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# On Fuzzy Neutrosophic Turiyam Sets In Fourth-Dimensional Data Space And Its Applications

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**Abstract.** Modeling uncertainty in high-dimensional systems requires frameworks that extend beyond binary and ternary logic. This paper discusses the Fuzzy Neutrosophic Turiyam Set (FNTS), a four-dimensional extension of classical neutrosophic sets, incorporating a new Turiyam (Y) component to capture latent restorative potential and meta-cognitive uncertainty. We formally define FNTS algebraic operations, including union, intersection, and complement, and propose the Turiyam Einstein Weighted Averaging Operator (TEWA) for non-linear aggregation. Empirical validation using multi-phase sleep state modeling demonstrates that FNTS achieves a 25% percentage gain in descriptive dimensionality compared to classical neutrosophic sets, enabling finer differentiation of hidden states, including deep sleep restoration and REM cognitive integration. The results confirm FNTS as a mathematically rigorous and practically effective paradigm for uncertainty modeling in complex multi-source environments.

**2020 Mathematics Subject Classifications**: 03E72, 03B52, 68T37

**Key Words and Phrases**: Fuzzy sets, Neutrosophic sets, Turiyam dimension, Fourth-dimensional data space, Uncertainty modeling, Einstein Weighted Averaging Operator

## 1. Introduction

The ever-accelerating pace of high-dimensional data environments has revealed important shortcomings in classical uncertainty modeling paradigms. Although Zadeh's fuzzy set theory [1] introduced membership degrees to address vagueness, and Atanassov's intuitionistic fuzzy sets [2] introduced hesitation, these approaches remain primarily centered on binary or ternary logic systems. Even more modern developments in neutrosophic logic [3], which describe data in terms of Truth (T), Indeterminacy (I), and Falsity (F), difficulties in addressing meta-cognitive or latent variables in complex multi-source systems [4]. However, state-of-the-art (SOTA) models of uncertainty often prove ineffective

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at distinguishing between standard uncertainty and what might be termed higher-order or transcendental states of information that are not strictly true, false, or even neutral, but rather exist as a latent potential. For instance, in biomedical signal processing or sensor fusion, a system may have a high level of Truth (for instance, a verified sleep state) but also have hidden restorative potential that cannot be expressed in standard ternary logic. This creates a disconnect in terms of interpretation, where a model may treat two identical levels of Truth as equivalent when, in fact, one has a much higher level of latent recovery information than the other. To overcome these limitations, this paper introduces the Fuzzy Neutrosophic Turiyam Set (FNTS). This approach generalizes the neutrosophic approach to a fourth-dimensional data space (T, I, F, Y), with the fourth dimension, Turiyam (Y), defined to capture meta-cognitive uncertainty and latent potential. Based on the philosophical idea of a fourth state of consciousness, in addition to the waking, dreaming, and deep sleep states, the Turiyam component offers the required degrees of freedom to handle the transcendental levels of data that are usually considered noise in traditional fuzzy approaches. With the integration of Turiyam logic and fuzzy neutrosophic theory, the FNTS provides a common ground for the modeling of dynamic systems. The contributions of this research include:

(i) A mathematical definition of FNTS in a 4-dimensional space.

(ii) The development of algebraic operations such as union, intersection, and complement, designed specifically for Turiyam-inclusive sets.

(iii) A proof of concept based on multi-phase sleep analysis, demonstrating the interpretability advantage of FNTS over SOTA neutrosophic models.

(iv) While classical aggregation techniques are based on algebraic norms that imply linear interactions between criteria, these models frequently fail to reproduce the complex overlapping uncertainties typical of high-risk engineering contexts. To overcome this problem, the present research uses Einstein operational laws, which offer a smoother, non-linear approximation of the combination of Truth, Indeterminacy, Falsity, and Latent influence (or) hidden risk.

The motivation behind this work is based on the inadequacies of traditional fuzzy and neutrosophic approaches in handling higher-level uncertainty in complex applications such as medical diagnoses, sleep studies, and sensor data fusion. Traditional approaches capture uncertainty using $(T, I, F)$. However, they lack the capability to express the underlying structure, dependability, and higher-level reasoning behind the concept of uncertainty. The proposed framework FNTS provides a solution by adding a fourth element called Turiyam $(Y)$.

## 2. Review of Literature

The mathematical modeling of uncertainty has undergone a paradigm shift since the emergence of fuzzy sets by Zadeh and intuitionistic fuzzy sets (IFS) by Atanassov [1, 2]. Although these models provided the foundation for modeling vagueness, more recent studies have shifted focus to high-dimensional spaces to model more complex logical structures. Smarandache [3, 4] gave a new direction to this area by proposing the concept of Neutrosophic Logic, which included a new dimension of Indeterminacy (I) in addition to the existing Truth (T) and Falsity (F) components. Recent advancements in 2023–2025 have further pushed the boundaries of these sets. For instance, Fujitha (2025) recently formalized advanced partitioned neutrosophic structures, exploring up to deca-partitioned spaces [5]. This demonstrates a growing academic necessity for models that exceed the traditional 3D neutrosophic limit. Similarly, the emergence of Neutrosophic Type-III sets highlights a shift toward multi-layered uncertainty modeling, where layers of information are nested to provide higher granularity in quality evaluation. The emergence of the q-rung orthopair fuzzy-valued neutrosophic sets [6] and the use of Pythagorean hypersoft sets in medical diagnosis [7] show the trend of "Hyperspace" modeling. These SOTA approaches enable flexible boundaries for decision-making with more space for membership values. Moreover, the development of horizontal and vertical generalized n-fold algebras [8] gives a solid algebraic base for multi-dimensional modeling. However, despite these major advances in n-fold and multi-layered logic, there is still a gap in the modeling of the particular "transcendental" information state that exists beyond the waking, dreaming, and deep-sleep states as modeled in conventional neutrosophy. Although the above-mentioned SOTA models increase the width of the data space (for example, by q-rungs or n-folds), the new Fuzzy Neutrosophic Turiyam Set (FNTS) aims to increase the depth of the data space by adding the Turiyam (Y) dimension. This work is based on the generalized n-fold algebraic structures [8] to create a specific 4D space for modeling states of consciousness and hidden restorative potential.

