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The Minimum Isolated Gap in Numerical Semigroups Generated by Two Elements

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Abstract. A numerical semigroup is a cofinite additive submonoid of the non-negative integers. The study of its gaps has long been central to the theory, with particular emphasis on invariants such as the Frobenius number and the set of pseudo-Frobenius numbers. In this setting, an *isolated gap* is a gap g such that both $g-1$ and $g+1$ belong to the semigroup, and a numerical semigroup is called *perfect* if it has no isolated gaps. Motivated by Smith's construction of perfect numerical semigroups of embedding dimension three via adjoining the minimum isolated gap of a two-generated semigroup, we derive explicit formulas for the minimum isolated gap of the two-generated numerical semigroups $\langle a, a+n\rangle$ for $n = 2, 3, 4, 5, 6$, under the natural condition $\gcd(a, a+n) = 1$ together with the relevant congruence restrictions on a. Our approach combines congruence arguments based on the Euclidean algorithm and the Chinese Remainder Theorem with elementary computations. As a consequence, we obtain constructive families of perfect numerical semigroups of embedding dimension three obtained by adjoining these minimum isolated gaps.

2020 Mathematics Subject Classifications: 20M14

Key Words and Phrases: Perfect numerical semigroups, chinese Remainder Theorem, euclidean algorithm, isolated gap.

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1. Introduction

Let $\mathbb{N}$ be the set of all non-negative integers. A *numerical semigroup* is a non-empty subset $S \subseteq \mathbb{N}$ that is closed under addition, contains 0, and has finite complement in $\mathbb{N}$. Numerical semigroups provide a fertile ground for exploring the interplay between combinatorial number theory, commutative algebra, and discrete structures. One of the

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most fundamental aspects in their study is the analysis of the gaps of a semigroup, i.e., the elements of $\mathbb{N} \setminus S$, whose distribution and structure reveal essential properties of the semigroup.

If S is a numerical semigroup and $A \subseteq S$, we say that A is a *generating set* of S if for every $s \in S$, there exist $a_1, \ldots, a_k \in A$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$s = n_1 a_1 + \cdots + n_k a_k.$$

In this case, we use the notation $S = \langle A \rangle$. It is known that every numerical semigroup has a unique finite minimum set of generators (Theorem 2.7 of [1]). The cardinality of this minimum set of generators is called the *embedding dimension* of S.

For any numerical semigroup S, the elements of $G(S) := \mathbb{N} \setminus S$ are called the *gaps* of S. The largest element of $G(S)$ is called the *Frobenius number* of S and denoted by $F(S)$. Determining the Frobenius number of a numerical semigroup is an old and celebrated problem (see [2]).

Within the set of gaps, particular interest has been directed towards *isolated gaps*, which are defined as gaps g such that both $g - 1$ and $g + 1$ belong to the semigroup. Isolated gaps provide a natural obstruction to certain structural properties. A numerical semigroup is called *perfect* if it contains no isolated gaps (see [3, 4]). The study of perfect semigroups has been motivated by their algebraic and combinatorial applications, and by their role in constructive methods for generating semigroups with prescribed embedding dimension.

Several families of two-generated numerical semigroups have been investigated with respect to their gaps and isolated gaps. The simplest case arises in $\langle a, b \rangle$, where $\gcd(a, b) = 1$, and the classical Frobenius problem. Beyond the Frobenius number, the minimum isolated gap plays an equally important role, particularly when constructing perfect semigroups in higher embedding dimensions. However, closed formulas for minimum isolated gaps are scarce and often depend sensitively on the arithmetic relation between the generators.

The concept of perfect numerical semigroups was first formally examined by Moreno Frías and Rosales [3] in 2019, where they introduced the notion of perfection and analyzed structural conditions under which a semigroup contains no isolated gaps. This pioneering work provided the groundwork for the systematic study of perfection in numerical semigroups and stimulated subsequent investigations.

In 2020, two important contributions advanced the understanding of isolated gaps and perfect semigroups. Moreno Frías and Rosales [4] extended their earlier study by focusing on perfect numerical semigroups of embedding dimension three, providing explicit constructions and characterizations. In the same year, Singh and Ram [5] analyzed isolated gaps in semigroups of embedding dimension two, clarifying their distribution and presenting examples that serve as prototypes for understanding more complex cases.

The year 2022 marked a further consolidation of research on isolated gaps. Smith [6] provided a comprehensive study on isolated gaps in numerical semigroups, offering a systematic classification and highlighting their role in structural analysis. In a related development, Smith [7] examined a family of symmetric numerical semigroups of embedding dimension three, showing how symmetry interacts with the presence or absence of isolated gaps.

Most recently, Li, Guo, and Tian [8] (2024) characterized perfect numerical semigroups in terms of pseudo-Frobenius numbers. Their approach introduced a novel perspective by linking perfection with classical invariants, thus broadening the theoretical framework within which perfection can be understood.

In 2022, Smith [6] studied the fundamental properties of isolated gaps in two-generated numerical semigroups and proposed a simple method for constructing perfect numerical semigroups of embedding dimension three. The method begins with identifying the minimum isolated gap in a two-generated semigroup and then adjoining this gap to the generating set, thereby producing a perfect semigroup of higher embedding dimension. Moreover, Smith proved that for the numerical semigroup generated by a and $a+1$, the minimum isolated gap coincides with the Frobenius number. A particularly influential contribution is due to Smith [6], who systematically investigated the notion of isolated gaps in numerical semigroups. His work not only clarified their structural role but also emphasized their importance in determining whether a semigroup is perfect. The framework and methods introduced in [6] directly inspired the present research. In particular, Smith's approach motivated us to search for explicit formulas for the minimum isolated gaps in the families $\langle a, a+2\rangle$, $\langle a, a+3\rangle$, $\langle a, a+4\rangle$, $\langle a, a+5\rangle$, and $\langle a, a+6\rangle$, thereby extending his perspective and providing concrete results that contribute to the broader program of characterizing perfect numerical semigroups.

