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New Results on Parameterized Mercer Type Inequalities: Analysis and Applications

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Abstract. In this paper, we establish a Mercer-parameterized integral identity for differentiable functions. Using this identity, we derive a family of new parameterized Hermite-Hadamard Mercer type inequalities for differentiable convex functions. The proposed framework provides a unified approach for generating several classical Mercer type bounds, and many well-known related inequalities can be obtained as particular cases through appropriate selections of the involved parameters. In addition, to validate our findings we present graphical illustrations. Moreover, the presented inequalities extend and generalize a number of existing results available in the literature. Finally, applications to the modified Bessel function, the q-digamma function, special means, and numerical quadrature schemes are presented.

2020 Mathematics Subject Classifications: 26D07, 26D10, 26D15, 26A33, 26A51

Key Words and Phrases: Mercer-midpoint type inequalities, convex function, power-mean inequality, Mercer Bullen type inequalities, Mercer Simspons type inequalities, modified Bessel function, q-digamma function, quadrature schemes

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1. Introduction

Convex analysis is a vital area of mathematical research, contributing significantly to fields such as numerical analysis, convex programming, optimization, and statistics. In particular, Convex functions have been widely investigated because of their fundamental role in these disciplines.

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Mathematical inequalities also hold great importance across various areas, including special functions, physics, fractals, and number theory, making them a key subject of interest for researchers. Their broad applicability has fueled ongoing investigations, leading to advancements in multiple branches of mathematics.

One of the most well-known inequalities in this context is the Hermite–Hadamard inequality, which has been extensively extended and generalized over the past decade. Due to its mathematical significance, it has been the focus of numerous studies. Further research into this inequality has led to the derivation of other important results, such as the midpoint, Simpson, Ostrowski, and trapezoidal inequalities, as well as a wide range of Hermite–Hadamard-type inequalities; see for example references [1–5]. Additionally, some studies have explored Hermite–Hadamard-like inequalities for functions whose third derivatives are convex in absolute value, providing deeper insights into this area of mathematical analysis [6, 7].

We begin by recalling the definition of convex functions, as outlined below [8].

Definition 1. *A function $F : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on the interval I of a real number and $\hbar_1, \hbar_2 \in I$ with $\varkappa \in [0,1]$:*

$$F(\varkappa\hbar_1 + (1-\varkappa)\hbar_2) \leq \varkappa F(\hbar_1) + (1-\varkappa) F(\hbar_2).$$

The well-established Hermite–Hadamard inequality in the setting of convex functions, as documented in the literature [9], is given by

$$F\left(\frac{\hbar_1 + \hbar_2}{2}\right) \leq \frac{1}{\hbar_2 - \hbar_1} \int_{\hbar_1}^{\hbar_2} F(x)\, dx \leq \frac{F(\hbar_1) + F(\hbar_2)}{2}, \tag{1}$$

where $F : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $\varkappa \in [0,1]$ is said to be convex on the interval I. Furthermore, if F is concave, then inequality (1) holds in the reverse direction. This result was first proposed by C. Hermite in [10] and later examined in depth by J. Hadamard in [9].

In 2003, Mercer introduced a notable variant of Jensen's inequality, commonly known as the Jensen-Mercer inequality, which is stated below [11].

Let F be a convex function and let u_i be non-negative weights such that $\sum_{i=1}^{n} u_i = 1$, the inequality is stated as:

$$F\left(\sum_{i=1}^{n} u_i x_i\right) \leq \sum_{i=1}^{n} u_i F(x_i),$$

This formulation extends the classical Jensen's inequality by incorporating a weighted average of points x_i.

Theorem 1. *For a convex mapping $F : [\hbar_1, \hbar_2] \rightarrow \mathbb{R}$, the following inequality holds for all values of $x_i \in [\hbar_1, \hbar_2]\ (i = 1, 2, 3, ..., n)$:*

$$F\left(\hbar_1 + \hbar_2 - \sum_{i=1}^{n} u_i x_i\right) \leq F(\hbar_1) + F(\hbar_2) - \sum_{i=1}^{n} u_i F(x_i),$$

where $u_i \in [0,1]$ $(i = 1,2,3,...,n)$ and $\sum_{i=1}^{n} u_i = 1$.

Moradi and Furuichi extended and refined Jensen–Mercer inequalities in [12]. Adil Khan [13] emphasized the usefulness of the Jensen–Mercer inequality within the framework of information theory where new approximations for Csiszár and related divergence measures were obtained. Moreover, the Jensen–Mercer inequality to derive novel boundaries for Zipf–Mandelbrot entropy. Kian et al. [14] applied the newly established Jensen inequality to derive novel formulations of the Hermite–Hadamard inequality, as presented below:

Theorem 2. *Let $F : [\hbar_1, \hbar_2] \to \mathbb{R}$ be a function. Then, for any $\ddot{y}_1, \ddot{y}_2 \in [\hbar_1, \hbar_2]$ with $\ddot{y}_1 < \ddot{y}_2$, the following inequality holds:*

$$\begin{aligned} F\left(\hbar_1 + \hbar_2 - \frac{\ddot{y}_1 + \ddot{y}_2}{2}\right) &\leq F(\hbar_1) + F(\hbar_2) - \frac{1}{\ddot{y}_2 - \ddot{y}_1} \int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du \\ &\leq F(\hbar_1) + F(\hbar_2) - F\left(\frac{\ddot{y}_1 + \ddot{y}_2}{2}\right), \end{aligned}$$

and

$$\begin{aligned} F\left(\hbar_1 + \hbar_2 - \frac{\ddot{y}_1 + \ddot{y}_2}{2}\right) &\leq \frac{1}{\ddot{y}_2 - \ddot{y}_1} \int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du \\ &\leq \frac{F(\hbar_1 + \hbar_2 - \ddot{y}_1) + F(\hbar_1 + \hbar_2 - \ddot{y}_2)}{2} \\ &\leq F(\hbar_1) + F(\hbar_2) - \frac{F(\ddot{y}_1) + F(\ddot{y}_2)}{2}. \end{aligned} \tag{2}$$

Remark 1. *If we put $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$ in inequality (2) this yields the classical Hermite–Hadamard inequality.*

In recent times, numerous researchers have extended and refined the Hermite–Hadamard-Mercer inequalities in various directions; see, for example, [15–17]. In particular, new versions of these inequalities for Riemann–Liouville fractional integrals were introduced in [18, 19], while [20] explored similar results for fractional integrals and for generalized fractional integrals involving differentiable functions. The study in [21] established improved bounds for Ostrowski–type inequalities via the Jensen–Mercer inequality, whereas [22] employed convexity concepts for interval-valued functions to derive fractional Hermite–Hadamard-Mercer-type inequalities. Additionally, [23] presented several such inequalities for harmonically convex functions. Further developments in this direction were reported by Butt et al. [24], who obtained Hermite-Jensen Mercer type inequalities for convex functions via generalized fractional integral operators. Moreover, Butt et al. [25] established several Ostrowski-Mercer type inequalities for differentiable convex functions using fractional integral operators with strong kernels.

