directed paths" from infinity and $\mathcal{F}_{1,3}$-continued fractions is established. The graph $\mathcal{F}_{1,2}$ is a tree, whereas $\mathcal{F}$ and $\mathcal{F}_{1,3}$ are connected but not trees. In this article, we study SRCFs with the help of the methodology in [10–12]. In Section 4, we establish a one to one correspondence between SRCFs (finite or infinite) and well directed paths from ∞ (respectively, finite or infinite).

Theorem 1. *There is a one to one correspondence between SRCFs and well directed paths in $\mathcal{F}$ from ∞. In fact,*

(i) every SRCF (finite or infinite) defines a well directed path in $\mathcal{F}$ from ∞ to its value;

(ii) every well directed path in $\mathcal{F}$ from ∞ to $x \in \mathbb{R}$ defines an SRCF whose value is x.

Certain variants of this result are available in the literature (e.g., [2, 13]). In sections 5 and 6, we use this correspondence to describe two different types of SRCF expansions of real numbers (Theorems 5 and 6). One can refer to [14] to find a result on generalized continued fraction expansions of real numbers, which is comparable to the main result in section 5.

2. Elementary Results on the Farey Graph

Let $x \in \mathbb{Q}$. Suppose that a path Θ_n joining ∞ and x is given by

$$\Theta_n \equiv \infty \sim P_0 \sim P_1 \sim \cdots \sim P_k \sim \cdots \sim P_n = x. \tag{1}$$

To emphasize that the path (1) is directed from ∞ to x, we replace $\sim$ by $\to$ at every stage. That is,

$$\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_k \to \cdots \to P_n = x.$$

The number of edges of a path is called its *length*. Thus, the length of Θ_n is $n+1$. The *distance* of x from ∞ along Θ_n is the length of Θ_n.
Here we state a few lemmas on the Farey graph.

Lemma 1. *Suppose $P \in \mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(i) $P \sim m$ if and only if $P = m + 1/n$ for some nonzero integer n;

(ii) $P \sim \infty$ if and only if $P \in \mathbb{Z}$.

Proof. Let $P = \frac{p}{q}$, where with $\gcd(p, q) = 1$. By the definition of the Farey graph, $p - qm = \pm 1$ so that $P = \frac{p}{q} = m + \frac{1}{n}$, where $n = \pm q$.

Lemma 2. *If $P \sim Q$, then $\gamma(P) \sim \gamma(Q)$ for every $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$.*

Proof. Let $P = \frac{p}{q}$ and $Q = \frac{p'}{q'}$. Then $P \sim Q$ implies that $pq' - qp' = \pm 1$ so that the determinant of $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ is ± 1. Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$. The $\gamma(P) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$ and $\gamma(Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$. The determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ is ± 1. Hence, $\gamma(P) \sim \gamma(Q)$.

Lemma 3. *If $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ fixes two distinct points of $\hat{\mathbb{Q}}$, then $\gamma = 1$.*

Proof. This is a standard consequence of the theory of Möbius transformations [6, Chapter 7]. For completeness, we include a proof.

Let $\gamma \in PSL(2,\mathbb{Z})$ be an element that fixes two distinct points $x, y \in \mathbb{Q} \cup \{\infty\}$. We wish to show that $\gamma = I$. Since $PSL(2,\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$, there exists an element $\delta \in PSL(2,\mathbb{Z})$ such that $\delta(x) = \infty$ and so that $\delta(y) = y'$. Define a new element $\eta \in PSL(2,\mathbb{Z})$ by conjugation

$$\eta = \delta\gamma\delta^{-1}.$$

Since $\gamma(x) = x$ and $\gamma(y) = y$, we get $\eta(\infty) = \delta\gamma\delta^{-1}(\infty) = \infty$ and $\eta(y') = y'$. We know that the stabilizer Γ_∞ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, therefore $\eta = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \in \mathbb{Z}$. Note that $\eta(y') = y' + n = y'$ which implies that $n = 0$. Consequently, the matrix $\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Thus, γ must be the identity element.

Lemma 4. *Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. If $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ is such that $\gamma(a) = \infty$ and $\gamma(a + b^{-1}) = 0$, then $\gamma(y) = b + (a - y)^{-1}$ for $y \in \mathbb{R}$.*

Proof. Since $a \sim a + b^{-1}$, there exists $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ such that $\gamma(a) = \infty$ and $\gamma(a + b^{-1}) = 0$. In fact, γ can be chosen to be the matrix

$$\begin{pmatrix} b & -1 - ab \\ 1 & -a \end{pmatrix}.$$

Use Lemma 3 to conclude that this is the only choice of γ.

Lemma 5. *Let $\infty \to P_0 \to P_1 \to P_2 \to \cdots \to P_n$ be a path in $\mathcal{F}$ from ∞ to P_n. Then there exist $\gamma_1, \gamma_2, \cdots, \gamma_{n-1} \in \mathrm{PSL}(2,\mathbb{Z})$ such that*

$$\gamma_j \circ \cdots \circ \gamma_2 \circ \gamma_1(P_{j-1}) = \infty, \tag{2}$$

$$\gamma_j \circ \cdots \circ \gamma_2 \circ \gamma_1(P_j) = 0, \tag{3}$$

for $1 \le j \le n-1$. In fact, for $2 \le j \le n$, there exist $a_j \in \mathbb{Z} \setminus \{0\}$ such that

$$\gamma_{j-1} \circ \cdots \circ \gamma_2 \circ \gamma_1(P_j) = 1/a_j, \tag{4}$$

$$\gamma_j(y) = a_j - \frac{1}{y}, \text{ for } y \in \mathbb{R}. \tag{5}$$

Proof. We apply induction. Since $\infty \sim P_0$, by Lemma 1, $P_0 \in \mathbb{Z}$. Let $P_0 = a_0$. Since $P_0 \sim P_1$, by Lemma 1, $P_1 = a_0 + 1/a_1$ where $a_1 \in \mathbb{Z} \setminus \{0\}$. By Lemma 4, $\gamma_1 \in \mathrm{PSL}(2,\mathbb{Z})$ as desired. Suppose $\gamma_1, \gamma_2, \ldots \gamma_j$ have been constructed satisfying Eqns (2) and (3). Then by Lemma 2, $0 = \gamma_j \circ \gamma_{j-1} \circ \cdots \circ \gamma_1(P_j) \sim \gamma_j \circ \gamma_{j-1} \circ \cdots \circ \gamma_1(P_{j+1})$ so that by Lemma 1, $\gamma_j \circ \cdots \circ \gamma_2 \circ \gamma_1(P_{j+1}) = 1/a_{j+1}$ where $a_{j+1} \in \mathbb{Z} \setminus \{0\}$. Again by Lemma 4, $\gamma_{j+1} \in \mathrm{PSL}(2,\mathbb{Z})$ such that $\gamma_{j+1}(0) = \infty$ and $\gamma_{j+1}(1/a_{j+1}) = 0$. This completes the proof.