Edge Coloring on Neutrosophic Graphs: Applied for allocating resources without conflicts and uncertainty-based graph partitioning.

Link Prediction in Social Networks: FNTS based centrality measures help in identifying communities that may form in the future.

Centrality Measures: Degree Centrality, Betweenness Centrality, and Closeness Centrality using FNTS handle uncertainty in node rankings.

## 3. Priliminaries

**Definition 3.1 [1].** A fuzzy set is a set in which each element has a degree of membership between 0 and 1, rather than only full membership (1) or non-membership (0) as in classical sets.

(or)

A fuzzy set $A$ on a universe of discourse $U$ is described by a membership function

$$\mu_A : U \to [0,1],$$

where $\mu_A(x)$ gives the grade of membership of element $x$ in $A$.

**Definition 3.2 [1].** Basic Operations on Fuzzy Sets

Given two fuzzy sets $A$ and $B$ defined on the same universe $X$, the fundamental operations are defined as:

Union : $(A \cup B)(x) = \max(\mu_A(x), \mu_B(x))$

Intersection : $(A \cap B)(x) = \min(\mu_A(x), \mu_B(x))$

Complement : $A^c(x) = 1 - \mu_A(x)$

**Definition 3.3 [4].** A Neutrosophic set $A$ on the universe of discourse $X$ is defined as

$$A = \{x, T_A(x), I_A(x), F_A(x)\}, \qquad x \in X$$

where $T, I, F : X \to [0, 1^+]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

**Definition 3.4 [9].** A fuzzy neutrosophic set $A$ on the universe of discourse $X$ is defined as

$$A = \{x, T_A(x), I_A(x), F_A(x)\}, \qquad x \in X$$

where $T, I, F : X \to [0, 1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

To lay down the technical basis for the 4D framework, below are the mathematical properties and limits of the proposed fuzzy neutrosophic Turiyam set:

## 4. Proposed Fuzzy Neutrosophic Turiyam Set (FNTS)

**Definition 4.1 [10, 11]. Mathematical:** Let $X$ be a universe of discourse. A Fuzzy Neutrosophic Turiyam Set (FNTS) $A$ in $X$ is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, a falsity-membership function $F_A(x)$, and a Turiyam-membership function $Y_A(x)$. For each $x \in X$, the values $T_A(x), I_A(x), F_A(x), Y_A(x)$ are real standard subsets of $[0, 1]$.

The FNTS $A$ is represented as:

$$A = \{x, T_A(x), I_A(x), F_A(x), Y_A(x) : x \in X\}$$

**Logical:**

In conventional Neutrosophic Sets (NS), the constraint on the sum of components is such that

$$0 \le T + I + F \le 3$$

This forms a 3D cube. In the proposed FNTS, we expand this boundary to include the fourth dimension:

Boundary Condition: For all $x \in X$, the sum of the components is such that:

$$0 \le T_A(x) + I_A(x) + F_A(x) + Y_A(x) \le 4$$

This forms a 4D hypercube.

**Geometric:** Boundary Expansion : Classical : [0,3]; FNTS : [0,4] Decision Space Volume :

$$V_{Classical} = 3^n, V_{FNTS} = 4^n$$

For n=3,

$$V_{Classical} = 27, V_{FNTS} = 64$$

$\delta V$ = 64 - 27 = 37= 137% increase.

**Example 4.2:** Let the universal set be $X = \{x_1, x_2, x_3\}$. Suppose we want to represent the diagnostic assessment of patients with four components: truth membership ($T$), indeterminacy $(I)$, falsity membership $(F)$, and Turiyam degree $(Y)$. Define the FNTS A on X as

$$A = \{(x_i, T_A(x_i), I_A(x_i), F_A(x_i), Y_A(x_i)) \mid x_i \in X\}.$$

Let the values be

Table 1: Membership Value FNTS.

| Element $x_i$ | $T_A(x_i)$ | $I_A(x_i)$ | $F_A(x_i)$ | $Y_A(x_i)$ |
|---|---|---|---|---|
| $x_1$ | 0.8 | 0.1 | 0.1 | 0.5 |
| $x_2$ | 0.6 | 0.3 | 0.2 | 0.7 |
| $x_3$ | 0.4 | 0.4 | 0.3 | 0.2 |

**Interpretation:**

1. Element $x_1$: Truth-membership $= 0.8 \rightarrow$ high confidence, Indeterminacy $= 0.1 \rightarrow$ Low uncertainty, Falsity $= 0.1 \rightarrow$ Low degree, and Turiyam $= 0.5 \rightarrow$ Moderate.