On the other hand, convex analysis has gained significant attention in both fundamental and advanced mathematics, playing a pivotal role in the generalization and extension of inequality theory. Among these, the Hermite–Hadamard inequality stands out as a fundamental result for convex functions, offering a straightforward geometric interpretation and numerous applications. For further insights into Hermite–Hadamard-type inequalities, see [26, 27]. Several researchers have contributed to the development of Hermite–Hadamard-type inequalities using different fractional integral operators. Sarikaya et al. [28] established new inequalities by applying Riemann–Liouville fractional integrals within the framework of convex analysis. Similarly, Set et al. [29] utilized Riemann–Liouville integrals to derive Ostrowski–type inequalities for differentiable functions. Recently, Hyder et al. [30] investigated generic and trapezoidal-shaped fractional inequalities. Xiaobin Wang et al. [31] explored Hermite–Hadamard-type inequalities for modified h-convex functions using the Caputo–Fabrizio fractional integral operator. Abbasi et al. [32] developed new variants of these inequalities for s-convex functions. Akdemir et al. [33] further contributed by employing Atangana-Baleanu fractional integral operators for concave and convex functions. In addition, Ali et al. [34] introduced new Ostrowski-type integral inequalities using a multi-step linear kernel approach, which provides refined bounds for integral inequalities. Recently, Özdemir et al. [35] derived several inequalities involving Caputo and Liouville fractional integral operators for *m*-convex functions, further enriching the literature on fractional integral inequalities.

Simpson-type inequalities play a pivotal role in various areas of mathematics. The classical form of the Simpson-type inequality is stated as follows:

Theorem 3. *Let* $F : [\hbar_1, \hbar_2] \to \mathbb{R}$ *be a four times continuously differentiable function on* $(\hbar_1, \hbar_2)$ *and* $\left\|F^{(4)}\right\|_\infty = \sup_{x\in(\hbar_1,\hbar_2)} \left|F^{(4)}(x)\right| < \infty$*, then the following inequality holds:*

$$\left|\left[\frac{1}{6}F(\hbar_1) + \frac{2}{3}F\left(\frac{\hbar_1+\hbar_2}{2}\right) + \frac{1}{6}F(\hbar_2)\right] - \frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2} F(x)\,dx\right| \le \frac{(\hbar_2-\hbar_1)^4}{2880}\left\|F^4\right\|_\infty .$$

H. Wang et al. [36] derived new inequalities utilizing the Riemann–Liouville fractional integral operator, along with various applications.

Theorem 4. *Let* $F : [\hbar_1, \hbar_2] \to \mathbb{R}$ *be a differentiable function on* $(\hbar_1, \hbar_2)$ *and* $0 < \alpha \le 1$*. If* $|F'|$ *is convex function on* $[\hbar_1, \hbar_2]$*, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\hbar_2-\hbar_1)^\alpha}\left[J^\alpha_{\hbar_1^+}F(\hbar_2) + J^\alpha_{\hbar_2^-}F(\hbar_1)\right] - F\left(\frac{\hbar_1+\hbar_2}{2}\right)\right| \\
\le\ & \frac{(\hbar_2-\hbar_1)}{4(\alpha+1)}\left(\frac{2^{\alpha-1}(\alpha-1)+1}{2^{\alpha-1}}\right)\left(\left|F'(\hbar_1)\right| + \left|F'(\hbar_2)\right|\right).
\end{aligned}$$

Theorem 5. *Let* $F : [\hbar_1, \hbar_2] \to \mathbb{R}$ *be a differentiable function on* $(\hbar_1, \hbar_2)$ *and* $0 < \alpha \le 1$*. If* $|F'|$ *is convex function on* $[\hbar_1, \hbar_2]$*, then the following inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{2(\hbar_2-\hbar_1)^\alpha}\left[J^\alpha_{\hbar_1^+}F(\hbar_2) + J^\alpha_{\hbar_2^-}F(\hbar_1)\right]\right.$$

$$
\begin{aligned}
&-\left\{\left(\frac{5^{\alpha}-1}{6^{\alpha}}\right)F\left(\frac{\hbar_1+\hbar_2}{2}\right)+\left(\frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}}\right)\frac{F(\hbar_1)+F(\hbar_2)}{2}\right\}\Bigg| \\
\leq\ & (\hbar_2-\hbar_1)\left(\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12\times 6^{\alpha}}\right)\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right).
\end{aligned}
$$

Bo-Yan Xi et al. [37] derived a Hermite–Hadamard-type inequality for the class of extended s-convex functions.

Theorem 6. *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o, $\hbar_1, \hbar_2 \in I$ with $\hbar_1 < \hbar_2$ and $F' \in L[\hbar_1, \hbar_2]$. If $|F'|^q$ is convex function on $[\hbar_1, \hbar_2]$ for $q \geq 1$, then the following inequality holds:*

$$
\left|\frac{1}{2}\left[\frac{F(\hbar_1)+F(\hbar_2)}{2}+F\left(\frac{\hbar_1+\hbar_2}{2}\right)\right]-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}F(x)\,dx\right|
$$

$$
\leq \frac{\hbar_2-\hbar_1}{16}\left[\left(\frac{3|F'(\hbar_1)|^q+|F'(\hbar_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|F'(\hbar_1)|^q+3|F'(\hbar_2)|^q}{4}\right)^{\frac{1}{q}}\right].
$$

Theorem 7. *Under the assumptions of Theorem 6, the following inequality holds:*

$$
\left|\frac{1}{3}\left[F(\hbar_1)+F(\hbar_2)+F\left(\frac{\hbar_1+\hbar_2}{2}\right)\right]-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}F(x)\,dx\right|
$$

$$
\leq \frac{5(\hbar_2-\hbar_1)}{72}\left[\left(\frac{37|F'(\hbar_1)|^q+8|F'(\hbar_2)|^q}{45}\right)^{\frac{1}{q}}+\left(\left(\frac{8|F'(\hbar_1)|^q+37|F'(\hbar_2)|^q}{45}\right)^{\frac{1}{q}}\right)\right].
$$

With the growing use of parameter-dependent inequalities in numerical integration and error analysis, there is a need for unified Mercer–type frameworks that allow continuous interpolation between classical bounds.

The present study is organized into four sections. The first section introduces the fundamental concepts and theoretical background that form the basis of the subsequent analysis. In the second section, we develop a Mercer-parameterized integral identity for differentiable functions and, based on this identity, derive a family of novel parameterized Hermite–Hadamard-Mercer–type inequalities. A key feature of these inequalities is their flexibility, which allows them to recover and extend classical midpoint, trapezoidal, Bullen–type, and Simpson–type inequalities as special cases. The third section is devoted to applications of the obtained results to the modified Bessel function, the q-digamma function, special means, and numerical quadrature schemes. Finally, in the fourth section, we present illustrative examples and graphical representations to illustrate the applicability of the proposed results. It is expected that this study will motivate further research in these directions.

2. Parameterized Mercer-type Integral Identity

In this section, we derive a new parameterized Mercer-type integral identity, which serves as a key auxiliary result.