3. Finite Well directed Paths from ∞ and SRCFs

Suppose $\Theta_n \equiv \infty = P_{-1} \to P_0 \to P_1 \to \cdots \to P_{n-1} \to P_n = x$ is such that no vertex is repeated (i.e., $P_i \neq P_j$ for $-1 \leq i \neq j \leq n$). If $Q \in \tilde{V}_n := \hat{\mathbb{Q}} \setminus \{P_i : -1 \leq i \leq n\}$ and $x \sim Q$, then we would like to consider $\Theta_{n+1} \equiv \infty = P_{-1} \to P_0 \to P_1 \to \cdots \to P_{n-1} \to P_n = x \to Q$.

Denote by $E^{(1)}_{\Theta_n}(x)$ the set of all edges from x to $Q \in \tilde{V}_n$ if $P_{n-1} < Q < x$ or $x < Q < P_{n-1}$. Elements of $E^{(1)}_{\Theta_n}(x)$ are called *direction changing edges* from x relative to Θ_n (see Figure 2).



Figure 2: Direction changing edge $P_n \sim Q$

Denote by $E^{(-1)}_{\Theta_n}$ the set of all edges from x to $Q \in \tilde{V}_n$ that are not in $E^{(1)}_{\Theta_n}$, that is, $P_{n-1} < x < Q$ or $P_{n-1} > x > Q$. The edges in $E^{(-1)}_{\Theta_n}$ are called *direction retaining edges* from x relative to Θ_n (see Figure 3).

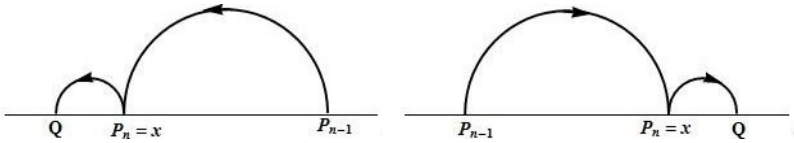

Figure 3: Direction retaining edge $P_n \sim Q$

Order both $E^{(1)}_{\Theta_n}$ and $E^{(-1)}_{\Theta_n}$ according to the size of the radii of the edges, such that the first edge has the longest radius. The pair $(e, a) \in \{-1, 1\} \times \mathbb{N}$ is referred to as the *label* and a as the *rank* of $P_n \sim Q$ relative to Θ_n if $P_n \sim Q$ is the a-th member of $E^{(e)}_{\Theta_n}$.

Let $\Theta_n \equiv \infty = P_{-1} \to P_0 \to P_1 \to \cdots \to P_{n-1} \to P_n = x$ be a path such that no vertex is repeated (i.e., $P_i \neq P_j$ for $-1 \leq i \neq j \leq n$). Then the path Θ_n is called *well directed* if it has the following property for each i $(1 \leq i \leq n)$: $P_i \sim P_{i+1}$ is direction changing relative to Θ_i whenever the rank of $P_{i-1} \sim P_i$ relative to Θ_{i-1} is 1 (every edge following a rank one edge is direction changing).

Lemma 6. *Let $a \in \mathbb{Z}$ and let $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ be such that $\gamma(a) = \infty$. Suppose $P, Q \in \mathbb{Q}\backslash\{a\}$ lie on the same side of a on the real line. Then*

$$P < Q \Rightarrow \gamma(P) < \gamma(Q).$$

Proof. If $\gamma = \begin{pmatrix} r & s \\ u & v \end{pmatrix}$ sends a to ∞, then $u = \pm 1, v = \mp a$ and $r, s \in \mathbb{Z}$ such that $ar + s = 1$. Then

$$\gamma(P) - \gamma(Q) = \frac{P - Q}{(P - a)(Q - a)}. \tag{6}$$

The lemma follows from (6).

Suppose $\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_{n-1} \to P_n$ is well directed and $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ satisfies $\gamma(P_{i-1}) = \infty$. We denote by $\dot{\gamma}(\Theta_n)$ the path $\infty \to \gamma(P_i) \to \gamma(P_{i+1}) \to \cdots \to \gamma(P_n)$ (see Lemma 2).

Proposition 1. *Let $a \in \mathbb{Z}$ and let $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ satisfy $\gamma(a) = \infty$. Suppose $\Theta_n \equiv \infty \to a \to P_1 \to P_2 \to \cdots \to P_n$ is well directed. Then*

(i) for $i \geq 1$, the rank of $P_i \sim P_{i+1}$ relative to Θ_i and the rank of $\gamma(P_i) \sim \gamma(P_{i+1})$ with respect to $\dot{\gamma}(\Theta_i)$ are the same.

(ii) for $i \geq 2$, the label of $P_i \sim P_{i+1}$ relative to Θ_i and the label of $\gamma(P_i) \sim \gamma(P_{i+1})$ with respect to $\dot{\gamma}(\Theta_i)$ are the same.

In particular, $\dot{\gamma}(\Theta_n)$ is well directed.

Proof. By Lemma 6, for $i \geq 2$, $P_{i-1} < P_i < P_{i+1} \Leftrightarrow \gamma(P_{i-1}) < \gamma(P_i) < \gamma(P_{i+1})$ and $P_{i-1} > P_i > P_{i+1} \Leftrightarrow \gamma(P_{i-1}) > \gamma(P_i) > \gamma(P_{i+1})$, which implies (ii), and (i) is a direct consequence of Lemma 6.