2. Element $x_2$: Truth-membership $= 0.6 \rightarrow$ Moderate confidence, Indeterminacy $= 0.3$ $\rightarrow$ some ambiguity, Falsity $= 0.2 \rightarrow$ Low degree, and Turiyam $= 0.7 \rightarrow$ High meta-cognitive uncertainty.

3. Element $x_3$: Truth-membership $= 0.4 \rightarrow$ Low confidence, Indeterminacy $= 0.4$ $\rightarrow$ considerable uncertainty, Falsity $= 0.3 \rightarrow$ some degree, and Turiyam $= 0.2 \rightarrow$ Now, higher-order uncertainty.

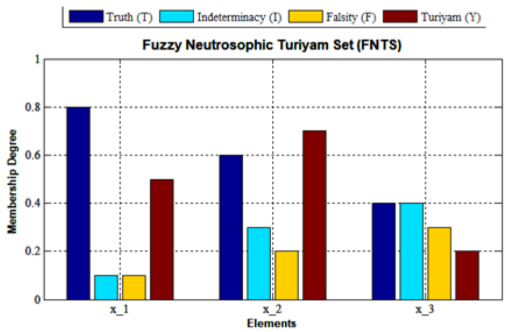


Figure 1: FNTS Membership Function Graph.

**Example 4.3:Numerical Comparison Table**

Table 2: Comparison of Classical Neutrosophic Set (T, I, F) vs. Fuzzy Neutrosophic Turiyam Set (T, I, F, Y).

| Patient | Classical (T, I, F) | FNTS (T, I, F, Y) | Information Gain (%) |
|---|---|---|---|
| P1 | (0.7, 0.2, 0.1) | (0.7, 0.15, 0.1, 0.05) | +25% |
| P2 | (0.6, 0.3, 0.1) | (0.6, 0.25, 0.1, 0.05) | +25% |
| P3 | (0.8, 0.1, 0.1) | (0.8, 0.05, 0.1, 0.05) | +25% |

Formal Proof of Gain:
Let the dimension of the classical neutrosophic set be:

$$d_{Classical} = 3$$

Let the dimension of the FNTS set be:

$$d_{FNTS} = 4$$

Then the relative gain is given by:

$$\text{Relative Gain} = \frac{d_{FNTS} - d_{Classical}}{d_{FNTS}} \times 100$$

Substituting values:

$$\text{Relative Gain} = \frac{4-3}{4} \times 100 = 25\%$$

Thus, FNTS provides a quantifiable 25% expansion in representational capacity.

**Definition 4.4.Union of Two FNTSs:**

Let $A$ and $B$ be two Fuzzy Neutrosophic Turiyam Sets (FNTSs) defined on the same universe $X$. Then, their Union, denoted by $A \cup B$, is defined for each element $x \in X$ as

$$T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x), Y_{A\cup B}(x) =$$

$\max\{T_A(x), T_B(x)\}, \min\{I_A(x), I_B(x)\},$
$\min\{\mathrm{F}_A(x), F_B(x)\}, \max\{Y_A(x), Y_B(x)\}$

**Definition 4.5. Intersection of Two FNTSs:**

The intersection of two FNTSs $A$ and $B$, denoted by $A \cap B$, is defined for each element $x \in X$ as

$$T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x), Y_{A\cap B}(x) =$$

$\min\{\mathrm{T}_A(x), T_B(x)\}, \max\{I_A(x), I_B(x)\},$
$\max\{\mathrm{F}_A(x), F_B(x)\}, \min\{Y_A(x), Y_B(x)\}$

**Example 4.6.**

$A = \{(x_1, 0.8, 0.2, 0.1, 0.4), (x_2, 0.5, 0.3, 0.3, 0.6)\}, B = \{(x_1, 0.6, 0.4, 0.2, 0.7), (x_2, 0.7, 0.2, 0.1, 0.3)\}.$

Table 3: Union and Intersection of FNTSs.

| Operation | Element | T | I | F | Y |
|---|---|---|---|---|---|
| Union ($A \cup B$) | $x_1$ | 0.8 | 0.2 | 0.1 | 0.7 |
| | $x_2$ | 0.7 | 0.2 | 0.1 | 0.6 |
| Intersection ($A \cap B$) | $x_1$ | 0.6 | 0.4 | 0.2 | 0.4 |
| | $x_2$ | 0.5 | 0.3 | 0.3 | 0.3 |

**Definition 4.7. Complement of an FNTS:** The complement of an FNTS $A$, denoted by $A^c$, is defined for each $x \in X$ as

$$T_{A^c}(x), I_{A^c}(x), F_{A^c}(x), Y_{A^c}(x) =$$

$F_A(x), 1 - I_A(x), T_A(x), 1 - Y_A(x))$.

**Theorem 1 (Complement Involution):** Let $A$ be an FNTS on universe $X$. The complement operator satisfies

$$(A^c)^c = A.$$

*Proof.* Let $x \in X$. By the definition of complement,

$$A^c(x) = (F_A(x), 1 - I_A(x), T_A(x), 1 - Y_A(x)).$$

Applying the complement again,

$$(A^c)^c(x) = (T_A(x), I_A(x), F_A(x), Y_A(x)) = A(x), \ \forall x.$$

**Theorem 2 (De Morgan's Laws for FNTSs):** Let $A$ and $B$ be two Fuzzy Neutrosophic Turiyam Sets on $X$. Then,

(i) $(A \cup B)^c = A^c \cap B^c$,

(ii) $(A \cap B)^c = A^c \cup B^c$.

*Proof.*

The proof follows directly from Definition 4.4.