Lemma 1. *Let* $F : [\hbar_1, \hbar_2] \to \mathbb{R}$ *be a differentiable function on* $[\hbar_1, \hbar_2]$. *Let* $\ddot{y}_1, \ddot{y}_2 \in [\hbar_1, \hbar_2]$ *with* $\ddot{y}_1 < \ddot{y}_2$ *and* $\beta, \xi \in [0,1]$. *Then the following equality holds:*

$$\begin{aligned}
&\frac{\ddot{y}_2-\ddot{y}_1}{8}\Bigg[\int_0^1 [(1-\varkappa)-\xi] F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\\
&+\int_0^1 [\beta-(1-\varkappa)] F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\\
&+\int_0^1 (\xi-\varkappa) F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)d\varkappa\\
&+\int_0^1 (\varkappa-\beta) F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa\Bigg]\\
=\ &\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\\
&-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right).
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\frac{\ddot{y}_2-\ddot{y}_1}{8}\Bigg[\int_0^1 [(1-\varkappa)-\xi] F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\\
&+\int_0^1 [\beta-(1-\varkappa)] F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\\
&+\int_0^1 (\xi-\varkappa) F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)d\varkappa\\
&+\int_0^1 (\varkappa-\beta) F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa\Bigg]\\
=\ &\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[I_1+I_2+I_3+I_4\right].
\end{aligned}$$

By integration by parts, we obtain

$$I_1 = \int_0^1 [(1-\varkappa)-\xi] F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa$$

$$
\begin{aligned}
&= \left.\frac{2\left((1-\varkappa)-\xi\right)F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)}{\ddot{y}_2-\ddot{y}_1}\right|_0^1 \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa \\
&= \frac{-2\xi}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\ddot{y}_1\right)-\frac{2\left(1-\xi\right)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa \\
&= \frac{-2\xi}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\ddot{y}_1\right)-\frac{2\left(1-\xi\right)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) \\
&\quad +\frac{4}{\left(\ddot{y}_2-\ddot{y}_1\right)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_1}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}} F(u)\,du. \qquad (3)
\end{aligned}
$$

In a similar manner, we obtain

$$
\begin{aligned}
I_2 &= \int_0^1 \left[\beta-(1-\varkappa)\right]F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa \\
&= \left.\frac{-2\left(\beta-(1-\varkappa)\right)F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)}{\ddot{y}_2-\ddot{y}_1}\right|_0^1 \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa \\
&= \frac{-2\beta}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\ddot{y}_2\right)-\frac{2}{\ddot{y}_2-\ddot{y}_1}\left(1-\beta\right)F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa \\
&= \frac{-2\beta}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\ddot{y}_2\right)-\frac{2}{\ddot{y}_2-\ddot{y}_1}\left(1-\beta\right)F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) \\
&\quad +\frac{4}{\left(\ddot{y}_2-\ddot{y}_1\right)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}} F(u)\,du, \qquad (4)
\end{aligned}
$$

$$
I_3 = \int_0^1 \left(\xi-\varkappa\right)F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)d\varkappa
$$

$$\begin{aligned}
&= \left.\frac{2(\xi-\varkappa)F\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)}{\ddot{y}_2-\ddot{y}_1}\right|_0^1 \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)d\varkappa \\
&= \frac{2(\xi-1)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)-\frac{2\xi}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_2) \\
&\quad +\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}^{\hbar_1+\hbar_2-\ddot{y}_2}F(u)\,du, \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1(\varkappa-\beta)F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa \\
&= \left.\frac{-2(\varkappa-\beta)F\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)}{\ddot{y}_2-\ddot{y}_1}\right|_0^1 \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa \\
&= \frac{-2(1-\beta)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)-\frac{2\beta}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_1) \\
&\quad +\frac{2}{\ddot{y}_2-\ddot{y}_1}\int_0^1 F\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa \\
&= \frac{-2(1-\beta)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)-\frac{2\beta}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_1) \\
&\quad +\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du. \qquad (6)
\end{aligned}$$

Adding the equalities (3), (4), (5), (6), we obtain

$$\begin{aligned}
&I_1+I_2+I_3+I_4 \\
&= \frac{-2\xi}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_1)-\frac{2(1-\xi)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) \\
&\quad +\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_1}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}F(u)\,du-\frac{2\beta}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_2) \\
&\quad -\frac{2}{\ddot{y}_2-\ddot{y}_1}(1-\beta)F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}F(u)\,du
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2(\xi-1)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)-\frac{2\xi}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_2)\\
&+\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}F(u)\,du-\frac{2(1-\beta)}{\ddot{y}_2-\ddot{y}_1}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\\
&-\frac{2\beta}{\ddot{y}_2-\ddot{y}_1}F(\hbar_1+\hbar_2-\ddot{y}_1)+\frac{4}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_1}^{\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}}F(u)\,du\\
=\;&\frac{-2}{\ddot{y}_2-\ddot{y}_1}(\xi+\beta)\left(F(\hbar_1+\hbar_2-\ddot{y}_1)+F(\hbar_1+\hbar_2-\ddot{y}_2)\right)\\
&-\frac{4}{\ddot{y}_2-\ddot{y}_1}(2-\beta-\xi)F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+\frac{8}{(\ddot{y}_2-\ddot{y}_1)^2}\int_{\hbar_1+\hbar_2-\ddot{y}_1}^{\hbar_1+\hbar_2-\ddot{y}_2}F(u)\,du. \qquad (7)
\end{aligned}
$$

By multiplying both sides of (7) by $\frac{\ddot{y}_2-\ddot{y}_1}{8}$, we obtain

$$
\begin{aligned}
&\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\int_0^1[(1-\varkappa)-\xi]F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\right.\\
&+\int_0^1[\beta-(1-\varkappa)]F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)d\varkappa\\
&+\int_0^1(\xi-\varkappa)F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)d\varkappa\\
&\left.+\int_0^1(\varkappa-\beta)F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)d\varkappa\right]\\
=\;&-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\\
&-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du.
\end{aligned}
$$

This completes the proof of Lemma 1.

Theorem 8. *Let the assumptions of Lemma 1 hold. If $F'\in L[\hbar_1,\hbar_2]$ is a convex function on $[\hbar_1,\hbar_2]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\right.\\
&\left.-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[2\left(1-\xi+\xi^2-\beta+\beta^2\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right. \\
& -\left(\frac{1}{6}\left(2-3\xi+6\xi^2-2\xi^3-3\beta+6\beta^2-2\beta^3\right)\left(\left|F'(\ddot{y}_1)\right|+\left|F'(\ddot{y}_2)\right|\right)\right. \\
& \left.\left.+\frac{1}{3}\left(4-3\xi+2\xi^3-3\beta+2\beta^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right].
\end{aligned}
$$

Proof. By using the Lemma 1, we obtain

$$
\begin{aligned}
& \left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\right. \\
& \left.-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right| \\
\leq\ & \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\int_0^1\left|(1-\varkappa)-\xi\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)\right|d\varkappa\right. \\
& +\int_0^1\left|\beta-(1-\varkappa)\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)\right|d\varkappa \\
& +\int_0^1\left|\xi-\varkappa\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)\right|d\varkappa \\
& \left.+\int_0^1\left|\varkappa-\beta\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_1\right)\right)\right|d\varkappa\right] \\
\leq\ & \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\int_0^1\left|(1-\varkappa)-\xi\right|\right. \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'(\ddot{y}_1)\right|+(1-\varkappa)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right)d\varkappa \\
& +\int_0^1\left|\beta-(1-\varkappa)\right| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'(\ddot{y}_2)\right|+(1-\varkappa)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right)d\varkappa \\
& +\int_0^1\left|\xi-\varkappa\right| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|+(1-\varkappa)\left|F'(\ddot{y}_2)\right|\right)\right)d\varkappa \\
& +\int_0^1\left|\varkappa-\beta\right| \\
& \left.\times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|+(1-\varkappa)\left|F'(\ddot{y}_1)\right|\right)\right)d\varkappa\right] \\
\leq\ & \frac{\ddot{y}_2-\ddot{y}_1}{8} \\
& \times\left[\left(\frac{1}{2}\left(1-2\xi+2\xi^2\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right.\right. \\
& -\left(\frac{1}{6}\left(1-3\xi+6\xi^2-2\xi^3\right)\left|F'(\ddot{y}_1)\right|+\frac{1}{6}\left(2-3\xi+2\xi^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\left(1-2\beta+2\beta^{2}\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right.\\
&\left.-\left(\frac{1}{6}\left(1-3\beta+6\beta^{2}-2\beta^{3}\right)\left|F'(\ddot{y}_2)\right|+\frac{1}{6}\left(2-3\beta+2\beta^{3}\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right)\\
&+\left(\frac{1}{2}\left(1-2\xi+2\xi^{2}\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right.\\
&\left.-\left(\frac{1}{6}\left(2-3\xi+2\xi^{3}\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|+\frac{1}{6}\left(1-3\xi+6\xi^{2}-2\xi^{3}\right)\left|F'(\ddot{y}_2)\right|\right)\right)\\
&+\left(\frac{1}{2}\left(1-2\beta+2\beta^{2}\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right.\\
&\left.\left.-\left(\frac{1}{6}\left(2-3\beta+2\beta^{3}\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|+\frac{1}{6}\left(1-3\beta+6\beta^{2}-2\beta^{3}\right)\left|F'(\ddot{y}_1)\right|\right)\right)\right]\\
\leq\ &\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[2\left(1-\xi+\xi^{2}-\beta+\beta^{2}\right)\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|\right)\right.\\
&-\left(\frac{1}{6}\left(2-3\xi+6\xi^{2}-2\xi^{3}-3\beta+6\beta^{2}-2\beta^{3}\right)\left(\left|F'(\ddot{y}_1)\right|+\left|F'(\ddot{y}_2)\right|\right)\right.\\
&\left.\left.+\frac{1}{3}\left(4-3\xi+2\xi^{3}-3\beta+2\beta^{3}\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\right)\right].
\end{aligned}
$$