Proposition 2. *Let $a \in \mathbb{Z}$ and let $\Theta_2 \equiv \infty \to a \to P_1 \to P_2$ be well directed. Suppose $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ satisfies $\gamma(a) = \infty$.*

(i) Suppose $a \sim P_1$ is direction retaining relative to $\infty \to a$. Then $P_1 \sim P_2$ is direction retaining relative to Θ_0 if and only if $\gamma(P_1) \sim \gamma(P_2)$ is direction retaining relative to $\dot{\gamma}(\Theta_0)$.

(ii) Suppose $a \sim P_1$ is direction changing relative to $\infty \to a$. Then $P_1 \sim P_2$ is direction retaining relative to Θ_0 if and only if $\gamma(P_1) \sim \gamma(P_2)$ is direction changing relative to $\dot{\gamma}(\Theta_0)$.

Proof. Since $\infty = \gamma(a)$, we have $\gamma(a) > \gamma(P_1)$. By Lemma 6, $P_1 > P_2 \Leftrightarrow \gamma(P_1) > \gamma(P_2)$. Let $P_1 \sim P_2$ be direction retaining relative to Θ_1. Then we have $\infty > a > P_1$. Thus, $\infty > a > P_1 > P_2 \Leftrightarrow \infty = \gamma(a) > \gamma(P_1) > \gamma(P_2)$. Next, let $P_1 \to P_2$ is direction changing. Then, $\infty > a < P_1$. Thus, $\infty > a < P_1 > P_2 \Leftrightarrow \infty = \gamma(a) > \gamma(P_1) > \gamma(P_2)$.

Theorem 2. *A well directed path in $\mathcal{F}$ from ∞ to $x \in \mathbb{Q}$ defines a unique finite SRCF of x. In fact, if $\Theta_i \equiv \infty \to P_0 \to P_1 \to \cdots \to P_i$ is well directed for each i $(0 \le i \le n)$ and the SRCF defined by Θ_n is*

$$b_0 + \frac{\epsilon_1}{b_1+} \cdots \frac{\epsilon_n}{b_n}, \tag{7}$$

then the label of the edge $P_i \sim P_{i+1}$ is $(\epsilon_{i+1}, b_{i+1})$ relative to Θ_i.

Proof. Suppose $\infty \to P_0 \to P_1 \to \cdots \to P_n$ is a well directed path from ∞ to $x = P_n$. Suppose $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are as in Lemma 5. Then, for $2 \le j \le n-1$,

$$\gamma_j^{-1}(y) = \frac{1}{a_j - y}; \tag{8}$$

$$\gamma_1^{-1}(y) = a_0 + \frac{1}{a_1 - y}. \tag{9}$$

Since $\gamma_{n-1} \circ \cdots \circ \gamma_2 \circ \gamma_1(P_n) = 1/a_n$ for some $a_n \in \mathbb{Z} \setminus \{0\}$, we have

$$P_n = \gamma_1^{-1} \circ \gamma_2^{-1} \circ \cdots \circ \gamma_{n-1}^{-1}(\frac{1}{a_n}).$$

Finally, by Equation(8) and Equation (9), we have

$$x = P_n = a_0 + \frac{1}{a_1-} \; \frac{1}{a_2-} \; \cdots \frac{1}{a_{n-1}-} \; \frac{1}{a_n} \tag{10}$$

Set $\epsilon_1 = \text{sign}(a_1)$ and $\epsilon_{i+1} = -\text{sign}(a_i)\text{sign}(a_{i+1})$. We get

$$x = P_n = a_0 + \frac{\epsilon_1}{|a_1|+} \; \frac{\epsilon_2}{|a_2|+} \; \cdots \frac{\epsilon_{n-1}}{|a_{n-1}|+} \; \frac{\epsilon_n}{|a_n|}, \tag{11}$$

where

$$\text{sign}(a_i) = (-1)^{i-1}\epsilon_1\epsilon_2 \cdots \epsilon_i \;\; (1 \le i \le n). \tag{12}$$

We can understand this sign convention by induction. For $n = 1$, $P_1 = a_0 + \frac{1}{a_1} = a_0 + \frac{\epsilon_1}{|a_1|}$ so that $\text{sign}(a_1) = \epsilon_1$. Now assume that $\text{sign}(a_k) = (-1)^{k-1}\epsilon_1\epsilon_2 \cdots \epsilon_k \quad$ for $k < n$. Since

$$\epsilon_{i+1} = -\,\text{sign}(a_i)\,\text{sign}(a_{i+1})$$

we have

$$\text{sign}(a_{i+1}) = -\text{sign}(a_i)\epsilon_{i+1}.$$

Substituting the induction hypothesis,

$$\begin{aligned} \text{sign}(a_{i+1}) &= -\text{sign}(a_i)\epsilon_{i+1} \\ &= -\left((-1)^{i-1}\epsilon_1\epsilon_2 \cdots \epsilon_i\right)\epsilon_{i+1} \\ &= (-1)^i\epsilon_1\epsilon_2 \cdots \epsilon_i\epsilon_{i+1}. \end{aligned}$$

Now, comparing Equations (7) and (11), we have $b_0 = a_0$, $b_i = |a_i|$ $(1 \le i \le n)$. Suppose $P_i \sim P_{i+1} \in E_{\Theta_i}^{(\epsilon'_{i+1})}$. Then by Proposition 1, for $i \ge 2$, $\gamma_1(P_i) \sim \gamma_1(P_{i+1}) \in E_{\gamma_1(\Theta_i)}^{(\epsilon'_{i+1})}$ and,

by Proposition 2, $\gamma_1(P_1) \sim \gamma_1(P_2) \in E^{(-\epsilon_1'\epsilon_2')}_{\gamma_1(\Theta_0)}$. Let $\sigma_1 = \gamma_1$ and let $\sigma_i = \gamma_i \circ \sigma_{i-1}$ for $i > 1$. Then by induction, the following statements hold for $j \leq n$,

$$\sigma_j(P_{i-1}) \sim \sigma_j(P_i) \quad \in \quad E^{(\epsilon_i')}_{\dot{\sigma}_j(\Theta_{i-1})} \text{ if } i-1 > j \tag{13}$$

$$\sigma_j(P_j) \sim \sigma_j(P_{j+1}) \quad \in \quad E^{((-1)^j\epsilon_1'\epsilon_2'\cdots\epsilon_{j+1}')}_{\dot{\sigma}_j(\Theta_j)}. \tag{14}$$