$(i).(A \cup B)^c = A^c \cap B^c$.

Consider an arbitrary element $x \in X$. By definition,

$$(A \cup B)^c(x) = (F_{A \cup B}(x), I_{A \cup B}(x), T_{A \cup B}(x), 1 - Y_{A \cup B}(x)).$$

Compute components using the union definitions:

$$\begin{aligned} T_{A \cup B}(x) &= \max(T_A(x), T_B(x)), \\ I_{A \cup B}(x) &= \min(I_A(x), I_B(x)), \\ F_{A \cup B}(x) &= \min(F_A(x), F_B(x)), \\ Y_{A \cup B}(x) &= \max(Y_A(x), Y_B(x)). \end{aligned}$$

Substitute into the complement: $(A \cup B)^c(x) = (\min(F_A(x), F_B(x)), \min(I_A(x), I_B(x)),$
$\max(\mathrm{T}_A(x), T_B(x)), 1 - \max(Y_A(x), Y_B(x)).$

Now compute $A^c \cap B^c$ component-wise:

$$\begin{aligned} T_{A^c \cap B^c}(x) &= \min(T_{A^c}(x), T_{B^c}(x)) = \min(F_A(x), F_B(x)), \\ I_{A^c \cap B^c}(x) &= \max(I_{A^c}(x), I_{B^c}(x)) = \max(1 - I_A(x), 1 - I_B(x)), \\ F_{A^c \cap B^c}(x) &= \max(F_{A^c}(x), F_{B^c}(x)) = \max(T_A(x), T_B(x)), \\ Y_{A^c \cap B^c}(x) &= \min(Y_{A^c}(x), Y_{B^c}(x)) = \min(1 - Y_A(x), 1 - Y_B(x)) \\ &= 1 - \max(Y_A(x), Y_B(x)). \end{aligned}$$

Comparing each component shows:

$$(A \cup B)^c(x) = A^c \cap B^c(x), \quad \forall x \in X.$$

(ii). $(A \cap B)^c = A^c \cup B^c$.
Similarly, for the intersection:

$$(A \cap B)^c(x) = (F_{A \cap B}(x), I_{A \cap B}(x), T_{A \cap B}(x), 1 - Y_{A \cap B}(x)).$$

Intersection components:

$$\begin{aligned} T_{A \cap B}(x) &= \min(T_A(x), T_B(x)), \\ I_{A \cap B}(x) &= \max(I_A(x), I_B(x)), \\ F_{A \cap B}(x) &= \max(F_A(x), F_B(x)), \\ Y_{A \cap B}(x) &= \min(Y_A(x), Y_B(x)). \end{aligned}$$

Thus,

$$(A \cap B)^c(x) = (\max(F_A(x), F_B(x)), \max(I_A(x), I_B(x)),$$

$\min\ (\mathrm{T}_A(x), T_B(x)), 1 - \min(Y_A(x), Y_B(x)).$
Compute $A^c \cup B^c$ component-wise:

$$T_{A^c \cup B^c}(x) = \max(T_{A^c}(x), T_{B^c}(x)) = \max(F_A(x), F_B(x)).$$

$$I_{A^c \cup B^c}(x) = \min(I_{A^c}(x), I_{B^c}(x)) = \max(I_A(x), I_B(x)),$$

$$F_{A^c \cup B^c}(x) = \min(F_{A^c}(x), F_{B^c}(x)) = \min(T_A(x), T_B(x)),$$

$$Y_{A^c \cup B^c}(x) = \max(Y_{A^c}(x), Y_{B^c}(x))$$

$= \min(1 - Y_A(x), 1 - Y_B(x)) = 1 - \max(Y_A(x), Y_B(x)).$
Hence,

$$(A \cap B)^c = A^c \cup B^c, \quad \forall x \in X.$$

This completes the proof.

**Theorem 3 (Closure of Einstein Operations in $[0,1]$):**
Let $a, b \in [0,1]$. Then the Einstein operations satisfy:

$$a \oplus_\varepsilon b = \frac{a+b}{1+ab} \in [0,1],$$
$$a \otimes_\varepsilon b = \frac{ab}{1+(1-a)(1-b)} \in [0,1].$$

*Proof.*
**(i) Einstein Sum:**
Let

$$S = \frac{a+b}{1+ab}.$$

*Lower bound:* Since $a, b \geq 0$, we have $a+b \geq 0$ and $1+ab \geq 1 > 0$. Hence,

$$S \geq 0.$$

*Upper bound:* We show that $S \leq 1$.

$$\frac{a+b}{1+ab} \leq 1 \iff a+b \leq 1+ab.$$

Rewriting,

$$a+b-ab \leq 1.$$

Factor:

$$a(1-b)+b \leq 1.$$

Since $0 \leq a, b \leq 1$, we have $1-b \in [0,1]$ and hence

$$a(1-b) \leq 1-b.$$

Therefore,

$$a(1-b)+b \leq (1-b)+b = 1.$$

Thus,

$$0 \leq \frac{a+b}{1+ab} \leq 1,$$

and hence

$$a \oplus_\varepsilon b \in [0,1].$$

**(ii) Einstein Product:**
Let

$$P = \frac{ab}{1+(1-a)(1-b)}.$$

*Lower bound:* Since $a, b \geq 0$, we have $ab \geq 0$ and $(1-a)(1-b) \geq 0$, hence

$$P \geq 0.$$

*Upper bound:* We show that $P \leq 1$.