This completes the proof.

Corollary 1. *By further simplification of Theorem 8, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\right.\\
&\left.-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\\
\leq\ &\frac{\ddot{y}_2-\ddot{y}_1}{8}\left(1-\xi+\xi^{2}-\beta+\beta^{2}\right)\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Corollary 2. *By setting $\beta=\xi=1$ in Corollary 1, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{F(\hbar_1+\hbar_2-\ddot{y}_1)+F(\hbar_1+\hbar_2-\ddot{y}_2)}{2}\right|\\
\leq\ &\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Remark 2. *By setting $\ddot{y}_1=\hbar_1$ and $\ddot{y}_2=\hbar_2$ in Corollary 2, this result was first established by Dragomir [38].*

Corollary 3. *By setting $\beta=\xi=0$ in Corollary 1, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\\
\leq\ &\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Remark 3. *By setting $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$ in Corollary 3, this result was first established by Kirmaci [39].*

Corollary 4. *By setting $\beta = \xi = \frac{1}{3}$ in Corollary 1, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du\right.\\
&\left.-\frac{1}{6}\left[F(\hbar_1+\hbar_2-\ddot{y}_1)+4F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+F(\hbar_1+\hbar_2-\ddot{y}_2)\right]\right|\\
\leq\ & \frac{5(\ddot{y}_2-\ddot{y}_1)}{72}\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Remark 4. *By setting $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$ in Corollary 4, this result was first established by Sarikaya [40].*

Corollary 5. *By setting $\beta = \xi = \frac{1}{2}$ in Corollary 1, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du\right.\\
&\left.-\frac{1}{2}\left[\frac{F(\hbar_1+\hbar_2-\ddot{y}_1)+F(\hbar_1+\hbar_2-\ddot{y}_2)}{2}+F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right]\right|\\
\leq\ & \frac{\ddot{y}_2-\ddot{y}_1}{16}\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Remark 5. *By setting $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$ in Corollary 5, this result was first established by H. Wang et al. [36].*

Corollary 6. *By setting $\beta = \xi = \frac{2}{3}$ in Corollary 1, we arrive at the following result:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du\right.\\
&\left.-\frac{1}{3}\left[F(\hbar_1+\hbar_2-\ddot{y}_1)+F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+F(\hbar_1+\hbar_2-\ddot{y}_2)\right]\right|\\
\leq\ & \frac{5(\ddot{y}_2-\ddot{y}_1)}{72}\left[F'(\hbar_1+\hbar_2-\ddot{y}_1)+F'(\hbar_1+\hbar_2-\ddot{y}_2)\right].
\end{aligned}
$$

Remark 6. *By setting $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$ in Corollary 6, this result was first established by Bo-Yan Xi et al. [37].*

Theorem 9. *Let the assumptions of Lemma 1 hold. If $F' \in L[\hbar_1, \hbar_2]$ and $|F'|^q$ is a convex function on $[\hbar_1, \hbar_2]$, where $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\right.\\
&\left.-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\\
\leq\;&\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\frac{1}{2}\left(1-2\xi+2\xi^2\right)+\frac{1}{2}\left(1-2\beta+2\beta^2\right)\right]^{1-\frac{1}{q}}\\
&\times\left[\left\{\left(1-\xi+\xi^2-\beta+\beta^2\right)\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q\right)\right.\right.\\
&-\left(\frac{1}{6}\left(2-3\beta+6\beta^2-2\beta^3-3\xi+6\xi^2-2\xi^3\right)\left|F'(\ddot{y}_1)\right|^q\right.\\
&\left.\left.+\frac{1}{6}\left(4-3\beta+2\beta^3-3\xi+2\xi^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right\}^{\frac{1}{q}}\\
&+\left\{\left(1-\xi+\xi^2-\beta+\beta^2\right)\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q\right)\right.\\
&-\left(\frac{1}{6}\left(2-3\beta+6\beta^2-2\beta^3-3\xi+6\xi^2-2\xi^3\right)\left|F'(\ddot{y}_2)\right|^q\right.\\
&\left.\left.\left.+\frac{1}{6}\left(4-3\beta+2\beta^3-3\xi+2\xi^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. By using the Lemma 1 and power-mean inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du-\frac{\beta+\xi}{4}\left[F(\hbar_1+\hbar_2-\ddot{y}_2)+F(\hbar_1+\hbar_2-\ddot{y}_1)\right]\right.\\
&\left.-\frac{2-\beta-\xi}{2}F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|\\
\leq\;&\frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\left(\int_0^1\left|(1-\varkappa)-\xi\right|d\varkappa\right)^{1-\frac{1}{q}}\right.\\
&\times\left\{\int_0^1\left|(1-\varkappa)-\xi\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_1+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)\right|^q d\varkappa\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\beta-(1-\varkappa)\right|d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\left\{\int_0^1\left|\beta-(1-\varkappa)\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\ddot{y}_2+(1-\varkappa)\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right)\right|^q d\varkappa\right\}^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\xi-\varkappa\right|d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\left\{\int_0^1\left|\xi-\varkappa\right|\left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\ddot{y}_2\right)\right)\right|^q d\varkappa\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
& +\left(\int_0^1 |\varkappa-\beta|\, d\varkappa\right)^{1-\frac{1}{q}} \\
& \times\left\{\int_0^1 |\varkappa-\beta| \left|F'\left(\hbar_1+\hbar_2-\left(\varkappa\frac{\ddot{y}_1+\ddot{y}_2}{2}+(1-\varkappa)\,\ddot{y}_2\right)\right)\right|^q\right\}^{\frac{1}{q}} d\varkappa\Bigg] \\
\leq\; & \frac{\ddot{y}_2-\ddot{y}_1}{8}\Bigg[\left(\int_0^1 |(1-\varkappa)-\xi|\, d\varkappa\right)^{1-\frac{1}{q}}\Bigg\{\int_0^1 |(1-\varkappa)-\xi| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'(\ddot{y}_1)\right|^q+(1-\varkappa)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)d\varkappa\Bigg\}^{\frac{1}{q}} \\
& +\left(\int_0^1 |\beta-(1-\varkappa)|\, d\varkappa\right)^{1-\frac{1}{q}}\Bigg\{\int_0^1 |\beta-(1-\varkappa)| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'(\ddot{y}_2)\right|^q+(1-\varkappa)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)d\varkappa\Bigg\}^{\frac{1}{q}} \\
& +\left(\int_0^1 |\xi-\varkappa|\, d\varkappa\right)^{1-\frac{1}{q}}\Bigg\{\int_0^1 |\xi-\varkappa| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q+(1-\varkappa)\left|F'(\ddot{y}_2)\right|^q\right)\right)d\varkappa\Bigg\}^{\frac{1}{q}} \\
& +\left(\int_0^1 |\varkappa-\beta|\, d\varkappa\right)^{1-\frac{1}{q}}\Bigg\{\int_0^1 |\varkappa-\beta| \\
& \times\left(\left|F'(\hbar_1)\right|+\left|F'(\hbar_2)\right|-\left(\varkappa\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q+(1-\varkappa)\left|F'(\ddot{y}_1)\right|^q\right)\right)d\varkappa\Bigg\}^{\frac{1}{q}}\Bigg] \\
=\; & \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\frac{1}{2}\left(1-2\xi+2\xi^2\right)+\frac{1}{2}\left(1-2\beta+2\beta^2\right)\right]^{1-\frac{1}{q}} \\
& \times\Bigg[\Bigg\{\left(1-\xi+\xi^2-\beta+\beta^2\right)\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q\right) \\
& -\Bigg(\frac{1}{6}\left(2-3\beta+6\beta^2-2\beta^3-3\xi+6\xi^2-2\xi^3\right)\left|F'(\ddot{y}_1)\right|^q \\
& +\frac{1}{6}\left(4-3\beta+2\beta^3-3\xi+2\xi^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\Bigg)\Bigg\}^{\frac{1}{q}} \\
& +\Bigg\{\left(1-\xi+\xi^2-\beta+\beta^2\right)\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q\right) \\
& -\Bigg(\frac{1}{6}\left(2-3\beta+6\beta^2-2\beta^3-3\xi+6\xi^2-2\xi^3\right)\left|F'(\ddot{y}_2)\right|^q \\
& +\frac{1}{6}\left(4-3\beta+2\beta^3-3\xi+2\xi^3\right)\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\Bigg)\Bigg\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