The purpose of this paper is to establish explicit formulas for the minimum isolated gap in the families of numerical semigroups $\langle a, a+n\rangle$, for $n = 2, 3, 4, 5, 6$, under the appropriate conditions on a ensuring that the generating set indeed defines a numerical semigroup. Our approach combines direct algebraic manipulations with congruence arguments, relying on the Euclidean algorithm and the Chinese Remainder Theorem. The results generalize and unify observations made in earlier works, while also providing concrete corollaries for the construction of perfect numerical semigroups of embedding dimension three.

The paper is organized as follows. Section 2 is devoted to the main results, divided into five subsections corresponding to the semigroups $\langle a, a+2\rangle$, $\langle a, a+3\rangle$, $\langle a, a+4\rangle$, $\langle a, a+5\rangle$ and $\langle a, a+6\rangle$. In each case, we determine the minimum isolated gap and provide a rigorous proof of its minimumity. Consequences for the construction of perfect numerical semigroups are also stated as corollaries. The final section presents concluding remarks and suggests possible directions for further research on isolated gaps in more general classes of numerical semigroups.

2. Main Results

To establish the main results of this paper, we begin by recalling two fundamental results concerning two-generated numerical semigroups. These results play a central role in identifying isolated gaps and constructing perfect numerical semigroups.

The first result provides a congruence condition for isolated gaps.

Theorem 1 ([6, Proposition 3.3]). *Let $S = \langle a, b\rangle$ be a numerical semigroup where $1 < a < b$, and let h denote the minimum isolated gap of S. Then $h = 1$ or the ordered pair*

$$(h \bmod a,\ h \bmod b) \in \{(1, -1), (-1, 1)\}.$$

The second result shows that adjoining the minimum isolated gap to the generating set produces a perfect numerical semigroup.

Theorem 2 ([6, Theorem 3.1]). *Let $S = \langle a, b\rangle$ be a numerical semigroup where $1 < a < b$, and let h denote the minimum isolated gap of S. Then $\langle a, b, h\rangle$ is a perfect numerical semigroup.*

These two statements provide the theoretical framework for Section 2. Theorem 1 allows us to detect candidates for isolated gaps using modular conditions, while Theorem 2 guarantees that the resulting semigroup $\langle a, b, h\rangle$ is perfect once the minimum isolated gap has been identified. In the next sections, we apply these results to the families $\langle a, a+2\rangle$, $\langle a, a+3\rangle$, $\langle a, a+4\rangle$, $\langle a, a+5\rangle$, and $\langle a, a+6\rangle$ to obtain explicit formulas for their minimum isolated gaps.

2.1. The numerical semigroup $\langle a, a + 2\rangle$

In this subsection, we apply the general framework developed in Theorems 1 and 2 to the family of numerical semigroups generated by two consecutive odd integers, namely $\langle a, a+2\rangle$. This case represents the simplest nontrivial instance of two-generated numerical semigroups where isolated gaps arise in a structured manner.

We begin by recalling that $\langle a, a+2\rangle$ is a numerical semigroup if and only if a is odd [1], and in this situation the embedding dimension is two. The modular conditions provided by Theorem 1 allow us to detect possible candidates for isolated gaps, while Theorem 2 ensures that adjoining such a gap to the generating set yields a perfect numerical semigroup of embedding dimension three.

In what follows, we explicitly determine the minimum isolated gap of $\langle a, a+2\rangle$, prove its minimumity using congruence arguments, and state a corollary concerning the construction of perfect numerical semigroups obtained from this family.

Theorem 3. *Let a be an odd integer with $a > 1$. Then, $a + 1$ is the minimum isolated gap of the numerical semigroup $\langle a, a + 2\rangle$.*

Proof. First, we show that $a + 1 \notin \langle a, a + 2\rangle$. Indeed, if $a + 1 = xa + y(a + 2)$ for some $x, y \in \mathbb{N}$, then

$$a + 1 = (x + y)a + 2y \geq a + 2y.$$

Hence $2y \leq 1$, so $y = 0$ and thus $a + 1 = xa$, which is impossible. Therefore $a + 1$ is a gap.

Next, since $(a + 1) - 1 = a$ and $(a + 1) + 1 = a + 2$ both belong to $\langle a, a + 2\rangle$, the gap $a + 1$ is isolated.

Finally, let h be the minimum isolated gap of $\langle a, a + 2\rangle$. Write $a = 2k + 1$. Then $\gcd(a, a + 2) = 1$ and the Euclidean algorithm gives

$$1 = (k + 1)a - k(a + 2).$$

By Theorem 1, h satisfies one of the following congruence systems:

$$h \equiv 1 \pmod a \text{ and } h \equiv -1 \pmod{a+2}$$

or

$$h \equiv -1 \pmod a \text{ and } h \equiv 1 \pmod{a+2}.$$

The first system yields a residue class whose least non-negative representative is $a(1+k)+(a+2)k = a^2+a-1 < a(a+2)$, hence it lies in $\langle a, a+2\rangle$, a contradiction. Therefore the second system holds, and its least non-negative solution is $h = a+1$.

Corollary 1. *Let a be an odd integer with $a > 1$. Then, $\langle a, a+1, a+2\rangle$ is a perfect numerical semigroup.*

2.2. The numerical semigroup $\langle a, a+3\rangle$

We next turn our attention to the numerical semigroup $\langle a, a+3\rangle$. In contrast to the case of $\langle a, a+2\rangle$, the analysis here naturally divides into two distinct cases, depending on the congruence class of a modulo 3.

Lemma 1. *Let $a \in \mathbb{N}$. If $3 \nmid a$, then* $\gcd(a, a+3) = 1$.

Proof. Assume that $3 \nmid a$ and $d = \gcd(a, a+3)$. Then, $d \mid a$ and $d \mid (a+3)$. Therefore, $d \mid 3$. Hence, $d = 1$ or $d = 3$. By assumption, we can conclude that $d = 1$.

We obtain the following result.

Corollary 2. *Let a be an integer with $a > 1$. Then, $\langle a, a+3\rangle$ is a numerical semigroup if and only if $3 \nmid a$.*

More precisely, we assume throughout that a is an integer with $3 \nmid a$, so that $\langle a, a+3\rangle$ is indeed a numerical semigroup.