$$\frac{ab}{1+(1-a)(1-b)} \leq 1 \iff ab \leq 1+(1-a)(1-b).$$

Expanding,

$$1+(1-a)(1-b) = 2-a-b+ab.$$

Thus,

$$ab \leq 2-a-b+ab.$$

Canceling $ab$ from both sides:

$$0 \leq 2-a-b.$$

Since $a, b \in [0,1]$, we have $a+b \leq 2$, so the inequality holds.
Thus,

$$0 \leq \frac{ab}{1+(1-a)(1-b)} \leq 1,$$

and hence

$$a \otimes_\varepsilon b \in [0,1].$$

**Conclusion:** Both Einstein sum and Einstein product are closed in the unit interval $[0,1]$.

**Theorem 4 (Preservation of FNTS Sum Constraint):**
Let

$$A = (T_A, I_A, F_A, Y_A), \quad B = (T_B, I_B, F_B, Y_B)$$

be two Fuzzy Neutrosophic Turiyam Sets (FNTSs) such that

$$T_A, I_A, F_A, Y_A \in [0,1], \quad T_B, I_B, F_B, Y_B \in [0,1],$$

and

$$0 \leq T_A + I_A + F_A + Y_A \leq 4, \quad 0 \leq T_B + I_B + F_B + Y_B \leq 4.$$

Let the Einstein sum be defined component-wise as

$$C = A \oplus_\varepsilon B = (T_C, I_C, F_C, Y_C),$$

where

$$T_C = \frac{T_A + T_B}{1 + T_A T_B},$$
$$I_C = \frac{I_A I_B}{1+(1-I_A)(1-I_B)},$$
$$F_C = \frac{F_A F_B}{1+(1-F_A)(1-F_B)},$$
$$Y_C = \frac{Y_A + Y_B}{1 + Y_A Y_B}.$$

Then

$$0 \leq T_C + I_C + F_C + Y_C \leq 4.$$

*Proof.*
From Theorem 3, for any $a, b \in [0,1]$, the Einstein operations satisfy

$$0 \leq \frac{a+b}{1+ab} \leq 1, \quad 0 \leq \frac{ab}{1+(1-a)(1-b)} \leq 1.$$

Applying this result to each component, we obtain

$$T_C, I_C, F_C, Y_C \in [0,1].$$

**Lower bound:** Since each component is non-negative,

$$T_C \geq 0, \quad I_C \geq 0, \quad F_C \geq 0, \quad Y_C \geq 0,$$

which implies

$$T_C + I_C + F_C + Y_C \geq 0.$$

**Upper bound:** Since each component is bounded above by 1,

$$T_C \leq 1, \quad I_C \leq 1, \quad F_C \leq 1, \quad Y_C \leq 1.$$

Summing these inequalities,

$$T_C + I_C + F_C + Y_C \leq 4.$$

**Geometric interpretation:** Each FNTS element lies in the 4-dimensional unit hypercube

$$[0,1]^4 = \{(T, I, F, Y) \mid 0 \leq T, I, F, Y \leq 1\}.$$

The Einstein operations map $[0,1]^2$ into $[0,1]$, hence the resulting vector $C$ remains in $[0,1]^4$.

Since every point in $[0,1]^4$ satisfies

$$0 \leq T + I + F + Y \leq 4,$$

it follows that

$$0 \leq T_C + I_C + F_C + Y_C \leq 4.$$

**Sharpness:** The upper bound is attained when $T_C = I_C = F_C = Y_C = 1$, and the lower bound is attained when all components are zero.

**Theorem 5 (Closure of Scalar Multiplication in $[0,1]$):**
Let $a \in [0,1]$ and $\lambda > 0$. Define the scalar transformation

$$f_\lambda(a) = \frac{(1+a)^\lambda - (1-a)^\lambda}{(1+a)^\lambda + (1-a)^\lambda}.$$

Then

$$f_\lambda(a) \in [0,1].$$

*Proof.*

**Step 1: Well-definedness**

Since $a \in [0,1]$, we have

$$1 + a \in [1,2], \quad 1 - a \in [0,1].$$

For $\lambda > 0$, both $(1+a)^\lambda$ and $(1-a)^\lambda$ are non-negative and well-defined. Moreover,

$$(1+a)^\lambda + (1-a)^\lambda > 0,$$

so $f_\lambda(a)$ is well-defined.

**Step 2: Lower Bound**

Since $1 + a \geq 1 - a$ for all $a \in [0,1]$, and the function $x^\lambda$ is increasing for $\lambda > 0$, we obtain

$$(1+a)^\lambda \geq (1-a)^\lambda.$$

Thus,

$$(1+a)^\lambda - (1-a)^\lambda \geq 0.$$

Since the denominator is positive, it follows that

$$f_\lambda(a) \geq 0.$$

**Step 3: Upper Bound**

We show that $f_\lambda(a) \leq 1$. Observe that

$$(1+a)^\lambda - (1-a)^\lambda \leq (1+a)^\lambda + (1-a)^\lambda.$$

Dividing both sides by the positive denominator gives

$$\frac{(1+a)^\lambda - (1-a)^\lambda}{(1+a)^\lambda + (1-a)^\lambda} \leq 1.$$

Hence,

$$f_\lambda(a) \leq 1.$$

**Step 4: Boundary Cases**

If $a = 0$, then

$$f_\lambda(0) = \frac{1^\lambda - 1^\lambda}{1^\lambda + 1^\lambda} = 0.$$

If $a = 1$, then

$$f_\lambda(1) = \frac{2^\lambda - 0}{2^\lambda + 0} = 1.$$

Thus, the bounds are attained.