This completes the proof.

Corollary 7. *By setting $\beta = \xi = 1$ in Theorem 9, we arrive at the following result:*

$$\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du - \frac{F(\hbar_1+\hbar_2-\ddot{y}_1)+F(\hbar_1+\hbar_2-\ddot{y}_2)}{2}\right|$$

$$\leq \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{2}{3}\left|F'(\ddot{y}_1)\right|^q + \frac{1}{3}\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{2}{3}\left|F'(\ddot{y}_2)\right|^q + \frac{1}{3}\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Corollary 8. *By setting $\beta = \xi = 0$ in Theorem 9, we arrive at the following result:*

$$\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du - F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|$$

$$\leq \frac{\ddot{y}_2-\ddot{y}_1}{8}\left[\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{1}{3}\left|(\ddot{y}_1)\right|^q + \frac{2}{3}\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{1}{3}\left|F'(\ddot{y}_2)\right|^q + \frac{2}{3}\left|F'\left(\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Corollary 9. *By setting $\beta = \xi = \frac{1}{3}$ in Theorem 9, we arrive at the following result:*

$$\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du\right.$$

$$\left.-\frac{1}{6}\left[F(\hbar_1+\hbar_2-\ddot{y}_1) + 4F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right) + F(\hbar_1+\hbar_2-\ddot{y}_2)\right]\right|$$

$$\leq \frac{5(\ddot{y}_2-\ddot{y}_1)}{72}\left[\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{61\left|F'(\ddot{y}_1)\right|^q + 29\left|F'(\ddot{y}_2)\right|^q}{90}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{29\left|F'(\ddot{y}_1)\right|^q + 61\left|F'(\ddot{y}_2)\right|^q}{90}\right)\right)^{\frac{1}{q}}\right].$$

Corollary 10. *By setting $\beta = \xi = \frac{1}{2}$ in Theorem 9, we arrive at the following result:*

$$\left|\frac{1}{\ddot{y}_2 - \ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1} F(u)\,du\right.$$

$$\left.-\frac{1}{2}\left[\frac{F(\hbar_1+\hbar_2-\ddot{y}_1)+F(\hbar_1+\hbar_2-\ddot{y}_2)}{2} + F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)\right]\right|$$

$$\leq \frac{\ddot{y}_2-\ddot{y}_1}{16}\left[\left(\left|F'(\hbar_1)\right|^q + \left|F'(\hbar_2)\right|^q - \left(\frac{3\left|F'(\ddot{y}_1)\right|^q + \left|F'(\ddot{y}_2)\right|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q-\left(\frac{|F'(\ddot{y}_1)|^q+3|F'(\ddot{y}_2)|^q}{4}\right)\right)^{\frac{1}{q}}\Bigg].$$

Corollary 11. *By setting $\beta=\xi=\frac{2}{3}$ in Theorem 8, we arrive at the following result:*

$$\begin{aligned}
&\Bigg|\frac{1}{\ddot{y}_2-\ddot{y}_1}\int_{\hbar_1+\hbar_2-\ddot{y}_2}^{\hbar_1+\hbar_2-\ddot{y}_1}F(u)\,du\\
&-\frac{1}{3}\left[F(\hbar_1+\hbar_2-\ddot{y}_1)+F\left(\hbar_1+\hbar_2-\frac{\ddot{y}_1+\ddot{y}_2}{2}\right)+F(\hbar_1+\hbar_2-\ddot{y}_2)\right]\Bigg|\\
\leq\;&\frac{5(\ddot{y}_2-\ddot{y}_1)}{72}\left[\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q-\left(\frac{37|F'(\ddot{y}_1)|^q+8|F'(\ddot{y}_2)|^q}{45}\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left|F'(\hbar_1)\right|^q+\left|F'(\hbar_2)\right|^q-\left(\frac{8|F'(\ddot{y}_1)|^q+37|F'(\ddot{y}_2)|^q}{45}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

3. Applications

3.1. Special Means:

For distinct positive real numbers $\hbar_1$ and $\hbar_2$, where $\hbar_1<\hbar_2$, we recall the following special means:

(1) The arithmetic mean:

$$A(\hbar_1,\hbar_2)=\frac{\hbar_1+\hbar_2}{2}\;\hbar_1,\hbar_2\in\mathbb{R}.$$

(2) The Logarithmic mean:

$$L(\hbar_1,\hbar_2)=\frac{\hbar_1,\hbar_2}{\ln\hbar_2-\ln\hbar_1}\;\;\hbar_1,\hbar_2\neq 0\;\;\hbar_1,\hbar_2\in\mathbb{R}.$$

(3) The generalized Logarithmic mean:

$$L_r(\hbar_1,\hbar_2)=\left[\frac{(\hbar_2)^{n+1}-(\hbar_1)^{n+1}}{(n+1)(\hbar_2-\hbar_1)}\right]^{\frac{1}{n}},\;n\in\mathbb{R}\backslash\{-1,0\}\;\hbar_1,\hbar_2>0.$$

Proposition 1. *For $q\geq 1$ and $\hbar_1,\hbar_2\in\mathbb{R}$ satisfying $0<\hbar_1<\hbar_2$, the following inequality is valid:*

$$\begin{aligned}
&\left|L_n^n(\hbar_1,\hbar_2)-A(\hbar_1,\hbar_2)\right|\\
\leq\;&\frac{n(\hbar_2-\hbar_1)}{8}\left[\left(\frac{2}{3}|\hbar_1|^{q(n-1)}+\frac{1}{3}\left|F'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^{q(n-1)}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\frac{2}{3}|\hbar_2|^{q(n-1)}+\frac{1}{3}\left|F'\left(\frac{\hbar_1+\hbar_2}{2}\right)\right|^{q(n-1)}\right)\Bigg]^{\frac{1}{q}}.$$

Proof. *The statement is a direct consequence of Corollary 7 when considering the function* $F(x) = x^n$ *and choosing* $\ddot{y}_1 = \hbar_1$ *and* $\ddot{y}_2 = \hbar_2$.