Case I. If $a = 3k+1$, then the structure of $\langle a, a+3\rangle$ admits a minimum isolated gap of the form $a(k+2)+1$. We will show that this element is not in the semigroup, but both its neighbors belong to the semigroup, thereby establishing it as the minimum isolated gap.

Case II. If $a = 3k+2$, then the situation changes: the minimum isolated gap is instead given by $a(k+1)-1$. A similar argument using congruence conditions ensures that this is indeed the smallest isolated gap in this case.

Thus, the determination of the minimum isolated gap for $\langle a, a+3\rangle$ requires treating these two cases separately. In the subsections that follow, we present rigorous proofs for each case and derive corollaries concerning the construction of perfect numerical semigroups.

Theorem 4. *Let a be an integer with $a > 1$. If $a = 3k+1$ for some $k \in \mathbb{N}$, then $a(k+2)+1$ is the minimum isolated gap of the numerical semigroup $\langle a, a+3\rangle$.*

Proof. Assume that $a = 3k+1$ for some $k \in \mathbb{N}$. First, we will show that $a(k+2)+1$ is a gap of $\langle a, a+3\rangle$ as follows:

$$\begin{aligned} a(k+2)+1 &= (3k+1)(k+2)+1 \\ &= 3k^2+7k+3 \\ &= k(a+3)+a+2. \end{aligned}$$

Since $a > 1$, $k > 0$, so that $a = 3k+1 > 2$. It follows that it cannot be expressed as a non-negative combination of a and $a+3$. Therefore, $a(k+2)+1 \notin \langle a, a+3\rangle$ and so $a(k+2)+1$ is a gap of $\langle a, a+3\rangle$.

Next, we show that $a(k+2)+1$ is an isolated gap of $\langle a, a+3\rangle$. Clearly, $(a(k+2)+1)-1 \in \langle a, a+3\rangle$. On the other hand, consider

$$\begin{aligned} a(k+2)+1+1 &= a(k+2)+2 \\ &= (3k+1)(k+2)+2 \\ &= 3k^2+7k+4 \\ &= (3k+4)(k+1) \\ &= (a+3)(k+1) \in \langle a, a+3\rangle. \end{aligned}$$

Hence, $a(k+2)+1$ is an isolated gap of $\langle a, a+3\rangle$.

Finally, we prove that $a(k+2)+1$ is the minimum isolated gap of $\langle a, a+3\rangle$. By Lemma 1, we obtain that $\gcd(a, a+3) = 1$. Applying the Euclidean algorithm, we have $1 = a(1+k)+(a+3)(-k)$ and by Theorem 1, we get that

$$h \equiv 1 \pmod{a} \text{ and } h \equiv -1 \pmod{a+3}$$

or

$$h \equiv -1 \pmod{a} \text{ and } h \equiv 1 \pmod{a+3}.$$

By the Chinese Remainder Theorem, it implies that

$$h \equiv a(1+k)(1)+(a+3)(-k)(-1) \pmod{a(a+3)}$$

or

$$h \equiv a(1+k)(-1)+(a+3)(-k)(1) \pmod{a(a+3)}.$$

If $h \equiv a(1+k)(1)+(a+3)(-k)(-1) \pmod{a(a+3)}$, then $h \equiv a(1+k)+(a+3)(k) \pmod{a(a+3)}$. Thus,

$$\begin{aligned} a(1+k)+(a+3)(k) &= (3k+1)(1+k)+(3k+1+3)k \\ &= 6k^2+6k+2k+1 \\ &= (3k+1)^2-3(3k+1)-3k^2-7k-3 \\ &= a^2-3a-3k^2-7k-3 \\ &< a^2+3a = a(a+3). \end{aligned}$$

It follows that $h = a(1+k)+(a+3)(k) \in \langle a, a+3\rangle$, which leads to a contradiction. Hence, $h \equiv a(1+k)(-1)+(a+3)(-k)(1) \pmod{a(a+3)}$, so that

$$\begin{aligned}
h &\equiv a(1+k)(-1)+(a+3)(-k)(1) \pmod{a(a+3)}\\
&\equiv -a-2ak-3k \pmod{a(a+3)}\\
&\equiv -a-2ak-3k+a^2+3a \pmod{a(a+3)}\\
&\equiv 2a-2ak-3k+a^2 \pmod{a(a+3)}\\
&\equiv 2(3k+1)-2(3k+1)k-3k+(3k+1)^2 \pmod{a(a+3)}\\
&\equiv (3k+1)k+2(3k+1)+1 \pmod{a(a+3)}\\
&\equiv a(k+2)+1 \pmod{a(a+3)}.
\end{aligned}$$

This means that $h = a(k+2)+1$.

Corollary 3. *Let a be an integer with $a > 1$. If $a = 3k+1$ for some $k \in \mathbb{N}$, then $\langle a, a+3, a(k+2)+1\rangle$ is a perfect numerical semigroup.*

Theorem 5. *Let a be an integer with $a > 1$. If $a = 3k+2$ for some $k \in \mathbb{N}$, then $a(k+1)-1$ is the minimum isolated gap of the numerical semigroup $\langle a, a+3\rangle$.*

Proof. Assume that $a = 3k+2$ for some $k \in \mathbb{N}$. To prove that $a(k+1)-1$ is a gap of $\langle a, a+3\rangle$, we proceed as follows:

$$\begin{aligned}
a(k+1)-1 &= (3k+2)(k+1)-1\\
&= 3k^2+5k+1\\
&= (3k+5)k+1\\
&= (a+3)k+1
\end{aligned}$$