Comparing Eqns. (12) and (14) for every i, we have, $\epsilon_i = \epsilon_i'$ as desired. Therefore, we have, $\sigma_j(P_{j+1}) = (-1)^{j-1}\epsilon_1\epsilon_2\cdots\epsilon_{j+1}/b_{j+1}$. Since, $\sigma_j(P_j) = 0$, the rank of $\sigma_j(P_j) \sim \sigma_j(P_{j+1})$ is b_{j+1}. Also observe that Lemma 6 and Proposition 2 imply that the rank of $P_j \sim P_{j+1}$ in $E^{(\epsilon_{j+1})}_{\Theta_j}$ and that of $\sigma_j(P_j) \sim \sigma_j(P_{j+1})$ in $E^{((-1)^{j-1}\epsilon_1\epsilon_2\cdots\epsilon_{j+1})}_{\dot{\sigma}_j(\Theta_j)}$ are equal. Thus, the label of $P_i \sim P_{i+1}$ with respect to Θ_i is $(\epsilon_{i+1}, b_{i+1})$. In particular, since each Θ_i is well directed, $b_i + \epsilon_{i+1} \geq 1$ for each $i \geq 1$.

Corollary 1. *Let $\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n$ be a well directed path and let $P_i = p_i/q_i$ $(1 \leq i \leq n)$. If (e, r) is the label of $P_n \sim P_{n+1}$ relative to Θ_n, then $q_{n+1} = rq_n + eq_{n-1}$ and $p_{n+1} = rp_n + ep_{n-1}$.*

Proof. Consider the continued fraction (7) defined by the path Θ_n in Theorem 2. Then, for $1 \leq i \leq n-1$, the $(i+1)$-th convergent $\frac{p_{i+1}}{q_{i+1}} = b_0 + \frac{\epsilon_1}{b_1+} \cdots \frac{\epsilon_{i+1}}{b_{i+1}}$ follows the following recurrence relations [8, pp. 2-4]

$$p_{i+1} = b_{i+1}p_i + \epsilon_{i+1}p_{i-1} \text{ and } q_{i+1} = b_{i+1}q_i + \epsilon_{i+1}q_{i-1}.$$

By the above theorem, $(\epsilon_{i+1}, b_{i+1})$ will be the label of the edge $P_i \sim P_{i+1}$. If the label of $P_n \sim P_{n+1}$ is given as (e, r), then $q_{n+1} = rq_n + eq_{n-1}$ and $p_{n+1} = rp_n + ep_{n-1}$.

4. Infinite Semi-regular Continued Fractions

A sequence $\{\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n\}$ of well directed paths in $\mathcal{F}$ from ∞ is said to be *coherent* if Θ_{n+1} is obtained by attaching the edge $P_n \sim P_{n+1}$ to the final vertex P_n of Θ_n for every n. If $\{\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n\}$ is a coherent sequence of paths in $\mathcal{F}$ from ∞ such that $\lim_{n\to\infty} P_n = x$, then the infinite path Θ in $\mathcal{F}$ whose $(n+1)$-th vertex is P_n is called an *infinite well directed path* from ∞ to x. We write $\Theta = \lim_{n\to\infty} \Theta_n$. By definition, if $\Psi = \lim_{n\to\infty} \Psi_n$, then we have

$$\Theta = \Psi \iff \Theta_n = \Psi_n \text{ for every } n \in \mathbb{N}.$$

Lemma 7. *If $P_1, P_2, P_3 \in \mathbb{Q}$ are such that $P_1 \sim P_2$, $P_2 \sim P_3$, $P_3 \sim P_1$ and $P_1 < P_2 < P_3$, then $p_2 = p_1 + p_3$ and $q_2 = q_1 + q_3$ where $P_i = p_i/q_i$ $(1 \leq i \leq 3)$ with $p_i, q_i \in \mathbb{Z}$, $q_i > 0$ and $(p_i, q_i) = 1$.*

Proof.

Using the given ordering $P_1 < P_2 < P_3$, we can translate the three adjacency relations into a system of three linear equations:

$$p_2q_1 - p_1q_2 = 1 \tag{15}$$
$$p_3q_2 - p_2q_3 = 1 \tag{16}$$
$$p_3q_1 - p_1q_3 = 1 \tag{17}$$

We rewrite equations (15) and (16) as a linear system with variables p_2 and q_2:

$$\begin{pmatrix} q_1 & -p_1 \\ -q_3 & p_3 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By solving this system using Cramer's Rule and using that the determinant of $\begin{pmatrix} q_1 & -p_1 \\ -q_3 & p_3 \end{pmatrix}$ is 1, we obtain $p_2 = p_1 + p_3$, $q_2 = q_1 + q_3$.

Lemma 8. *Let $P_1 < P_2 < P_3, Q \in \mathbb{Q}$ be such that $P_1 \sim P_2$, $P_1 \sim P_3$. If $P_2 \sim P_3$ and $P_2 < Q < P_3$, then $P_1 \not\sim Q$.*

Proof. If $P_2 \sim P_3$, then P_1, P_2 and P_3 form a triangle in the Farey graph. We know that two edges do not cross which fails if $P_1 \sim Q$.

Theorem 3. *Suppose $\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n$ is well directed and $P_n \not\sim P_{n+2}$ for every $n \geq 0$. Let $P_n = p_n/q_n$. Then*

(i) $q_n < q_{n+1}$ for each $n \geq 1$;

(ii) the sequence $\{P_n\}$ converges.

Proof. Since $\infty \sim P_0$, $P_0 \in \mathbb{Z}$. Without loss of generality, assume that $P_0 = 0$ and $P_0 < P_1$. Then $P_1 = 1/m$ for some $m \in \mathbb{N}$. Since $P_1 \sim P_2$, $q_2 = p_2m + e$ for $e = \pm 1$. If $m = 1$, then $e = 1$ so that $\infty \to 0 \to 1 \to P_2$ is well directed. Thus, for every m, $q_2 = p_2m + e > m = q_1$.

