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Special Properties of Rings of Crossed Products Type

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Abstract. Let R be a ring and M be a strictly ordered monoid equipped with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to \mathrm{Aut}(R)$. In this paper, we introduce the concept of strongly CM-reversible rings and investigate properties of crossed products of R with M. We specifically focus on the CM-Armendariz condition and its extension from polynomial rings to these crossed product structures. We prove that, under some additional conditions, $R * M$ is semicommutative (symmetric, Dedekind finite, clean, uniquely clean, or exchange) if and only if R satisfies the corresponding property. Furthermore, we prove that a ring R is strongly CM-reversible if and only if it is a right Ore ring with a classical right quotient ring Q. Finally, we discuss several related properties and characterizations of strongly CM-reversibility.

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1. Introduction

Let R be a ring with unity and M be a multiplicative monoid. The crossed product $R*M$ of M over R is an associative ring whose structure is determined by R, M, and a set of defining parameters, namely a monoid action ω and a twisting map f. Let $\omega : M \to Aut(R)$ be a monoid homomorphism. For $g \in M$, we denote the automorphism ω_g by $\omega(g)$. More precisely, the crossed product $R * M$ over R is defined as the set of all finite sums of the form $R * M = \{y_g g | y_g \in R, g \in M\}$, where addition is defined component-wise, and multiplication is defined using the distributive law alongside two rules known as action and twisting. Specifically, for $s, g \in M$ and $y \in R$, we have $gy = \omega_g(y)g$ and $s\,g = f(s,g)s\,g$, where $f : M \times M \to U(R)$ is a twisted function and $U(R)$ denotes the set of units of R. Here, the twisted function f and the action ω of M on R satisfy the following conditions: $\omega_s(\omega_g(y)) = f(s,g)\omega_s(\omega_g(y)f(s,g)^{-1})$, $\omega_s(f(g,k))f(s,gk) = f(s,g)f(s\,g,k)$, $f(1,s) = f(s,1) = 1$ for all $s, g, k \in M$. It is worth noting that the monoid crossed product is a general ring construction.

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Given a monoid crossed product $R * M$ with a twisting function f and an action ω, if the twisting f is trivial (i.e., $f(b,c) = 1$) for all $b, c \in M$, then $R * M$ is the skew monoid ring $R * M$. If both the twisting function f and the action ω are trivial, then $R * M$ becomes a monoid ring, denoted by $R[M]$ (see [1] and [2]). A monoid M is called a $u.p.$-monoid (unique product monoid) if, for any two nonempty finite subsets X and Y of M, there exists a unique element $h \in M$ that can be expressed as $h = uv$ with $u \in X$ and $v \in Y$. An ordered monoid $(M, \preceq)$ is termed strictly ordered if the following condition holds: whenever $g, k, h \in M$ with $g \prec k$, it follows that $gh \prec kh$ and $hg \prec hk$. The concept of a reversible ring was first introduced by Cohn in [3], defining a ring R as reversible if $bc = 0$ implies $cb = 0$ for all $b, c \in R$. Anderson and Camillo [4] referred to this property as ZC_2. The research presented in [5] demonstrated that reversible rings do not necessarily lead to reversible polynomial rings. A ring is considered reduced if it contains no non-zero nilpotent elements, meaning that $d^2 = 0$ implies $d = 0$ for all $d \in R$ (as defined in [5]). In [6], a strongly reversible ring is defined as one in which polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$ implies $g(x)f(x) = 0$. It is important to note that while every reduced ring is strongly reversible, the converse is not necessarily true. The concept of an Armendariz ring was introduced by Rege and Chhawchharia [7]. They defined a ring R to be Armendariz if the product of any two polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ in $R[x]$ equals zero only if the product of any corresponding coefficients a_i and b_j equals zero. A ring is said to be semicommutative if for all elements x and y in the ring, their product being zero implies that both their right and left annihilators are also zero. According to [8], a ring R is termed M-Armendariz if for any nonzero element $\mu = c_1s_1 + c_2s_2 + \cdots + c_ns_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$ in $R[M]$ satisfy $\mu\psi = 0$, then for all i, j, it follows that $c_ia_j = 0$. Furthermore, a ring R is strongly M-reversible if for any nonzero elements $\psi, \mu \in R[M]$ such that $\mu\psi = 0$ and $c_ia_j = 0$, it holds that $\psi\mu = 0$ for all $\psi, \mu \in R[M]$. This is discussed in detail in [9]. Numerous publications have explored various extensions of reversible rings, including strongly reversible rings, strongly M-reversible rings, Armendariz rings, reversible rings, and reflexive on skew monoid rings. These discussions are documented, and additional findings are also available in [8, 10–14] and [6]. According to [15], a ring R is referred to as an M-Armendariz ring of crossed product type relative to the given twisting f and action ω, (simply CM-Armendariz ring) if, for any $\mu = \sum_{i=1}^{n} c_is_i, \psi = \sum_{j=1}^{m} a_jh_j \in R * M$ such that $\mu\psi = 0$, it follows that $c_i\omega_{s_i}(a_j) = 0$ for all i, j and $s_i, h_j \in M$.