Proposition 2. *For* $q \geq 1$ *and* $\hbar_1, \hbar_2 \in \mathbb{R}$ *satisfying* $0 < \hbar_1 < \hbar_2$*, the following inequality is valid:*

$$\begin{aligned}&\left|L^{-1}(\hbar_1,\hbar_2)-A^{-1}(\hbar_1,\hbar_2)\right|\\ \leq\ & \frac{(\hbar_2-\hbar_1)}{8}\left[\left(\frac{1}{3}|\hbar_1|^q+\frac{2}{3}\left|A^{-2}(\hbar_1,\hbar_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}|\hbar_2|^q+\frac{2}{3}\left|A^{-2}(\hbar_1,\hbar_2)\right|^q\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. The statement is a direct consequence of Corollary 8 when considering the function $F(x) = \frac{1}{x}$ and choosing $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

3.2. Quadrature schemes:

The following subsection focuses on deriving novel error bounds for Bullen-type quadrature schemes.

Suppose the partition $\zeta : \hbar_1 = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_i < \omega_{i+1} < \ldots < \omega_n = \hbar_2$ of the interval $[\hbar_1, \hbar_2]$, where $[\omega_i, \omega_{i+1}]$ is an arbitrary subinterval of $[\hbar_1, \hbar_2]$.

$$\begin{aligned}G(F,\zeta) :=\ & \sum_{i=0}^{n-1}\frac{(\omega_{i+1}-\omega_i)^2}{2}\left[\frac{F(\omega_i)+F(\omega_{i+1})}{2}+F\left(\frac{\omega_i+\omega_{i+1}}{2}\right)\right],\\ \int_{\hbar_1}^{\hbar_2}F(z)\,dz =\ & G(F,\zeta)+\mathbb{Z}(F,\zeta),\text{ where }\mathbb{Z}(F,\zeta)\text{ represents the error terms.}\end{aligned}$$

Proposition 3. *From Corollary 5, it follows that*

$$|\mathbb{Z}(F,\zeta)| \leq \sum_{i=0}^{n-1}\frac{(\omega_{i+1}-\omega_i)^2}{16}\left[\left|F'(\omega_i)\right|+\left|F'(\omega_{i+1})\right|\right].$$

Proof. To obtain the desired result, we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$, in Corollary 5, and then apply this result to the subinterval $[\omega_i, \omega_{i+1}]$, and summing them for $i = 0, 1, \ldots, n-1$ using triangle inequality.

Proposition 4. *From Corollary 10, it follows that*

$$|\mathbb{Z}(F,\zeta)| \leq \sum_{i=0}^{n-1}\frac{(\omega_{i+1}-\omega_i)^2}{16}\left[\left(\frac{3|F'(\omega_i)|^q+|F'(\omega_{i+1})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|F'(\omega_i)|^q+3|F'(\omega_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right].$$

Proof. To obtain the desired result, we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$, in Corollary 10, and then apply this result to the subinterval $[\omega_i, \omega_{i+1}]$ and summing them for $i = 0, 1, \ldots, n-1$ using triangle inequality.

3.3. q-Digamma Function:

The q-digamma function, represented as Ψ_q, is the q-analogue of the classical digamma function. Its explicit formula can be found in references [41, 42] and is given by:

$$\Psi_q(\gamma) = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+\gamma}}{1-q^{k+\gamma}},$$

alternatively, it can be expressed as:

$$\Psi_q(\gamma) = -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{k\gamma}}{1-q^{k}}.$$

This relation remains valid for $q > 1$ and $\gamma > 0$. Additionally, the q-digamma function Ψ_q can be rewritten as follows:

$$\Psi_q(\gamma) = -\ln(1-q) + \ln q \left[\gamma - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+\gamma)}}{1-q^{-(k+\gamma)}}\right],$$

alternatively, it can be expressed as:

$$\Psi_q(\gamma) = -\ln(1-q) + \ln q \left[\gamma - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-k\gamma}}{1-q^{-k}}\right]$$

Proposition 5. *From Corollary 2, it follows that*

$$\begin{aligned} &\left|\frac{\Psi_q(\hbar_2) - \Psi_q(\hbar_1)}{\hbar_2 - \hbar_1} - \frac{\Psi'_q(\hbar_1) + \Psi'_q(\hbar_2)}{2}\right| \\ \leq\ & \frac{(\hbar_2 - \hbar_1)}{8}\left[\Psi''_q(\hbar_1) + \Psi''_q(\hbar_2)\right]. \end{aligned}$$

Proof. The result is obtained immediately from Corollary 2, by considering $F(x) \to \Psi_q(x)$ which is completely monotone over the interval $(0, \infty)$ for all $a > 0$. Consequently, defining $F(x) = \Psi'_q(x)$ ensures that it is a convex function and we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

Proposition 6. *From Corollary 3, it follows that*

$$\begin{aligned} &\left|\frac{\Psi_q(\hbar_2) - \Psi_q(\hbar_1)}{\hbar_2 - \hbar_1} - \Psi'_q\left(\frac{\hbar_1 + \hbar_2}{2}\right)\right| \\ \leq\ & \frac{(\hbar_2 - \hbar_1)}{8}\left[\Psi''_q(\hbar_1) + \Psi''_q(\hbar_2)\right]. \end{aligned}$$

Proof. The result is obtained immediately from Corollary 3, by considering $F(x) \to \Psi_q(x)$ which is completely monotone over the interval $(0, \infty)$ for all $a > 0$. Consequently, defining $F(x) = \Psi'_q(x)$ ensures that it is a convex function and we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

3.4. Modified Bessel function:

Let recall the modified Bessel function belonging to the first kind I_σ, which have the following series representation ([43], p.77):

$$I_\rho(\Omega) = \sum_{n\geq 0} \frac{\left(\frac{\Omega}{2}\right)^{\rho+2n}}{n!\Gamma(\rho+n+1)},$$

where $\Omega \in \mathbb{R}$ and $\rho > -1$. The modified Bessel function belonging to the second kind$\mathbb{C}_\sigma$ ([43], p.78) is defined as:

$$\mathbb{C}_\sigma(\Omega) = \frac{\pi}{2}\frac{I_{-\rho}(\Omega) - I_\rho(\Omega)}{\sin\rho\pi}.$$

Let the function $\Psi_\rho(\Omega) : \mathbb{R} \to [1,\infty)$ defined by

$$\Psi_\rho(\Omega) = 2^\rho\Gamma(\rho+1)\,\Omega^{-\rho}\mathbb{C}_\rho(\Omega),$$