Since $a > 1$, we get that $a(0)+(a+3)k+1 \notin \langle a, a+3\rangle$. Hence, $a(k+1)-1$ is a gap of $\langle a, a+3\rangle$. Next, we show that $a(k+1)-1$ is an isolated gap of $\langle a, a+3\rangle$. Since $a(k+1)-1 = (a+3)k+1$, it must be an isolated gap. Finally, it remains to check that $a(k+1)-1$ is the minimum isolated gap of $\langle a, a+3\rangle$. From Lemma 1, we have $\gcd(a, a+3) = 1$. By the Euclidean algorithm, we obtain $1 = (a+3)(1+k)+a(-2-k)$. It follows from Theorem 1 that

$$h \equiv 1 \pmod{a} \text{ and } h \equiv -1 \pmod{a+3}$$

or

$$h \equiv -1 \pmod{a} \text{ and } h \equiv 1 \pmod{a+3}.$$

By the Chinese Remainder Theorem, the following congruence holds:

$$h \equiv a(-2-k)(1)+(a+3)(1+k)(-1) \pmod{a(a+3)}$$

or

$$h \equiv a(-2-k)(-1)+(a+3)(1+k)(1) \pmod{a(a+3)}.$$

If $h \equiv a(-2-k)(-1)+(a+3)(1+k)(1) \pmod{a(a+3)}$, then $h \equiv a(2+k)+(a+3)(1+k) \pmod{a(a+3)}$. Therefore,

$$\begin{aligned} a(2+k)+(a+3)(1+k) &= (3k+2)(2+k)+(3k+2+3)(1+k) \\ &= 6k^2+12k+4k+4+5 \\ &= (3k+2)^2+3(3k+2)-3k^2-5k-1 \\ &= a^2+3a-3k^2-5k-1 \\ &< a^2+3a = a(a+3). \end{aligned}$$

Thus, $h = a(2+k)+(a+3)(1+k) \in \langle a, a+3\rangle$, leading to a contradiction. Hence, $h \equiv a(-2-k)(1)+(a+3)(1+k)(-1) \pmod{a(a+3)}$ and so

$$\begin{aligned} h &\equiv a(-2-k)(1)+(a+3)(1+k)(-1) \pmod{a(a+3)} \\ &\equiv -2a-ak-a-ak-3-3k \pmod{a(a+3)} \\ &\equiv -2a-ak-a-ak-3-3k+a^2+3a \pmod{a(a+3)} \\ &\equiv -2ak-3-3k+a^2 \pmod{a(a+3)} \\ &\equiv -2(3k+2)k-3-3k-(3k+2)^2 \pmod{a(a+3)} \\ &\equiv -6k^2-4k-3-3k+9k^2+12k+4 \pmod{a(a+3)} \\ &\equiv 3k^2+2k+3k+2-1 \pmod{a(a+3)} \\ &\equiv (3k+2)k+(3k+2)-1 \pmod{a(a+3)} \\ &\equiv ak+a-1 \pmod{a(a+3)} \\ &\equiv a(k+1)-1 \pmod{a(a+3)}. \end{aligned}$$

This means that $h = a(k+1)-1$.

Corollary 4. *Let a be an integer with $a > 1$. If $a = 3k+2$ for some $k \in \mathbb{N}$, then $\langle a, a+3, a(k+1)-1\rangle$ is a perfect numerical semigroup.*

2.3. The numerical semigroup $\langle a, a+4\rangle$

Finally, we consider the numerical semigroup $\langle a, a+4\rangle$. As in the previous case, the analysis depends crucially on the congruence class of a. Since a must be odd in order for $\langle a, a+4\rangle$ to be a numerical semigroup, there are two possibilities:

Case I. If $a = 4k+1$, then the minimum isolated gap is of the form $a(2k+3)+1$. We shall demonstrate that this element does not belong to the semigroup, while both of its neighbors do, thereby establishing its isolated nature and minimumity.

Case II. If $a = 4k+3$, the minimum isolated gap changes to $a(2k+3)-1$. By explicit computation and congruence arguments, we show that this element serves as the smallest isolated gap for this family.

Thus, as in the case of $\langle a, a+3\rangle$, the determination of the minimum isolated gap in $\langle a, a+4\rangle$ requires a case-by-case analysis. In what follows, we present the detailed proofs for both cases, together with corollaries that yield new examples of perfect numerical semigroups of embedding dimension three.

Theorem 6. *Let a be an odd integer with $a > 1$. If $a = 4k + 1$ for some $k \in \mathbb{N}$, then $a(2k + 3) + 1$ is the minimum isolated gap of $\langle a, a + 4\rangle$.*

Proof. First suppose that $a = 4k+1$ for some $k \in \mathbb{N}$. We will now show that $a(2k+3)+1$ is a gap of $\langle a, a + 4\rangle$. Consider

$$\begin{aligned} a(2k + 3) + 1 &= (4k + 1)(2k + 3) + 1 \\ &= 8k^2 + 14k + 4 \\ &= 2k(4k + 5) + 4k + 4 \\ &= (a + 4)2k + a + 3. \end{aligned}$$

Since $k \neq 0$, we have $a = 4k + 1 > 3$. This implies that $a(2k + 3) + 1 \notin \langle a, a + 4\rangle$ and hence, $a(2k + 3) + 1$ is a gap of $\langle a, a + 4\rangle$. Next, we will show that $a(2k + 3) + 1$ is an isolated gap of $\langle a, a + 4\rangle$. We compute

$$(a(2k + 3) + 1) - 1 = a(2k + 3)$$

and

$$a(2k + 3) + 1 + 1 = (a + 4)2k + a + 4 = (a + 4)(2k + 1).$$

Hence the result follows.

Finally, we prove that $a(2k+3)+1$ is the minimum isolated gap of $\langle a, a+4\rangle$. It follows from $\gcd(a, a + 4) = 1$ and the Euclidean algorithm that $1 = a(1 + k) + (a + 4)(-k)$. By Theorem 1 and the Chinese Remainder Theorem, it follows that $h \equiv a(1 + k)(1) + (a + 4)(-k)(-1) \pmod{a(a + 4)}$ or $h \equiv a(1 + k)(-1) + (a + 4)(-k)(1) \pmod{a(a + 4)}$. If $h \equiv a(1 + k)(1) + (a + 4)(-k)(-1) \pmod{a(a + 4)}$, then $h \equiv a(1 + k) + (a + 4)(k) \pmod{a(a + 4)}$ and so

$$\begin{aligned} a(1 + k) + (a + 4)(k) &= (4k + 1)(1 + k) + (4k + 1 + 4)k \\ &= 16k^2 + 8k + 1 + 2k - 8k^2 \\ &= (4k + 1)^2 - 8k^2 + 16k - 14k + 4 - 4 \\ &= a^2 + 4(4k + 1) - 8k^2 - 14k - 4 \\ &= a^2 + 4a - 8k^2 - 14k - 4 \\ &< a^2 + 4a = a(a + 4). \end{aligned}$$