Now suppose $q_i < q_{i+1}$ for every $i < n$. By Corollary 1, $q_{n+1} = rq_n + eq_{n-1}$. Observe, if $P_n \sim P_{n+1} \in E^{(-1)}_{\Theta_n}$ then r, the rank of $P_n \sim P_{n+1}$ in $E^{(-1)}_{\Theta_n}$, is at least 2. Thus, $q_{n+1} \geq 2q_n - q_{n-1} > 2q_n - q_n = q_n$.

Since $|P_{n+1} - P_n| = 1/(q_nq_{n+1}) < 1/(n(n+1))$ and $P_n = (P_n - P_{n-1}) + P_{n-1} - P_{n-2} + \cdots + (P_2 - P_1) + P_1$, the sequence $\{P_n\}$ converges by comparison test.

Theorem 4. *If, for $n \geq 0$, $\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n$ is well directed and $P_n = p_n/q_n$, then $\{q_n\} \to \infty$ and $\{P_n\}$ is convergent.*

Proof. Suppose $P_n \sim P_{n+2}$ for finitely many values of n. Then, there is $N \in \mathbb{N}$ such that $P_n \not\sim P_{n+2}$ for every $n \geq N$. Then, it is clear from the proof of Theorem 3 that $\{q_{N+n}\}$ is strictly increasing and hence the result follows.

Thus, if the sequence $\{q_n\}$ eventually gets increasing, the sequence $\{P_n\}$ converges. Now, we assume the situation when $\{q_n\}$ fails to be increasing. By Theorem 3, it is only possible when $P_n \sim P_{n+2}$ with $q_{n+2} < q_{n+1}$ for infinitely many values of n, namely, for $n = n_k$ for $k \in \mathbb{N}$ such that $P_{n_k} \sim P_{n_k+2}$ and q_{n_k+2} are the points which make the sequence q_n non increasing. By Lemma 7, $q_{n_k+1} = q_{n_k} + q_{n_k+2}$ so that the label of $P_{n_k+1} \sim P_{n_k+2}$ is $(-1, 1)$, and hence, the rank of the next edge will be at least 2 as the path is well directed. Thus, $n_{k+1} > n_k + 1$. Now consider the subsequence $\{q_{n_k+2}\}_{k=1}^{\infty}$, let $k_1 < k_2$, the claim is $q_{n_{k_1}+2} < q_{n_{k_2}+2}$. We know that $n_{k_2} > n_{k_1} + 1$ so that $n_{k_2} + 2 \geq n_{k_1} + 4$. It is enough to see the claim for $n_{k_2} = n_{k_1} + 2$. We have $q_{n_{k_1}+4} = q_{n_{k_1}+3} - q_{n_{k_1}+2} > (2q_{n_{k_1}+2} + q_{n_{k_1}+1}) - q_{n_{k_1}+2}$ and we get $q_{n_{k_1}+4} > q_{n_{k_1}+2}$. Since q_{n_k+2} are the points which make the sequence q_n non increasing. We split the sequence in two disjoint subsequences, one is $\{q_{n_k+2}\}_{k=1}^{\infty}$ and other is obtained by deleting the subsequence $\{q_{n_k+2}\}_{k=1}^{\infty}$ from $\{q_n\}_{k=1}^{\infty}$. Since the path is well directed, both the sequences are strictly increasing and tends to infinity. Hence, $q_n \to \infty$, so that $\{P_n\}$ is convergent.

The Pringsheim Theorem ([15]) for SRCFs follows from Theorem 2 and Theorem 4.

Example 1. *Let $P_0 = 1, P_1 = 1/2, P_2 = 1/3$ and define $P_k = p_k/q_k$ by*

$$\begin{aligned} p_{2k} &= 2p_{2k-1} + p_{2k-2}, \quad q_{2k} = 2q_{2k-1} + q_{2k-2}, \\ p_{2k+1} &= p_{2k} - p_{2k-1}, \quad q_{2k+1} = q_{2k} - q_{2k-1}. \end{aligned}$$

If $\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n$, then $\lim_{n\to\infty} \Theta_n$ is a path with $P_{2k-1} \sim P_{2k+1}$ for every $k \in \mathbb{N}$. The path Θ_n is well-directed by the way it is defined in the question, characterized by edge labels that alternate systematically between $(1, 2)$ and $(-1, 1)$. It follows that any edge of rank 1 is invariably followed by a direction changing edge. Note that the sequence $\{q_n\}_{n=0}^{\infty}$ is not increasing as $q_{2k+1} < q_{2k}$ for every $k \in \mathbb{N}$. The subsequences $\{q_{2k+1}\}$, and $\{q_{2k}\}$ partition the sequence $\{q_n\}$. Thus the sequence $\{q_n\}$ tends to infinity as both the subsequences tends to infinity and hence $\{P_n\}$ is convergent.

Proof of Theorem 1. It follows from Theorems (2) and (4) □.

5. $\mathcal{E}$-Continued Fractions

Lemma 9. *Let Θ_n be a well directed path in $\mathcal{F}$ from ∞ to P_n. Suppose, $P_n \sim B(r) \in E_{\Theta_n}^{(e)}$ has rank r (≥ 1) and the path obtained by attaching $P_n \sim B(r)$ to Θ_n is well directed. If $r \geq 2$, there are exactly two infinite well directed paths from ∞ to $B(r)$ that extends Θ_n, namely via $B(r-1)$ and $B(r+1)$. There is a unique infinite well directed path from ∞ to $B(1)$ that extends Θ_n, namely, via $B(2)$.*

Proof. Let $\gamma \in \mathrm{PSL}(2, \mathbb{Z})$ be such that $\gamma(B(r \pm 1)) = 0$ and $\gamma(B(r)) = 1$. Observe that $\infty \to 0 \to \frac{1}{2} \to \frac{2}{3} \to \cdots \to \frac{k-1}{k} \to \cdots$ is a well directed path to 1. Set $\gamma^{-1}(\frac{k-1}{k}) = P_{n+k}$.

Then for every $k \geq 1$, extend Θ_n to Θ_{n+k} by attaching $P_{n+1} \to P_{n+2} \to \cdots \to P_{n+k}$ so that $\{\Theta_m\}$ is a coherent sequence of well directed paths from ∞ to P_m. Then the limit of $\{\Theta_m\}$ is an infinite path from ∞ to $B(r)$. The result follows.