This paper consider various classes of nonreduced CM-Armendariz rings and investigating strongly CM-reversible rings, which are defined as a reversible-like property for the monoid crossed product $R * M$ in relation to the given twisting map f and action map ω. We prove that if, R is a semiprime, then $R * M$ is reversible if and only if $R * M$ is semicommutative. Additionally, we show that $R * M$ is semicommutative (symmetric, Dedekind finite, clean, uniquely clean, or exchange) if and only if R has the corresponding property under some conditions. Also, we prove that a ring R is strongly CM-reversible if and only if it is a right Ore ring with a classical right quotient ring Q. Finally, we discuss example and present some results related to this topic.

For a ring R, we denote by $U(R)$ group of units. An element $\mu \in R * M$ is a unit if there exists an element $\phi \in R * M$ such that $\mu\phi = 1$, where 1 is the identity element in $R * M$. The elements of the skew group ring $R * M$ can be expressed as finite sums of the form $\sum_{j=1}^{n} a_j h_j$, where $a_j \in R$ and $h_j \in M$. The action $\omega : M \to \mathrm{Aut}(R)$ and the twisting function $f : M \times M \to U(R)$ provide the necessary structure for the skew group ring.

2. Reversible ring of Crossed products type

In this section, we will primarily delve into the concept of strongly reversible properties in the context of a crossed product monoid $R * M$. For a ring R and a monoid M with a twisting map $f : M \times M \to U(R)$ a monoid homomorphism.

Definition 1. *Let $R*M$ be the crossed product of a ring R and a monoid M with respect to an action ω and a twisting map f. The ring R is said to be strongly M-reversible of crossed product type (or simply strongly CM-reversible) if, for any elements $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j$ in $R * M$ satisfying the condition that $\mu\psi = 0$ implies $c_i \omega_{s_i}(\omega_g(a_j)) = 0$, and $\psi\mu = 0$ implies $a_j \omega_{h_j}(\omega_g(c_i)) = 0$, for all i, j and for every $g \in M$.*

Remark 1. (1) *When a ring R is strong CM-reversibility with a trivial twisting map f, we classify the corresponding monoid M as a strongly M-reversible ring. Similarly, if R is strong CM-reversibility with a trivial action map ω, then R as a strongly TM-reversible (or twisted strongly M-reversible). It is worth noting that when both f and ω are trivial, R is simply considered strongly M-reversible. In particular, if $M = (\mathbb{N} \cup \{0\}, +)$ and both f and ω are trivial, then R is strongly CM-reversible if and only if R is strongly reversible.*

(2) *Suppose R is a strongly CM-reversible ring with a trivial twisting map f. In this case, if we consider any (M, ω)-invariant subring D of R (meaning $\omega_g(D) \subseteq D$ for all $g \in M$), then D is strongly CM-reversible.*

It is worth noting that the category of ω-rigid rings and ω-compatible rings is a limited one, and it is evident that every ω-compatible ring falls under the category of ω-weakly rigid rings. However, there exist several classes of ω-weakly rigid rings that do not belong to the category of ω-compatible rings. By [16], R is ω-rigid if and only if R is ω-compatible and reduced. According to [17], any prime ring that has an automorphism ω is considered to be ω-weakly rigid. If a monoid homomorphism $\omega : M \to Aut(R)$ is weakly-rigid (compatible), it means that the ring R is also weakly rigid (compatible) with respect to each $g \in M$ under the automorphism ω_g.