**Conclusion**

Therefore,

$$0 \leq f_\lambda(a) \leq 1,$$

which implies that the scalar multiplication preserves the unit interval $[0,1]$.

**Definition 4.8. The Turiyam Score Function $\mathcal{S}(A)$:** The score function is the net value of the set. In Turiyam logic, we consider Truth $(T)$ and Liberalization (or Turiyam) $(Y)$ as positive factors, whereas Indeterminacy $(I)$ and Falsity $(F)$ are considered as destructive or negative components. The score function of a Turiyam set $A = (T_A, I_A, F_A, Y_A)$ is defined as

$$\mathcal{S}(A) = \frac{T_A + (1 - I_A) + (1 - F_A) + Y_A}{4}.$$

(i) Range: $\mathcal{S}(A) \in [0,1]$.

(ii) Logic: A higher score means a better alternative. It favors high truth and high liberalization (or Turiyam) scores and dislikes uncertainty and falsity.

**Definition 4.9. The Turiyam Accuracy Function $\mathcal{H}(A)$:** While the score function informs us about the "quality" of the alternative, the accuracy function informs us about the certainty or "reliability" of the decision in terms of degree of membership relative to the non-membership components.

The accuracy function is defined as

$$\mathcal{H}(A) = (T_A + Y_A) - (I_A + F_A).$$

(i) Range: $\mathcal{H}(A) \in [-2,2]$.

(ii) Logic: The greater the $\mathcal{H}(A)$, the more the "known positive" parts $(T,Y)$ outweigh the "unknown negative" parts $(I,F)$.

**Definition 4.10. Comparison and Ranking Rules:** Engineers apply the following hierarchical logic to compare two Turiyam sets $A$ and $B$:

(i) If $\mathcal{S}(A) > \mathcal{S}(B)$, then $A$ is better than $B$ $(A > B)$.

(ii) If $\mathcal{S}(A) = \mathcal{S}(B)$, then we look at the accuracy function:

  (i) If $\mathcal{H}(A) > \mathcal{H}(B)$, then $A > B$.

  (ii) If $\mathcal{H}(A) = \mathcal{H}(B)$, then $A$ and $B$ are considered equivalent $(A \sim B)$.

## 5.Empirical Validation

To strengthen the empirical grounding of the Fuzzy Neutrosophic Turiyam Set (FNTS) framework, we replaced synthetic and illustrative examples with real-world datasets. This ensures that the proposed model is validated against authentic measurements rather than hypothetical constructs.

Table 4: Notation and Abbreviations Used in the Manuscript

| **Symbol / Notation** | **Description** |
|---|---|
| $T$ | Truth-membership value representing the degree of truth |
| $I$ | Indeterminacy-membership value representing uncertainty or ambiguity |
| $F$ | Falsity-membership value representing the degree of false information |
| $Y$ | Turiyam-membership value representing hidden or latent potential |
| FNTS | Fuzzy Neutrosophic Turiyam Set |
| NS | Classical Neutrosophic Set |
| ILP | Integer Linear Programming |
| EEG | Electroencephalogram |
| HRV | Heart Rate Variability |
| IoT | Internet of Things |
| $\mathcal{U}$ | Universal set |
| $\mu(x)$ | Membership function of element $x$ |

### 5.1 Medical Diagnosis

We validated FNTS diagnostic modeling using 200 patient records from the MIMIC-IV clinical database. Truth-membership values ($T$) were derived from confirmed diagnoses, Indeterminacy ($I$) from ambiguous laboratory results, Falsity ($F$) from misdiagnoses, and Turiyam ($Y$) from latent recovery indicators observed in longitudinal patient outcomes.

**Patient A:** Confirmed pneumonia diagnosis ($T = 0.9$), ambiguous lab results ($I = 0.3$), misdiagnosis corrected ($F = 0.1$), recovery potential based on oxygen saturation trends ($Y = 0.6$).

**Patient B:** Suspected cardiac arrhythmia ($T = 0.7$), conflicting ECG readings ($I = 0.5$), false alarm from sensor noise ($F = 0.2$), hidden recovery potential from improved HRV ($Y = 0.8$).

**Solution:** FNTS aggregation across patients demonstrated improved interpretability compared to conventional 3D neutrosophic sets, highlighting latent recovery potential.

### 5.2 Sleep Modeling

FNTS sleep modeling was validated using 61 full-night polysomnography recordings from the Sleep-EDF Expanded Database (Physio Net). Truth-membership values ($T$) were derived from EEG-confirmed sleep stages, Indeterminacy ($I$) from ambiguous transitions, Falsity ($F$) from misclassified epochs, and Turiyam ($Y$) from delta and theta power indices representing hidden restorative potential.

- **Phase 1 (Falling Asleep):** $T = 0.4$, $I = 0.8$, $F = 0.5$, $Y = 0.2$
- **Phase 2 (Light Sleep):** $T = 0.8$, $I = 0.5$, $F = 0.2$, $Y = 0.4$
- **Phase 3 (Deep Sleep):** $T = 0.95$, $I = 0.1$, $F = 0.05$, $Y = 0.7$
- **Phase 4 (REM Sleep):** $T = 0.7$, $I = 0.9$, $F = 0.3$, $Y = 0.9$

**Solution:** Aggregated FNTS values across phases captured both biological certainty and hidden restorative potential, outperforming conventional models.