Given a sequence $\mathcal{E} = \{\epsilon_n\}_{n=1}^{\infty}$ where $\epsilon_n = \pm 1$, an SRCF

$$a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_j}{a_j} \text{ (finite) or}$$

$$a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_j}{a_j+} \cdots \text{ (infinite)}$$

is called an *$\mathcal{E}$-continued fraction* if either its last partial denominator a_j is bigger than 1 or $\frac{-1}{2+}\ \frac{-1}{2+}\ \cdots$ is not a tail.

Proposition 3. *We have* $1 = \frac{1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots$ *and* $-1 = \frac{-1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots$.

Proof. Consider the continued fraction $\frac{1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots$. Let $\{P_n\}_{n\geq 0}$ be the sequence of partial quotients. Then $P_1 = \frac{1}{2}$, $P_2 = \frac{2}{3}$ and for $n \geq 3$, $P_n = \frac{n}{n+1} \to 1$. Now consider $\frac{-1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots$. Note that $\frac{-1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots = -(\frac{-1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+}\ \cdots)$. The claim is true by the first identity.

Theorem 5. *Suppose $\mathcal{E} = \{\epsilon_n\}_{n=1}^{\infty}$ is a sequence with $\epsilon_n = \pm 1$. Then every real number has a unique $\mathcal{E}$-continued fraction expansion.*

Proof. Existence. Let $x \in \mathbb{R}$. We construct a coherent sequence $\{\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n\}$ of paths, finite or infinite, such that $P_N = x$ if Θ_N is the last member of the sequence or $\{P_n\}$ converges to x such that $P_n \sim P_{n+1} \in E_{\Theta_n}^{(\epsilon_{n+1})}$ for every n. Set,

$$P_0 = \begin{cases} \lfloor x \rfloor, \text{ if } \epsilon_1 = 1 \\ \lceil x \rceil, \text{ if } \epsilon_1 = -1. \end{cases}$$

If $x = P_0$, we stop. Suppose $P_1, P_2, \ldots, P_n$ with $P_i \neq x$ for $1 \leq i \leq n-1$ have been defined such that there is a pair of vertices A, B of $\mathcal{F}$ such that $A \leq x \leq B$, P_n is adjacent to both A and B. For $n \geq 0$, suppose

$$A_n(1) = \max\{P \in \mathbb{Q} : \ P \leq x, \ P_n \sim P\},$$

$$A_n(-1) = \min\{P \in \mathbb{Q} : \ P \geq x, \ P_n \sim P\}.$$

Let $e_n = \text{sign}(P_n - x)$. Then, define $P_{n+1} = A_n(e_n\epsilon_{n+2})$. Observe, $A_n(1) = A_n(-1)$ if and only if $P_{n+1} = x$, and in this case, the path ends at P_{n+1}, and we have a path of length $(n+1)$. Suppose $x \neq P_{n+1}$. Then $A_n(1) \neq A_n(-1)$ and x lies between them. Then, by Lemma 8, $A_n(-1) \sim A_n(1)$, and hence, we can find $A \leq x \leq B$ such that $P_{n+1} \sim A$ and $P_{n+1} \sim B$.

Uniqueness. Suppose $A_n(1) \neq A_n(-1)$. If $P_{n+1} \in \hat{\mathbb{Q}} \setminus \{A_n(1), A_n(-1)\}$, then the points lying between $A_n(1)$ and $A_n(-1)$ (but not $A_n(1)$ and $A_n(-1)$) cannot be connected by a

well directed path from ∞ via P_{n+1}. Hence, the choice of P_{n+1} and hence, Θ_{n+1} is unique. Thus, if $A_n(1) \neq A_n(-1)$ for every n, there is a unique $\mathcal{E}$-continued fraction expansion of x.

Suppose $A_n(1) = A_n(-1) = x$ for some n. Let the label of $P_n \sim x$ relative to Θ_n be (e, r). Then P_{n+1} can be chosen differently from x. Let, for $j \geq 1$, $P_n \sim B(j) \in E^{(e)}_{\Theta_n}$ have rank j. If

$$P_{n+1} = \begin{cases} B(r+1) \text{ if } \epsilon_{n+2} = -1, \\ B(r-1) \text{ if } \epsilon_{n+2} = 1, \end{cases}$$

and $P_{n+2} = x$, then the label of $P_{n+1} \sim P_{n+2}$ relative to Θ_{n+1} is $(\epsilon_{n+2}, 1)$, and it defines another possible well directed extension of Θ_n to x. Such a change while choosing P_{n+1} produces the following possibilities for the tail at the $(n-1)$-th stage:

$$a_{n-1} + \frac{\epsilon_n}{r} = a_{n-1} + \frac{\epsilon_n}{(r-1)+} \frac{1}{1} \tag{18}$$

$$= a_{n-1} + \frac{\epsilon_n}{(r+1)+} \frac{-1}{1} \tag{19}$$

Thus, in this case, if $r \neq 1$ or $\epsilon_{n+1} = -1$, the choice of P_{n+1} and hence the choice of finite paths from ∞ to x has exactly two possibilities. Further, by Lemma 9, in these cases, there will be one infinite path from ∞ to x via P_{n+1} when $P_{n+1} \neq x$. The infinite tail given by this possibility is obtained from (18) and (19) by using Proposition 3, that is, $1 = \frac{1}{2+} \frac{-1}{2+} \cdots$ and $-1 = \frac{-1}{2+} \frac{-1}{2+} \cdots$ respectively. But these SRCFs do not satisfy the properties of an $\mathcal{E}$-continued fraction. Hence, uniqueness follows.

Apply Theorem 5 to the sequences (i) $\epsilon_n = 1 \; \forall \, n \in \mathbb{N}$ and (ii) $\epsilon_n = -1 \; \forall \, n \in \mathbb{N}$ to get the following (well known) results, respectively.