Lemma 1. *[18, Lemma 1.1] If M is a unique product monoid, then it is cancellative. This means that for any elements $\mu, h, \gamma \in M$, if $\mu\gamma = h\gamma$ or $\gamma\mu = \gamma h$, it follows that $\mu = h$.*

Lemma 2. *[19, Lemma 2] Let R be a ring and M be a unique product monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. If R is an (M, ω)-rigid ring, then the monoid ring $R * M$ is a reduced ring.*

Theorem 1. *Let R be a reduced ring and M be a unique product monoid (u.p.-monoid). Let $f : M \times M \to U(R)$ be a twisting map and $\omega : M \to Aut(R)$ be an action map. If R is (M, ω)-compatible, then R is strongly CM-reversible.*

Proof. Let $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j \in R * M$ satisfying $\mu\psi = 0$. Since R is reduced by Proposition 2.2 [15], R is CM-Armendariz, which implies $c_i \omega_{s_i}(a_j) = 0$ for all i, j. We now show the stronger condition $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all $g \in M$. For any $g \in M$, we have:

$$(c_1 s_1 + c_2 s_2 + \cdots + c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0. \tag{2.1}$$

We proceed by induction on n, to show that $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. This can be achieved by utilizing the fact that M is a compatible monoid. If we take $n = 1$, then we have $(c_1 s_1)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0$. Therefore, for each $1 \leq j \leq m$, we have $c_1 \omega_{s_1}(\omega_g(a_j)) f(s_1, h_j))(s_1 h_j) = 0$. Since M is cancellative by Lemma 1, the elements $s_1 h_j$ are distinct, $s_1 h_i \neq s_1 h_j$ for any i and j with $1 \leq j \leq m$. Thus, $c_1 \omega_{s_1}(a_j) f(s_1, h_j) = 0$ for each j. Since $f(s_1, h_j)$ is a unit, $c_1 \omega_{s_1}(a_j) = 0$. In a reduced (M, ω)-compatible ring, $ab = 0 \implies a\omega_g(b) = 0$ for any $g \in M$. Therefore, $c_1 \omega_{s_1}(a_j) = 0$ implies $c_1 \omega_{s_1}(\omega_g(a_j)) = 0$ for all j and all $g \in M$. Inductive step ($n \geq 2$). Assume the result holds for all μ with support of size less than n. Since M is a $u.p$-monoid, for the finite subsets $K = \{s_1, \ldots, s_n\}$ and $H = \{h_1, \ldots, h_m\}$, there exists $s_d h_t$ that is uniquely represented. Without loss of generality, let $d = 1$ and $t = 1$. From Eq. (2.1), the coefficient of the unique product $s_1 h_1$ must vanish, $c_1 \omega_{s_1}(a_1) f(s_1, h_1) = 0$ implies $c_1 \omega_{s_1}(a_1) = 0$. By (M, ω)-compatibility and the reduced property, $c_1 \omega_{s_1}(a_1) = 0$ implies $c_1 \omega_{s_1}(\omega_g(a_j)) = 0$ for all j and g. This allows us to "annihilate" the first row of coefficients. Specifically, $c_1 a_j = 0$ implies $c_1 s_1 \psi = 0$. Subtracting this from Eq. (2.1) yields

$$(c_2 s_2 + \cdots + c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m). \tag{2.2}$$