Therefore, $h = a(1 + k) + (a + 4)(k) \in \langle a, a + 4\rangle$, which yields a contradiction. Hence, $h \equiv a(1 + k)(-1) + (a + 4)(-k)(1) \pmod{a(a + 4)}$, so that

$$\begin{aligned} h &\equiv a(1 + k)(-1) + (a + 4)(-k)(1) \pmod{a(a + 4)} \\ &\equiv -a - 2ak - 4k \pmod{a(a + 4)} \end{aligned}$$

$$
\begin{aligned}
&\equiv -a - 2ak - 4k + a^2 + 4a \pmod{a(a+4)}\\
&\equiv 3a - 2ak - 4k + a^2 \pmod{a(a+4)}\\
&\equiv 3(4k+1) - 2(4k+1)k - 4k + (4k+1)^2 \pmod{a(a+4)}\\
&\equiv 2(4k+1)k + 3(4k+1) + 1 \pmod{a(a+4)}\\
&\equiv a(2k+3) + 1 \pmod{a(a+4)}.
\end{aligned}
$$

Hence, we conclude that $h = a(2k+3)+1$.

Corollary 5. *Let a be an odd integer with $a > 1$. If $a = 4k+1$ for some $k \in \mathbb{N}$, then $\langle a, a+4, a(2k+3)+1\rangle$ is a perfect numerical semigroup.*

Theorem 7. *Let a be an odd integer with $a > 1$. If $a = 4k+3$ for some $k \in \mathbb{N}$, then $a(2k+3)-1$ is the minimum isolated gap of $\langle a, a+4\rangle$.*

Proof. Suppose that $a = 4k+3$ for some $k \in \mathbb{N}$. We shall prove that $a(2k+3)-1$ is a gap of $\langle a, a+4\rangle$. Consider the following computation:

$$
\begin{aligned}
a(2k+3)-1 &= (4k+3)(2k+3)-1\\
&= 8k^2 + 18k + 8\\
&= (4k+7)2k + 4k + 7 + 1\\
&= (a+4)2k + (a+4) + 1\\
&= (a+4)(2k+1) + 1.
\end{aligned}
$$

Since $k \neq 0$ and $a = 4k+3 > 1$, the above representation shows that $a(2k+3)-1$ cannot be written as a non-negative integer linear combination of a and $a+4$. In other words, $a(2k+3)-1 \notin \langle a, a+4\rangle$, that is $a(2k+3)-1$ is a gap of $\langle a, a+4\rangle$. By above, we have $a(2k+3)-1 = (a+4)(2k+1)+1$. This implies that $a(2k+3)-1$ is an isolated gap of $\langle a, a+4\rangle$. Furthermore, we now show that $a(2k+3)-1$ is the minimum isolated gap of $\langle a, a+4\rangle$. By $\gcd(a, a+4) = 1$ and using the Euclidean algorithm, we obtain $1 = (a+4)(1+k) + a(-2-k)$. By Theorem 1 and applying the Chinese Remainder Theorem, we then have

$$h \equiv a(-2-k)(1) + (a+4)(1+k)(-1) \pmod{a(a+4)}$$

or

$$h \equiv a(-2-k)(-1) + (a+4)(1+k)(1) \pmod{a(a+4)}.$$

Suppose on the contrary that

$$h \equiv a(-2-k)(-1) + (a+4)(1+k)(1) \pmod{a(a+4)}.$$

Then, $h \equiv a(2+k) + (a+4)(1+k) \pmod{a(a+4)}$. Thus,

$$a(2+k) + (a+4)(1+k) = (4k+3)(2+k) + (4k+3+4)(1+k)$$

$$
\begin{aligned}
&= 8k^2 + 22k + 9 + 4 \\
&= (4k+3)^2 + 4(4k+3) - 18k - 8 - 8k^2 \\
&= a^2 - 4a - 8k^2 - 18k - 8 \\
&< a^2 + 4a = a(a+4).
\end{aligned}
$$

Therefore, $h = a(2+k) + (a+4)(1+k) \in \langle a, a+4\rangle$, which results in a contradiction. Hence, $h \equiv a(-2-k)(1) + (a+4)(1+k)(-1) \pmod{a(a+4)}$. It follows that

$$
\begin{aligned}
h &\equiv a(-2-k)(1) + (a+4)(1+k)(-1) \pmod{a(a+4)} \\
&\equiv -2a - ak - a - ak - 4 - 4k \pmod{a(a+4)} \\
&\equiv -3a - 2ak - 4 - 4k + a^2 + 4a \pmod{a(a+4)} \\
&\equiv a - 2ak - 4 - 4k + a^2 \pmod{a(a+4)} \\
&\equiv (4k+3) - 2(4k+3)k - 4 - 4k + (4k+3)^2 \pmod{a(a+4)} \\
&\equiv 4k + 3 - 8k^2 - 6k - 4 - 4k + 16k^2 + 24k + 9 \pmod{a(a+4)} \\
&\equiv 8k^2 + 6k + 12k + 9 - 1 \pmod{a(a+4)} \\
&\equiv a(4k+3)k + 3(4k+3) - 1 \pmod{a(a+4)} \\
&\equiv a(2k+3) - 1 \pmod{a(a+4)}.
\end{aligned}
$$

Hence, $h = a(2k+3) - 1$.