Corollary 2. *Every real number has a regular continued fraction expansion.*

Corollary 3. *Every real number has a backward (also called negative) continued fraction expansion.*

Example 2. *Suppose* $\mathcal{E} = \{(-1)^n\}_{n=1}^{\infty}$. *Then* $\mathcal{E}$*-continued fraction expansion of* $\sqrt{2}$ *is given by*

$$2 + \frac{-1}{1+} \frac{1}{2+} \frac{-1}{1+} \frac{1}{2+} \frac{-1}{1+} \cdots \frac{-1}{1+} \frac{1}{2+} \cdots .$$

Note that $P_0 = a_0$. *Since* $\epsilon_1 = -1$, *by the above theorem,* $P_0 = \lceil \sqrt{2} \rceil = 2$. *To find* P_1, *observe that* $A_0(1) = \max\{P \in \mathbb{Q} : \; P \leq \sqrt{2}, \; P_0 \sim P\} = 1$, *and* $A_0(-1) = \min\{P \in \mathbb{Q} : \; P \geq x, \; P_0 \sim P\} = \frac{3}{2}$. *Here,* $e_0 = 1$ *and* $\epsilon_2 = 1$, *hence,* $P_1 = A_0(1) = 1$ *so that* $a_1 = 1$. *Now we have* $P_1 = 1$ *and* $A_1(1) = \max\{P \in \mathbb{Q} : \; P \leq \sqrt{2}, \; P_1 \sim P\} = \frac{4}{3}$, *and* $A_1(-1) = \min\{P \in \mathbb{Q} : \; P \geq x, \; P_1 \sim P\} = \frac{3}{2}$. *Here,* $e_1 = -1$ *and* $\epsilon_3 = -1$, *hence* $P_2 = A_1(1) = \frac{4}{3}$ *so that* $a_2 = 2$. *Similarly, we get* $a_{2n+1} = 1$ *and* $a_{2n+2} = 2$ *for* $n \geq 1$. *Now observe*

$$2 + \frac{-1}{1+} \frac{1}{2+} \frac{-1}{1+} \frac{1}{2+} \frac{-1}{1+} \cdots \frac{-1}{1+} \frac{1}{2+} \cdots = 2 + y,$$

where

$$y = \frac{-1}{1+}\,\frac{1}{2+y} \in (-1, 0).$$

Therefore, $y^2 + 4y + 2 = 0$ *so that* $y = -2 + \sqrt{2}$*. Thus,*

$$\sqrt{2} = 2 + \frac{-1}{1+}\,\frac{1}{2+}\,\frac{-1}{1+}\,\frac{1}{2+}\,\frac{-1}{1+}\cdots\,\frac{-1}{1+}\,\frac{1}{2+}\cdots.$$

The regular continued fraction expansion of $\sqrt{2}$ *is*

$$1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$$

whereas its backward continued fraction is

$$2 + \frac{-1}{2+}\,\frac{1}{4+}\,\frac{-1}{2+}\,\frac{1}{4+}\,\frac{-1}{2+}\cdots.$$

6. SRCFs of $\mathcal{A}$-type Parity

Suppose $\mathcal{A} = \{A_n\}_{n=0}^{\infty}$ satisfies $A_n = 0$ or 1. An SRCF of the form

$$a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_j}{a_j} \text{ (finite) or}$$

$$a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_j}{a_j+}\cdots \text{ (infinite)}$$

is called an *SRCF of* $\mathcal{A}$*-type parity* if the n-th partial denominator a_n has the same parity as A_n for each $n \geq 0$.

Theorem 6. *Suppose* $\mathcal{A} = \{A_n\}_{n=0}^{\infty}$ *is such that* $A_n = 0$ *or* 1*. Then every real number has an SRCF of* $\mathcal{A}$*-type parity. In fact, there are at most two such SRCF expansions for a rational number, whereas it is unique for an irrational number.*

Proof. It is enough to show that given any $x \in \mathbb{R}$, there is a coherent sequence $\{\Theta_n \equiv \infty \to P_0 \to P_1 \to \cdots \to P_n\}$ of paths, finite or infinite, such that $P_N = x$ if Θ_N is the last member of the sequence (in the finite case) or $\{P_n\}$ converges to x (in the infinite case) such that the rank of $P_i \sim P_{i+1}$ relative to Θ_i has the same parity as A_i for every i ($0 \leq i \leq n$ in the finite case and $i \geq 0$ for the infinite case). We define Θ_n by induction.

Define P_0 to be the integer nearest to x having parity of A_0 so that P_0 is unique except when x is an integer having a different parity than A_0. Observe, $x \in [P_0 - 1, P_0 + 1]$. If $x = P_0$, we stop.

Suppose $P_1, P_2, \ldots, P_{n-1}$ with $P_i \neq x$ for $1 \leq i \leq n-1$ have been defined such that there is a couple of vertices A, B of $\mathcal{F}$ such that $A \leq x \leq B$, $P_{n-1} \sim A$ and $P_{n-1} \sim B$. Let

$$X_n = \{P \in \mathbb{Q} : P_{n-1} \sim P \text{ and } (P_{n-1} - x)(P_{n-1} - P) > 0\}. \tag{20}$$

Suppose $B_n(r) \in X_n$ is such that the rank of $P_{n-1} \sim B_n(r)$ with respect to Θ_{n-1} is r. By induction hypothesis, there exists $r \in \mathbb{N}$ such that x lies in the closed interval whose endpoints are $B_n(r)$ and $B_n(r+1)$. Clearly, if $x \in X_n$, then there are two choices of r.

The parity of A_n matches exactly one of r and $r+1$. Let $\delta \in \{0,1\}$ be such that $r - A_n \equiv \delta \pmod 2$. Define $P_n = B_n(r+\delta)$. We stop if $P_n = x$. If $P_n \neq x$, then x lies between P_n and $B_n(r+1-\delta)$. Then the rank of $P_n \sim B_n(r+1-\delta)$ relative to Θ_n is 1. Hence, there are vertices A, B such that $A \leq x \leq B$ with $P_n \sim A$, $P_n \sim B$.