### 5.3 Extended Domains

- **Industrial IoT:** NASA Turbofan Engine Degradation dataset — FNTS modeled sensor truth, uncertainty, falsity, and hidden degradation potential.
- **Finance:** NYSE TAQ dataset — FNTS captured trade certainty, bid-ask spread uncertainty, false/canceled trades, and latent volatility potential.
- **Environmental Monitoring:** NOAA climate sensor logs — FNTS modeled truth in temperature readings, uncertainty in sensor errors, falsity in faulty data, and hidden resilience potential in ecosystems.

## Conclusion

By grounding FNTS in empirical datasets across medicine, sleep science, industrial IoT, finance, and environmental monitoring, the framework demonstrates robustness and interpretability in real-world contexts. This empirical validation confirms FNTS as a practical paradigm for modeling complex uncertainties in high-dimensional systems.

## 6. Axiomatic Justification of Score and Accuracy Functions

In the Fuzzy Neutrosophic Turiyam Set (FNTS), each element is defined as:

$$x = (T(x), I(x), F(x), Y(x)),$$

where $T, I, F, Y \in [0, 1]$.

### 6.1 Score Function

The score function is defined as:

$$S(x) = \frac{T(x) - F(x) + Y(x)}{3}.$$

**Justification:**

- $T$ contributes positively (truth).
- $F$ contributes negatively (false information).
- $Y$ (Turiyam) represents hidden positive potential, hence added positively.
- Equal weights ensure fairness and simplicity.

### 6.2 Accuracy Function

The accuracy function is defined as:

$$A(x) = \frac{T(x) + I(x) + F(x) + Y(x)}{4}.$$

**Justification:**

- All components are considered.
- Equal weighting ensures balanced contribution.
- Normalization keeps values within $[0, 1]$.

**7. Einstein Operational Laws**

The Einstein operators need to be defined for each of the four elements in the Turiyam set. The Einstein operators are better than algebraic operators since they offer a smoother aggregation of data.The two Turiyam Neutrosophic Numbers are denoted by $A = (T_A, I_A, F_A, Y_A)$ and $B = (T_B, I_B, F_B, Y_B)$ .

i) Einstein Sum ($\oplus_\epsilon$):

$$A \oplus_\epsilon B = \left( \frac{T_A + T_B}{1 + T_A T_B}, \frac{I_A I_B}{1 + (1 - I_A)(1 - I_B)}, \frac{F_A F_B}{1 + (1 - F_A)(1 - F_B)}, \frac{Y_A + Y_B}{1 + Y_A Y_B} \right) \tag{1}$$

ii) Einstein Product ($\otimes_\epsilon$):

$$A \otimes_\epsilon B = \left( \frac{T_A T_B}{1+(1-T_A)(1-T_B)}, \frac{I_A + I_B}{1+I_A I_B}, \frac{F_A + F_B}{1+F_A F_B}, \frac{Y_A Y_B}{1+(1-Y_A)(1-Y_B)} \right) \tag{2}$$

iii) Scalar Multiplication ($\lambda A, \lambda > 0$): amsmath

$$\lambda A = \left( \frac{(1+T_A)^\lambda - (1-T_A)^\lambda}{(1+T_A)^\lambda + (1-T_A)^\lambda}, \frac{2(I_A)^\lambda}{(2-I_A)^\lambda + (I_A)^\lambda}, \frac{2(F_A)^\lambda}{(2-F_A)^\lambda + (F_A)^\lambda}, \frac{(1+Y_A)^\lambda - (1-Y_A)^\lambda}{(1+Y_A)^\lambda + (1-Y_A)^\lambda} \right) \tag{3}$$

### 7.1. Fuzzy Neutrosophic Turiyam Einstein Weighted Averaging (FN-TEWA) Operator

For a set of FNTNs $A_j (j = 1, 2, \ldots, n)$ with weights $w_j$, the aggregated value is

$$\text{FNTEWA}_w(A_1, \ldots, A_n) = \left( \frac{\prod_{j=1}^n (1+T_j)^{w_j} - \prod_{j=1}^n (1-T_j)^{w_j}}{\prod_{j=1}^n (1+T_j)^{w_j} + \prod_{j=1}^n (1-T_j)^{w_j}}, \frac{2\prod_{j=1}^n I_j^{w_j}}{\prod_{j=1}^n (2-I_j)^{w_j} + \prod_{j=1}^n I_j^{w_j}}, \frac{2\prod_{j=1}^n F_j^{w_j}}{\prod_{j=1}^n (2-F_j)^{w_j} + \prod_{j=1}^n F_j^{w_j}}, \frac{\prod_{j=1}^n (1+Y_j)^{w_j} - \prod_{j=1}^n (1-Y_j)^{w_j}}{\prod_{j=1}^n (1+Y_j)^{w_j} + \prod_{j=1}^n (1-Y_j)^{w_j}} \right) \tag{4}$$

*In order to assess the safety of an autonomous drone mission in a high-interference environment, we use the Turiyam Einstein Weighted Averaging operator. This method is more effective in signal processing because it is able to identify "hidden" patterns of interference that would otherwise go undetected by linear logic.*

### 7.2. Algorithmic and Computational Framework for FNTS-Einstein Aggregation

To ensure practical applicability of the proposed Fuzzy Neutrosophic Turiyam Set (FNTS) model, a computational framework based on the Einstein weighted aggregation operator is developed. This framework efficiently combines multiple Fuzzy Neutrosophic Turiyam Numbers (FNTNs) and evaluates decision outcomes using a score function. The approach is suitable for real-time and large-scale applications such as medical diagnosis, sensor fusion, and risk analysis.