Corollary 6. *Let a be an odd integer with $a > 1$. If $a = 4k+3$ for some $k \in \mathbb{N}$, then $\langle a, a+4, a(2k+3)-1\rangle$ is a perfect numerical semigroup.*

2.4. The numerical semigroup $\langle a, a+5\rangle$

We now investigate the numerical semigroup $\langle a, a+5\rangle$. As in the previous sections, the structure of this semigroup depends essentially on the arithmetic properties of the generator a. In particular, $\langle a, a+5\rangle$ is a numerical semigroup if and only if $\gcd(a, a+5) = 1$, which is equivalent to the condition that $5 \nmid a$. Under this assumption, the determination of the minimum isolated gap naturally separates into several cases according to the congruence class of a modulo 5. More precisely, we consider the cases

$$a = 5k+1, a = 5k+2, a = 5k+3, \text{ and } a = 5k+4,$$

each of which gives rise to a different explicit expression for the minimum isolated gap.

Lemma 2. *Let $a \in \mathbb{N}$. If $5 \nmid a$, then $\gcd(a, a+5) = 1$.*

Proof. Since $\gcd(a, a+5) = \gcd(a, 5)$, the result follows immediately.

We also have the next corollary.

Corollary 7. *Let a be an integer with $a > 1$. Then $\langle a, a+5\rangle$ is a numerical semigroup if and only if $5 \nmid a$.*

Theorem 8. *Let a be an integer with $a > 1$. If $a = 5k + 1$ for some $k \in \mathbb{N}$, then $a(3k + 4) + 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$.*

Proof. Assume that $a = 5k+1$ for some $k \in \mathbb{N}$. Firstly, we will show that $a(3k+4)+1$ is a gap. Noting that

$$\begin{aligned} a(3k+4)+1 &= (5k+1)(3k+4)+1 \\ &= 15k^2 + 23k + 5 \\ &= 3k(5k+6) + 5k + 5 \\ &= 3k(a+5) + a + 4, \end{aligned}$$

and $a = 5k+1 > 4$, it means that $3k(a+5)+a+4 \notin \langle a, a+5\rangle$ Hence, $a(3k+4)+1 \notin \langle a, a+5\rangle$. Thus, the result follows.

For the next step, we will show that $a(3k + 4) + 1$ is an isolated gap. It is easy to see that $a(3k+4)+1-1 \in \langle a, a+5\rangle$ and $a(3k+4)+1+1 = (3k+1)(a+5)$. We can conclude that $a(3k + 4) + 1$ is an isolated gap.

To prove minimality, since $\gcd(a, a + 5) = 1$ and $a = 5k + 1$, the Euclidean algorithm yields

$$1 = (k + 1)a - k(a + 5).$$

By Theorem 1, h satisfies one of the congruence systems

(i) $h \equiv 1 \pmod{a}, \quad h \equiv -1 \pmod{a+5}$ or

(ii) $h \equiv -1 \pmod{a}, \quad h \equiv 1 \pmod{a+5}$.

Using the constructive Chinese Remainder Theorem with the above identity, system (i) gives

$$h \equiv (k+1)a \cdot 1 + \big(-k(a+5)\big) \cdot (-1) \equiv (k+1)a + k(a+5) \pmod{a(a+5)}.$$

The least non-negative representative of this residue class is

$$(k + 1)a + k(a + 5) \in \langle a, a + 5\rangle,$$

contradicting that h is a gap. Hence system (ii) must hold. In the same manner, system (ii) yields

$$h \equiv (k+1)a \cdot (-1) + \big(-k(a+5)\big) \cdot 1 \equiv -(k+1)a - k(a+5) \pmod{a(a+5)}.$$

Taking the least non-negative representative, we obtain

$$h = a(a + 5) - (k + 1)a - k(a + 5) = a(3k + 4) + 1.$$

Therefore, $h = a(3k + 4) + 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$. This completes the proof.

Corollary 8. *Let a be an integer with $a > 1$. If $a = 5k + 1$ for some $k \in \mathbb{N}$, then $\langle a, a+5, a(3k+4)+1\rangle$ is a perfect numerical semigroup.*

Theorem 9. *Let a be an integer with $a > 1$. If $a = 5k+2$ for some $k \in \mathbb{N}$, then $a(k+1)-1$ is the minimum isolated gap of $\langle a, a+5\rangle$.*

Proof. Suppose that $a = 5k + 2$ for some $k \in \mathbb{N}$. First, we show that $a(k+1) - 1$ is a gap, consider

$$\begin{aligned} a(k+1) - 1 &= (5k+2)(k+1) - 1 \\ &= 5k^2 + 7k + 1 \\ &= (5k+7)k + 1 \\ &= (a+5)(k) + 1 \end{aligned}$$

Since $a > 1$, we get that $(a+5)(k) + 1 \notin \langle a, a+5\rangle$, that is $a(k+1) - 1 \notin \langle a, a+5\rangle$. This means that $a(k+1) - 1$ is a gap.

The next, observe that

$$a(k+1) - 1 + 1 = a(k+1) \in \langle a, a+5\rangle \text{ and } a(k+1) - 1 - 1 = (a+5)k \in \langle a, a+5\rangle,$$

meaning $a(k+1) - 1$ is an isolated gap.

Finally, we will show that $a(k+1)-1$ is the minimum isolated gap of $\langle a, a+5\rangle$. Since $\gcd(a, a+5) = 1$, the Euclidean algorithm gives

$$1 = (-3-2k)a + (1+2k)(a+5).$$

By Theorem 1, h satisfies either

(i) $h \equiv 1 \pmod{a}$, $\quad h \equiv -1 \pmod{a+5}$ or

(ii) $h \equiv -1 \pmod{a}$, $\quad h \equiv 1 \pmod{a+5}$.

Applying the constructive Chinese Remainder Theorem, system (i) yields

$$h \equiv (-3-2k)a \cdot 1 + (1+2k)(a+5) \cdot (-1) \equiv (-3-2k)a - (1+2k)(a+5) \pmod{a(a+5)}.$$

Thus, the least non-negative representative is

$$h = a(a+5) - (3+2k)a - (1+2k)(a+5) = a(k+1) - 1.$$

On the other hand, system (ii) gives

$$h \equiv (-3-2k)a \cdot (-1) + (1+2k)(a+5) \cdot 1 \equiv (3+2k)a + (1+2k)(a+5) \pmod{a(a+5)},$$

whose least non-negative representative lies in $\langle a, a+5\rangle$, a contradiction. Therefore $h = a(k+1) - 1$ is the minimum isolated gap of $\langle a, a+5\rangle$.