If $x = P_n$ for some n, then there is no other well directed path from ∞ to x. Observe, when $B_n(r) = x$, $B_n(r-1)$ is a possible choice of P_n if and only if $B_n(r+1)$ is a choice of P_n. Thus, if $B_n(r) = x$ and $r \not\equiv A_n \pmod 2$, there are two SRCFs of $\mathcal{A}$-type parity. In fact, if $A_i = 0$ for every $i > n$, the tails at the n-th stage when $B_n(r) = x$, are $(r-1)+\frac{1}{2-}\,\frac{1}{2-}\,\cdots\frac{1}{2-}\cdots$ and $(r+1)-\frac{1}{2-}\,\frac{1}{2-}\,\cdots\frac{1}{2-}\cdots$ that have the same value. If $A_i = 0$ for $n \leq i < m$ and $A_m = 1$ then the two SRCFs of $\mathcal{A}$-type parity have length m (both). Indeed, the tails at the n-th stage of the SRCF of $\mathcal{A}$-type parity are $(r-1)+\frac{1}{2-}\,\frac{1}{2-}\,\cdots\frac{1}{2-}\,\frac{1}{1}$ and $(r+1)-\frac{1}{2-}\,\frac{1}{2-}\,\cdots\frac{1}{2-}\,\frac{1}{1}$ having the same value.

Use Theorem 6 to the sequences (i) $A_n = 0\ \forall\, n \in \mathbb{N}$ and (ii) $A_n = 1\ \forall\, n \in \mathbb{N}$ to get the following statements (which are well known).

Corollary 4. *Every real number has an even continued fraction expansion.*

Corollary 5. *Every real number has an odd continued fraction expansion.*

Example 3. *Suppose $\mathcal{A} = \{1+(-1)^n\}_{n=0}^{\infty}$. Then continued fraction expansion of $\sqrt{2}$ of $\mathcal{A}$-type parity is given by*

$$2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{2+}\,\frac{-1}{1+}\cdots\,\frac{-1}{1+}\,\frac{1}{2+}\cdots.$$

Note that $P_0 = a_0$. Since $A_0 = 0$, by the above theorem, $P_0 = 2$ as it is the nearest integer of the same parity. To find P_1, observe that $X_1 = \{P \in \mathbb{Q} :\ P \sim 2,\ (2-\sqrt{2})(2-P) > 0\} = \{1, \frac{3}{2}, \frac{5}{3}, \dots\}$, and $\sqrt{2} \in [B_1(1), B_1(2)]$, where $B_1(1) = 1$ and $B_1(2) = \frac{3}{2}$. Since $A_1 = 1$, $P_1 = B_1(1) = 1$ so that $a_1 = 1$. Now we have $A_2 = 0$ and $\sqrt{2} \in [B_2(1), B_2(2)]$, where $B_2(1) = \frac{3}{2}$ and $B_2(2) = \frac{4}{3}$. Therefore, $P_2 = \frac{4}{3}$ with $a_2 = 2$. Similarly, we get $a_{2n+1} = 1$ and $a_{2n+2} = 2$ for $n \geq 1$. Thus,

$$\sqrt{2} = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{2+}\,\frac{-1}{1+}\cdots\,\frac{-1}{1+}\,\frac{1}{2+}\cdots.$$

The continued fraction expansion of $\sqrt{2}$ with even partial quotients [3], that is, $\mathcal{A} = \{0\}_{n=0}^{\infty}$, is

$$2 + \frac{-1}{2+}\,\frac{1}{4+}\frac{-1}{2+}\,\frac{1}{4+}\,\frac{-1}{2+}\cdots,$$

whereas the continued fraction expansion of $\sqrt{2}$ with odd partial quotients [3] is given by

$$1 + \frac{1}{3+}\,\frac{-1}{1+}\frac{1}{1+}\,\frac{1}{3+}\,\frac{-1}{1+}\frac{1}{1+}\cdots.$$

Let $x = \frac{7}{5}$, the continued fraction expansion of $x = \frac{7}{5}$ of $\mathcal{A}$-type parity is given by

$$2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{2}.$$

Note that x is in the well directed path $2 \to 1 \to \frac{4}{3} \to \frac{3}{2} \to \frac{7}{5}$ so that there is a unique continued fraction of $\mathcal{A}$-type parity.

Now take $x = \frac{19}{15}$, then

$$x = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{3},$$

which does not follow the parity condition. Here, $B_4(3) = x$ and $r \not\equiv A_4 \pmod 2$, in this case, there are two SRCFs of $\mathcal{A}$-type parity given by

$$x = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{2}\,\frac{+1}{1},$$

and

$$x = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{4}\,\frac{-1}{1}.$$

Now consider the case where $\mathcal{A} = \{1, 0, 1, 0, 0, 0 \ldots, 0, \ldots\}$. Then $x = \frac{19}{15}$ has two infinite continued fraction expansions of $\mathcal{A}$-type parity given by

$$x = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{-1}{2+}\,\frac{1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\frac{-1}{2+}\,\frac{-1}{2+} \cdots$$

and

$$x = 2 + \frac{-1}{1+}\,\frac{1}{2+}\frac{-1}{1+}\,\frac{1}{4+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\frac{-1}{2+}\,\frac{-1}{2+} \cdots .$$

Remark 1. *The nearest integer continued fraction expansions of* $3/7$ *and* $3/8$ *are* $0 + \frac{1}{2+}\,\frac{1}{3}$ *and* $0 + \frac{1}{3-}\,\frac{1}{3}$ *respectively, and they neither have the same sequence of partial numerators nor have the partial denominators with the same parity. Let the nearest integer continued fraction is the same as an* $\mathcal{E}$*-continued fraction for some* $\mathcal{E} = \{\epsilon_n\}_{n=1}^{\infty}$*. Then the expansion of* $3/7$ *says* $\epsilon_1 = 1$ *and* $\epsilon_2 = 1$*, whereas the expansion of* $3/8$ *does not follow the same sequence of signs. For a given sequence of signs* $\mathcal{E}$*,* $\epsilon_1 = -1$*, then the nearest integer continued fraction expansion of* $3/7$ *and* $3/8$ *does not follow* $\mathcal{E}$*-continued fraction expansion. Similarly, if we assume that the nearest integer continued fraction expansion is same as an SRCF of* $\mathcal{A}$*-type parity for some* $\mathcal{A} = \{A_n\}_{n=0}^{\infty}$*, by the expansion of* $3/7$*,* $a_0 = 0, a_1 = 0$ *and* $a_2 = 1$ *which is not followed by the nearest integer continued fraction expansion of* $3/8$*. Thus, the nearest integer continued fraction expansion of a real number is neither an* $\mathcal{E}$*-continued fraction nor an SRCF of* $\mathcal{A}$*-type parity for whatever choices of* $\mathcal{E}$ *or* $\mathcal{A}$*.*

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