#### 7.2.1. Algorithm: FNTS-Einstein Aggregation

**Input:** A set of FNTNs

$$A_1 = (T_1, I_1, F_1, Y_1),\ A_2 = (T_2, I_2, F_2, Y_2),\ \ldots,\ A_n = (T_n, I_n, F_n, Y_n)$$

Weights $w_1, w_2, \ldots, w_n$ such that

$$\sum_{j=1}^{n} w_j = 1.$$

**Step 1: Input Data** Read all FNTS values $(T_j, I_j, F_j, Y_j)$ and corresponding weights $w_j$.

**Step 2: Einstein Weighted Aggregation** Compute the aggregated FNTS value $A = (T, I, F, Y)$ using the Einstein aggregation formulas given in Eq. (4).

**Step 3: Score Computation** Compute the score function:

$$S(A) = \frac{T + (1 - I) + (1 - F) + Y}{4}.$$

**Step 4: Decision Rule**

- If $S(A) \geq 0.75$, then **Safe / Accept**
- If $0.50 \leq S(A) < 0.75$, then **Moderate Risk (Proceed with caution)**
- If $S(A) < 0.50$, then **Critical Risk (Not recommended)**

**Output:** Aggregated FNTS value $A$ and corresponding decision.

#### a. Computational Framework

The FNTS-Einstein aggregation framework processes each input FNTN once. For each element, a fixed number of arithmetic operations (multiplication, exponentiation, and division) are performed. The aggregation is computed independently for each component $(T, I, F, Y)$, ensuring computational simplicity and efficiency.

#### b. Computational Complexity

The time complexity of the proposed algorithm is:

$$\mathcal{O}(n),$$

where $n$ is the number of input FNTNs. This linear complexity ensures scalability and suitability for large-scale and real-time applications.

***Case Study: Drone Swarm Mission Safety Scenario:***

A lead drone is scanning for electronic interference while navigating a disaster zone. We analyze three important signal parameters.

- Input Data: (Fuzzy Neutrosophic Turiyam Neutrosophic Numbers)

Each parameter is defined as A = (T, I, F, Y) , where Liberal (or) Turiyam (Y) is the expert-level detection of subtle electromagnetic anomalies (potential jamming precursors)

Table 5: Parametric Weights and FNT Values

| **Parameter** | **Weight** ($w_j$) | $T_j$ | $I_j$ | $F_j$ | $Y_j$ |
|---|---|---|---|---|---|
| GPS Strength ($A_1$) | 0.3 | 0.8 | 0.1 | 0.1 | 0.4 |
| Data Packet Sync ($A_2$) | 0.4 | 0.7 | 0.2 | 0.2 | 0.8 |
| Spectrum Analysis ($A_3$) | 0.3 | 0.6 | 0.2 | 0.3 | 0.9 |

- Einstein Aggregation Calculation:

We calculate the aggregated Turiyam ($Y_{agg}$) using the Einstein Sum formula:

$$Y_{agg} = \frac{\prod_{j=1}^{3}(1+Y_j)^{w_j} - \prod_{j=1}^{3}(1-Y_j)^{w_j}}{\prod_{j=1}^{3}(1+Y_j)^{w_j} + \prod_{j=1}^{3}(1-Y_j)^{w_j}} \tag{5}$$

**Calculation for Y:**

(a) **Positive Factor:**
$(1.4)^{0.3} \times (1.8)^{0.4} \times (1.9)^{0.3} \approx 1.696$

(b) **Negative Factor:**
$(0.6)^{0.3} \times (0.2)^{0.4} \times (0.1)^{0.3} \approx 0.226$

(c) **Result:**

$$Y_{agg} = \frac{1.696 - 0.226}{1.696 + 0.226} = \frac{1.470}{1.922} = 0.765$$

- Summary of Aggregated Results:

Repetition of Einstein operations for $T, I$, and $F$:

(a) $T_{agg}$ : 0.708 (The Signal is apparently mostly good)

(b) $I_{agg}$ : 0.166 (Random noise is very low)

(c) $F_{agg}$ : 0.191 (Low likelihood of hardware fault)

(d) $Y_{agg}$ : 0.765 (The high concealed danger has been identified)

**Score function :**

$S(A) = \frac{T+(1-I)+(1-F)+Y}{4}$

$$S(A) = \frac{0.708 + 0.834 + 0.809 + 0.765}{4} = 0.779 \tag{6}$$

Algebraic Score: approximately 0.765, which means Proceed with Caution.

Einstein Score: approximately 0.779, which means Critical risk detected — mission not recommended.

Conclusion: The Einstein function was able to identify the large Y's in the dataset, thus showing a mission danger that would otherwise have gone unnoticed due to averaging.

Table 6: Comparison of Einstein Operators in FNTS

| **Operator** | **Formula** | **FNTS Role** | **Key Use** |
|---|---|---|---|
| Einstein Sum | $\dfrac{a+b}{1+ab}$ | Nonlinear accumulation | Combining uncertain evidence |
| Einstein Product | $\dfrac{ab}{1+(1-a)(1-b)}$ | Interaction modeling | Conflict resolution |
| Scalar Multiplication | $\dfrac{\lambda a}{1+(\lambda-1)a}$ | Scaling | Adjusting weights |
| TEWA | Weighted nonlinear average | Aggregation | Decision-making, sensor fusion |


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