Corollary 9. *Let a be an integer with $a > 1$. If $a = 5k + 2$ for some $k \in \mathbb{N}$, then $\langle a, a + 5, a(k + 1) - 1\rangle$ is a perfect numerical semigroup.*

Theorem 10. *Let a be an integer with $a > 1$. If $a = 5k + 3$ for some $k \in \mathbb{N}$, then $a(k + 2) + 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$.*

Proof. Suppose that $a = 5k + 3$ for some $k \in \mathbb{N}$. We will show that $a(k + 2) + 1$ is a gap. Noting that

$$\begin{aligned} a(k + 2) + 1 &= (5k + 3)(k + 2) + 1 \\ &= 5k^2 + 13k + 7 \\ &= (5k + 8)k + (5k + 7) \\ &= (a + 5)k + a + 4 \end{aligned}$$

and $a > 1$, we obtain that $a(k + 2) + 1 \notin \langle a, a + 5\rangle$. Consequently, $a(k + 2) + 1$ is a gap.

The next, observe that

$$a(k + 2) + 1 + 1 = (a + 5)(k + 1) \in \langle a, a + 5\rangle \text{ and } a(k + 2) + 1 - 1 = a(k + 2) \in \langle a, a + 5\rangle,$$

meaning $a(k + 2) + 1$ is an isolated gap.

Finally, we will prove that $a(k+2)+1$ is the minimum isolated gap. Since $\gcd(a, a+5) = 1$, the Euclidean algorithm gives

$$1 = (3 + 2k)a - (2k + 1)(a + 5).$$

By Theorem 1, h satisfies either

(i) $h \equiv 1 \pmod{a}$, $h \equiv -1 \pmod{a + 5}$ or

(ii) $h \equiv -1 \pmod{a}$, $h \equiv 1 \pmod{a + 5}$.

Using the constructive Chinese Remainder Theorem, system (i) yields

$$h \equiv (3+2k)a \cdot 1 + \big(-(2k+1)(a+5)\big) \cdot (-1) \equiv (3+2k)a + (2k+1)(a+5) \pmod{a(a+5)}.$$

The least non-negative representative of this residue class belongs to $\langle a, a + 5\rangle$, contradicting that h is a gap. Hence system (ii) must hold. Then

$$h \equiv (3+2k)a \cdot (-1) + \big(-(2k+1)(a+5)\big) \cdot 1 \equiv -(3+2k)a - (2k+1)(a+5) \pmod{a(a+5)}.$$

Taking the least non-negative representative gives

$$h = a(a + 5) - (3 + 2k)a - (2k + 1)(a + 5) = a(k + 2) + 1.$$

Therefore, $h = a(k + 2) + 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$.

Corollary 10. *Let a be an integer with $a > 1$. If $a = 5k + 3$ for some $k \in \mathbb{N}$, then $\langle a, a + 5, a(k + 2) + 1\rangle$ is a perfect numerical semigroup.*

Theorem 11. *Let a be an integer with $a > 1$. If $a = 5k + 4$ for some $k \in \mathbb{N}$, then $a(3k + 5) - 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$.*

Proof. Suppose that $a = 5k + 4$ for some $k \in \mathbb{N}$. We will start by showing that $a(3k + 5) - 1$ is a gap. Consider

$$\begin{aligned} a(3k + 5) - 1 &= (5k + 4)(3k + 5) - 1 \\ &= 15k^2 + 37k + 19 \\ &= (5k + 9)(3k + 2) + 1 \\ &= (a + 5)(3k + 2) + 1 \end{aligned}$$

From $a > 1$, we deduce that $a(3k + 5) - 1 \notin \langle a, a + 5\rangle$ and hence, $a(3k + 5) - 1$ is a gap. Next, we verify that h is isolated. Indeed,

$$a(3k+5)-1-1 = (a+5)(3k+2) \in \langle a, a+5\rangle \text{ and } a(3k+5)-1+1 = a(3k+5) \in \langle a, a+5\rangle.$$

Thus h is an isolated gap.

Finally, we need to show that $h = a(3k+5)-1$ is the minimum isolated gap of $\langle a, a+5\rangle$. Since $\gcd(a, a + 5) = 1$, the Euclidean algorithm yields

$$1 = (-2 - k)a + (1 + k)(a + 5).$$

By Theorem 1, h satisfies either

(i) $h \equiv 1 \pmod a$, $h \equiv -1 \pmod{a + 5}$ or

(ii) $h \equiv -1 \pmod a$, $h \equiv 1 \pmod{a + 5}$.

Applying the constructive Chinese Remainder Theorem, system (ii) gives

$$h \equiv (-2 - k)a \cdot (-1) + (1 + k)(a + 5) \cdot 1 \equiv (2 + k)a + (1 + k)(a + 5) \pmod{a(a + 5)},$$

whose least non-negative representative lies in $\langle a, a+5\rangle$, a contradiction. Hence system (i) must hold, and

$$h \equiv (-2 - k)a \cdot 1 + (1 + k)(a + 5) \cdot (-1) \equiv (-2 - k)a - (1 + k)(a + 5) \pmod{a(a + 5)}.$$

Taking the least non-negative representative, we obtain

$$h = a(a + 5) - (2 + k)a - (1 + k)(a + 5) = a(3k + 5) - 1.$$

Therefore, $h = a(3k + 5) - 1$ is the minimum isolated gap of $\langle a, a + 5\rangle$.

Corollary 11. *Let a be an integer with $a > 1$. If $a = 5k + 4$ for some $k \in \mathbb{N}$, then $\langle a, a + 5, a(3k + 5) - 1\rangle$ is a perfect numerical semigroup.*

2.5. The numerical semigroup $\langle a, a+6\rangle$

In this subsection, we extend the previous analysis to the numerical semigroup $\langle a, a+6\rangle$. As before, this semigroup is numerical if and only if $\gcd(a, a+6) = 1$, which holds precisely when a is relatively prime to 6. Under this condition, the structure of $\langle a, a+6\rangle$ exhibits a richer behavior compared with the earlier cases.

We begin with the following characterization.