A formula for the first-order derivative of $\Psi_\rho(\Omega)$ is presented in [43] as:

$$\Psi'_\rho(\Omega) = \frac{\Omega}{2(\rho+1)}\Psi_{\rho+1}(\Omega), \tag{8}$$

and second derivative can be readily derived from equation (8) as:

$$\Psi''_\rho(\Omega) = \frac{\Omega^2}{4(\rho+1)(\rho+2)}\Psi_{\rho+2}(\Omega) + \frac{1}{2(\rho+1)}\Psi_{\rho+1}(\Omega).$$

Proposition 7. *Let $\rho > -1$, $\hbar_1, \hbar_2 \in \mathbb{R}$ with $0 < \hbar_1 < \hbar_2$, then we have*

$$\begin{aligned}&\left|\frac{\Psi_\rho(\hbar_2) - \Psi_\rho(\hbar_1)}{\hbar_2 - \hbar_1} - \frac{\Psi'_\rho(\hbar_1) + \Psi'_\rho(\hbar_2)}{2}\right| \\ \leq\ & \frac{(\hbar_2 - \hbar_1)}{8}\left[\Psi''_\rho(\hbar_1) + \Psi''_\rho(\hbar_2)\right].\end{aligned}$$

Proof. The result is obtained immediately from Corollary 2, by considering $F(\Omega) = \Psi'_\rho(\Omega)$, $\Omega > 0$ which is convex function $(0,\infty)$, $F'(\Omega) = \Psi''_\rho(\Omega)$ and we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

Proposition 8. *Let $\rho > -1$, $\hbar_1, \hbar_2 \in \mathbb{R}$ with $0 < \hbar_1 < \hbar_2$, then we have*

$$\begin{aligned}&\left|\frac{\Psi_\rho(\hbar_2) - \Psi_\rho(\hbar_1)}{\hbar_2 - \hbar_1} - \Psi'_\rho\left(\frac{\hbar_1 + \hbar_2}{2}\right)\right| \\ \leq\ & \frac{(\hbar_2 - \hbar_1)}{8}\left[\Psi''_\rho(\hbar_1) + \Psi''_\rho(\hbar_2)\right].\end{aligned}$$

Proof. The result is obtained immediately from Corollary 3, by considering $F(\Omega) = \Psi'_\rho(\Omega)$, $\Omega > 0$ which is convex function $(0,\infty)$, $F'(\Omega) = \Psi''_\rho(\Omega)$ and we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

Proposition 9. *Let $\rho > -1$, $\hbar_1, \hbar_2 \in \mathbb{R}$ with $0 < \hbar_1 < \hbar_2$, then we have*

$$\begin{aligned} &\left| \frac{\Psi_\rho(\hbar_2) - \Psi_\rho(\hbar_1)}{\hbar_2 - \hbar_1} - \frac{1}{6}\left[\Psi'_\rho(\hbar_1) + 4\Psi'_\rho\left(\frac{\hbar_1 + \hbar_2}{2}\right) + \Psi'_\rho(\hbar_2) \right] \right| \\ \leq\ & \frac{5(\hbar_2 - \hbar_1)}{72}\left[\Psi''_\rho(\hbar_1) + \Psi''_\rho(\hbar_2) \right]. \end{aligned}$$

Proof. The result is obtained immediately from Corollary 4, by considering $F(\Omega) = \Psi'_\rho(\Omega)$, $\Omega > 0$ which is convex function $(0, \infty)$, $F'(\Omega) = \Psi''_\rho(\Omega)$ and we set $\ddot{y}_1 = \hbar_1$ and $\ddot{y}_2 = \hbar_2$.

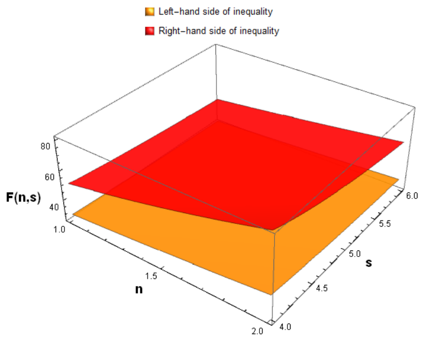
4. Simulations

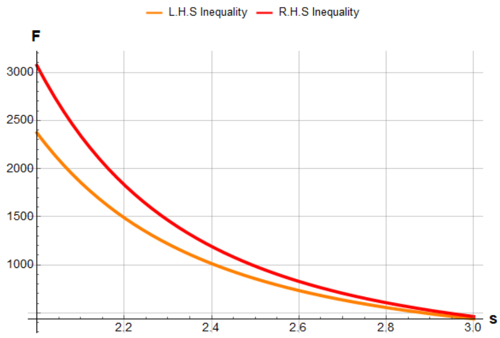
In the section that follows, we will verify our key findings using a range of graphical representations.

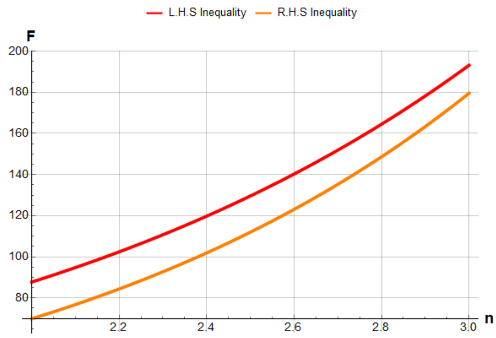
Example 1. *Let the assumptions of Corollary 2 are valid; moreover, we assume the function $F(z) = \frac{s}{n+2s} z^{\frac{n}{s}+2}$ defined on $\mathbb{R}^+$ with $n \geq 1$ and $s > 1$ which is a convex function. In addition, let $F'(z) = z^{\frac{n}{s}+1}$ also be a convex function. By specifying the values $\hbar_1 = 4$, $\hbar_2 = 5$, $\ddot{y}_1 = 2$ and $\ddot{y}_2 = 3$, we proceed as follows:*

$$\begin{aligned} &\left| \frac{n^2}{(n+2s)(n+3s)}\left[7^{\frac{n}{s}+3} - 6^{\frac{n}{s}+3} \right] - \frac{s}{(n+2s)}\left[\frac{7^{\frac{n}{s}+2} + 6^{\frac{n}{s}+2}}{2} \right] \right| \\ \leq\ & \frac{1}{8}\left[2\left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) - \left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) \right]. \end{aligned} \tag{9}$$

The LHS of (9) in Example 1 is strictly below the RHS.







In Fig 1, we select $n \in [1,2]$ and $s \in [4,6]$ to graphically compare the LHS and RHS of Corollary 2, thereby illustrating their relationship.

In Fig 2, we select $s \in [2,3]$ to graphically compare the LHS and RHS of Corollary 2, thereby illustrating their relationship.

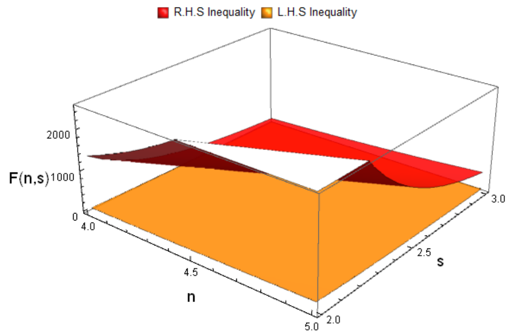
In Fig 3, we select $n \in [2,3]$ to graphically compare the LHS and RHS of Corollary 2, thereby illustrating their relationship.