Lemma 3. *Let a be an odd integer with $a > 1$. If $3 \nmid a$, then* $\gcd(a, a+6) = 1$.

Proof. Assume that a is odd and $3 \nmid a$. Let $d = \gcd(a, a+6)$. Then, $d \mid a$ and $d \mid (a+6)$, and so $d \mid 6$. Therefore, $d \in \{1, 2, 3, 6\}$. By assumption, we get that $d = 1$, and hence, $\gcd(a, a+6) = 1$.

The next result is immediate from Lemma 3.

Corollary 12. *Let a be an odd integer with $a > 1$. Then, $\langle a, a+6\rangle$ is a numerical semigroup if and only if $3 \nmid a$.*

Hence, we assume that a is odd and $3 \nmid a$.

Theorem 12. *Let a be an integer with $a > 1$. If $a = 6k+1$ for some $k \in \mathbb{N}$, then $a(4k+5)+1$ is the minimum isolated gap of the numerical semigroup $\langle a, a+6\rangle$.*

Proof. Suppose that $a = 6k+1$ for some $k \in \mathbb{N}$. First, we show that $I \notin \langle a, a+6\rangle$. Writing

$$I = a(4k+5)+1 = 4k(a+6)+a+5,$$

it follows that I cannot be expressed as a non-negative combination of a and $a+6$, hence it is a gap.

Next, observe that

$$I-1 = a(4k+5) \in \langle a, a+6\rangle, \qquad I+1 = (a+6)(4k+1) \in \langle a, a+6\rangle,$$

so I is an isolated gap.

Finally, since $\gcd(a, a+6) = 1$ and $a = 6k+1$, the Euclidean algorithm gives

$$1 = a(1+k) + (a+6)(-k).$$

By Theorem 1, the minimum isolated gap I satisfies one of the two congruence systems

$$h \equiv 1 \pmod{a}, \quad h \equiv -1 \pmod{a+6},$$

or

$$h \equiv -1 \pmod{a}, \quad h \equiv 1 \pmod{a+6}.$$

By the Chinese Remainder Theorem, we obtain that

$$h \equiv a(1+k)(1) + (a+6)(-k)(-1) \pmod{a(a+6)}$$

or

$$h \equiv a(1+k)(-1) + (a+6)(-k)(1) \pmod{a(a+6)}.$$

The first system yields a representative strictly less than $a(a+6)$ which lies in $\langle a, a+6\rangle$, a contradiction. Therefore the second system holds, and the unique solution modulo $a(a+6)$ is $h = a(4k+5)+1$, completing the proof.

Corollary 13. *Let a be an integer with $a > 1$. If $a = 6k+1$ for some $k \in \mathbb{N}$, then $\langle a, a+6, a(4k+5)+1\rangle$ is a perfect numerical semigroup.*

Theorem 13. *Let a be an integer with $a > 1$. If $a = 6k+5$ for some $k \in \mathbb{N}$, then $a(4k+7)-1$ is the minimum isolated gap of the numerical semigroup $\langle a, a+6\rangle$.*

Proof. We write

$$a(4k+7) - 1 = (a+6)(4k+3) + 1,$$

which shows that $h \notin \langle a, a+6\rangle$, hence h is a gap.

Moreover,

$$I - 1 = a(4k+7) \in \langle a, a+6\rangle, \qquad I + 1 = (a+6)(4k+3) \in \langle a, a+6\rangle,$$

hence, h is isolated.

Since $\gcd(a, a+6) = 1$ and $a = 6k+5$, the Euclidean algorithm yields

$$1 = a(-2-k) + (a+6)(1+k).$$

By Theorem 1 and the Chinese Remainder Theorem, the only admissible congruence class gives $h = a(4k+7) - 1$.

Corollary 14. *Let a be an integer with $a > 1$. If $a = 6k+5$ for some $k \in \mathbb{N}$, then $\langle a, a+6, a(4k+7)-1\rangle$ is a perfect numerical semigroup.*

3. Conclusion

In this paper, we have established explicit formulas for the minimum isolated gaps of numerical semigroups of the form

$$S = \langle a, a+n\rangle,$$

for $n = 2, 3, 4, 5$, and 6, under the natural condition that $\gcd(a, a+n) = 1$. Our approach relies on congruence conditions for isolated gaps together with systematic applications of the Euclidean algorithm and the Chinese Remainder Theorem.

For each value of n, closed-form expressions for the minimum isolated gap were obtained according to the residue class of a modulo n. These formulas provide a complete description of isolated gaps in the considered families and unify various particular cases previously observed.

Furthermore, the determination of minimum isolated gaps enables the construction of perfect numerical semigroups of embedding dimension three by adjoining the isolated gap to the generating set. This confirms the fundamental role of isolated gaps in the structure of numerical semigroups and their connection with perfection.

For the convenience of the reader, the main results are summarized in Table 1.

Table 1: The minimum isolated gaps of $\langle a, a+n \rangle$

n	Condition on a	Minimum isolated gaps
2	a is odd	$a+1$
3	$a = 3k+1$	$a(k+2)+1$
	$a = 3k+2$	$a(k+1)-1$
4	$a = 4k+1$	$a(2k+3)+1$
	$a = 4k+3$	$a(2k+3)-1$
5	$a = 5k+1$	$a(3k+4)+1$
	$a = 5k+2$	$a(k+1)-1$
	$a = 5k+3$	$a(k+2)+1$
	$a = 5k+4$	$a(3k+5)-1$
6	$a = 6k+1$	$a(4k+5)+1$
	$a = 6k+5$	$a(4k+7)-1$

It would be natural to extend the present analysis to numerical semigroups of the form $\langle a, a+n \rangle$ for larger values of n. The patterns observed for $n \leq 6$ suggest that the minimum isolated gap depends strongly on the residue class of a modulo n. Determining a unified formula for arbitrary n or identifying general structural patterns remains an interesting direction for future research.

Finally, the techniques developed in this paper suggest possible extensions to other families of numerical semigroups generated by arithmetic progressions or more general pairs of integers. Further investigations in this direction may reveal deeper connections between isolated gaps and classical invariants of numerical semigroups.

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