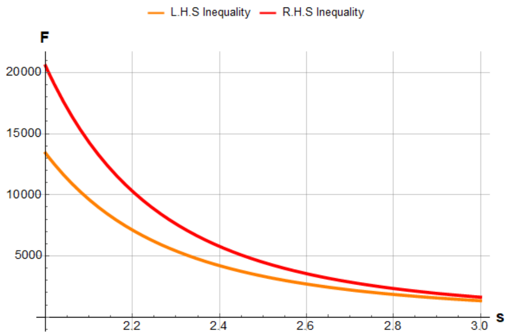
Example 2. *Let the assumptions of Corollary 3 are valid; moreover,we assume the function* $F(z) = \frac{s}{n+2s} z^{\frac{n}{s}+2}$ *defined on* $\mathbb{R}^+$ *with* $n \geq 1$ *and* $s > 1$ *which is a convex function. In addition, let* $F'(z) = z^{\frac{n}{s}+1}$ *also be a convex function. By specifying the values* $\hbar_1 = 4$, $\hbar_2 = 5$, $\ddot{y}_1 = 2$ *and* $\ddot{y}_2 = 3$, *we proceed as follows:*

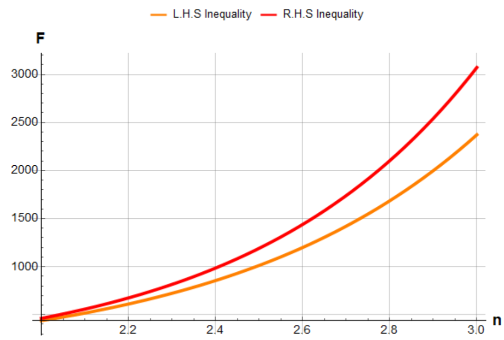
$$\left| \frac{n^2}{(n+2s)(n+3s)} \left[7^{\frac{n}{s}+3} - 6^{\frac{n}{s}+3} \right] - \frac{s}{(n+2s)} \left(\frac{13}{2} \right)^{\frac{n}{s}+2} \right| \tag{10}$$

$$\leq \frac{1}{8} \left[2 \left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) - \left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) \right].$$

The LHS of (10) in Example 2 is strictly below the RHS.







In Fig 5, we select $n \in [4, 5]$ and $s \in [2, 3]$ to graphically compare the LHS and RHS of Corollary 3, thereby illustrating their relationship.

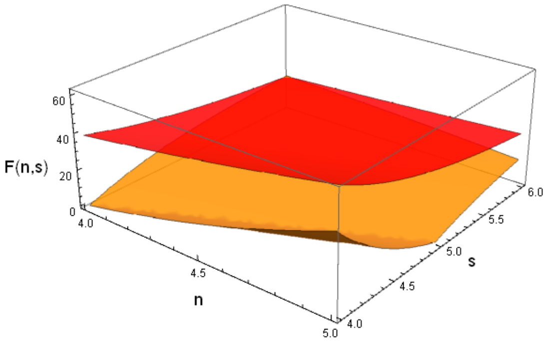
In Fig 5, we select $s \in [2, 3]$ to graphically compare the LHS and RHS of Corollary 3, thereby illustrating their relationship.

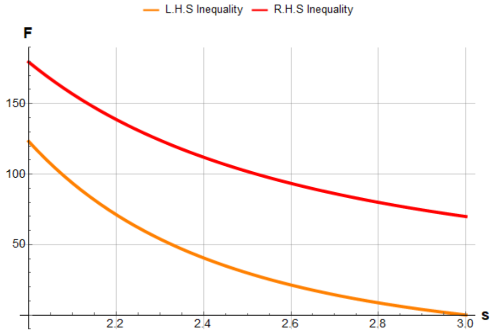
In Fig 6, we select $n \in [2, 3]$ to graphically compare the LHS and RHS of Corollary 3, thereby illustrating their relationship.

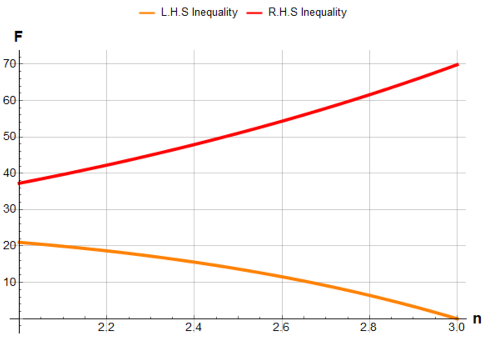
Example 3. *Let the assumptions of Corollary 4 are valid; moreover, we assume the function* $F(z) = \frac{s}{n+2s} z^{\frac{n}{s}+2}$ *defined on* $\mathbb{R}^+$ *with* $n \geq 1$ *and* $u > 1$ *which is a convex function. In addition, let* $F'(z) = z^{\frac{n}{s}+1}$ *also be a convex function. By specifying the values* $\ddot{y} = \alpha = 1$, $\hbar = 6$, $y = 4$, $\beta(0) = \beta(1) = 1$ *and* $w = 5$, *we proceed as follows:*

$$\left| \frac{n^2}{(n+2s)(n+3s)} \left[7^{\frac{n}{s}+3} - 6^{\frac{n}{s}+3} \right] - \frac{s}{6(n+2s)} \left[7^{\frac{n}{s}+2} + 6^{\frac{n}{s}+2} + 4 \left(\frac{13}{2} \right)^{\frac{n}{s}+2} \right] \right| \tag{11}$$
$$\leq \frac{5}{72} \left[2 \left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) - \left(7^{\frac{n}{s}+1} + 6^{\frac{n}{s}+1} \right) \right].$$

The LHS of (11) in Example 3 is strictly below the RHS.







In Fig 7, we select $n \in [4,5]$ and $s \in [4,6]$ to graphically compare the LHS and RHS of Corollary 4, thereby illustrating their relationship.

In Fig 8, we select $s \in [2,3]$ to graphically compare the LHS and RHS of Corollary 4, thereby illustrating their relationship.

In Fig 9, we select $n \in [2,3]$ to graphically compare the LHS and RHS of Corollary 4, thereby illustrating their relationship.

5. Conclusions and Future work

In this paper, we introduced a novel Mercer parameterized integral identity for differentiable convex functions and employed it to establish a comprehensive family of parameterized Hermite-Hadamard-Mercer type inequalities. The proposed framework unifies and significantly extends many well-known classical inequalities, including midpoint, trapezoidal, Bullen type, and Simpson-type inequalities, which are recovered as special cases through appropriate choices of the involved parameters. In addition, to validate our findings, we perform various graphical simulations. Moreover, the presented inequalities extend and generalize a number of existing results available in the literature. Furthermore, applications to the modified Bessel function, the q-digamma function, special means, and numerical quadrature schemes are presented. Finally, the Mercer-parameterized integral identity introduced here can be extended to several fractional operators, including Caputo, Caputo-Fabrizio, Riemann-Liouville, and Atangana-Baleanu operators, leading to new families of fractional Hermite-Hadamard–Mercer type inequalities. Finally, the framework may be extended to broader classes of functions, such as s-convex, harmonically convex, and strongly convex functions, providing further opportunities for applications in optimization, control theory, and numerical inverse problems.

The obtained error bounds for quadrature formulas highlight the practical importance of the proposed inequalities in numerical analysis and approximation theory. Beyond their role in convex analysis and inequality theory, these results also provide useful analytical tools for inverse problems and fractional differential equations. In many inverse problems, unknown parameters or source terms are recovered from indirect observations involving integral averages or nonlocal operators. The Mercer-type inequalities established in this study provide sharp bounds for such integral expressions and can therefore be used to derive stability estimates for inverse parameter identification problems. This is particularly relevant for fractional partial differential equations, where forward operators involve fractional integrals and memory-dependent terms. The parameterized structure of the inequalities further enables the analysis of the sensitivity of inverse reconstructions to data perturbations.

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