The support of this new element $\mu' = \sum_{i=2}^{n} c_i s_i$ is $n - 1$. By the induction hypothesis, $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all $2 \leq i \leq n$ and $1 \leq j \leq m$. Combining the base case and the inductive step, we have $c_i \omega_{s_i}(\omega_g(a_j)) = 0 \quad \text{for all } 1 \leq i \leq n, \quad 1 \leq j \leq m, \quad \forall g \in M$. To conclude that R is strongly CM-reversible, we must show that $\psi\mu = 0$ implies the corresponding coefficient vanishing. Suppose $\psi\mu = 0$. Since R is reduced and compatible, the symmetry of the Armendariz property in reduced rings implies $c_i a_j = 0 \iff a_j c_i = 0$. Applying the same inductive argument to $\psi\mu = 0$, we can see that

$$a_j \omega_{h_j}(\omega_g(c_i)) = 0 \quad \text{for all } i, j \text{ and } g \in M.$$

Thus, R is strongly CM-reversible. □

The following example illustrates the existence of a ring R over a field F which is not strongly CM-reversible.

Example 1. *If M is a monoid with a minimum of two elements, and $D = M_2(F)$ is the matrix ring over a field F equipped with a twisting map $f : M \times M \to U(R)$, then D is not strongly CM-reversible.*

Solution. Take $e \neq s \in M$, we define $\omega : M \to Aut(D)$ by

$$\omega_s\left(\left(\begin{array}{cc} x & y \\ 0 & t \end{array}\right)\right) = \left(\begin{array}{cc} x & -y \\ 0 & t \end{array}\right).$$

If the twisting map f is trivial, meaning that $f(b, c) = 1$ for all $b, c \in M$, then it follows that the ring D is not strongly CM-reversible. To prove this, for any elements $\mu = E_{12}e + E_{11}s$ and $\psi = E_{11}e + E_{12}s \in D * M$. We can verify that $\mu\psi = 0$. But, $\psi\mu \neq 0$, which implies that the ring D is not strong CM-reversibility □

Suppose I is an ideal of R and $\omega : M \to \mathrm{Aut}(R)$ is a monoid homomorphism. We define $\bar{\omega} : M \to \mathrm{Aut}(R/I)$ as $\bar{\omega}_s(d + I) = \omega_s(d) + I$, where $d \in R$ and $s \in M$. It can be shown that $\bar{\omega}$ is a monoid homomorphism.

Additionally, the twisting map $f : M \times M \to U(R)$ induces a twisting map $\bar{f} : M \times M \to U(R/I)$ given by $\bar{f}(x, y) = f(x, y) + I$. Furthermore, for every $\mu = \sum_{i=1}^{n} c_i s_i \in R * M$, we denote $\bar{\mu} = \sum_{i=1}^{n} \bar{c}_i s_i \in (R/I) * M$, where $\bar{c}_i = c_i + I$ for $1 \leq i \leq n$. It can be easily verified that the mapping $\theta : R \times M \to (R/I) \times M$ defined as $\theta(\mu) = \bar{\mu}$ is a ring homomorphism.

In [9], it was shown that when I is a reduced ideal of R and R/I is strongly M-reversible, then R is strongly M-reversible. Similarly, we have the following result.

Theorem 2. *Let M be a u.p.-monoid and I an ideal of R with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If I is a reduced and R/I is strongly CM-reversible, then R is strongly CM-reversible.*

Proof. Let $\mu = \Sigma_{i=1}^{n} c_i s_i, \psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\mu\psi = 0$. We will show that $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for any i and j. Note that in $(R/I) * M$, $\bar{\mu} = \Sigma_{i=1}^{n} \bar{c}_i s_i, \bar{\psi} = \Sigma_{j=1}^{m} \bar{a}_j h_j \in (R/I) * M$, we have

$$\begin{aligned} \bar{0} = \;& \bar{\mu}\bar{\psi} \\ = \;& (\bar{c}_1 s_1 + \bar{c}_2 s_2 + \cdots + \bar{c}_n s_n)\omega_{s_i}(\omega_g(\bar{a}_1 h_1 + \bar{a}_2 h_2 + \cdots + \bar{a}_m h_m))f(s_i, h_j)s_i h_j \\ = \;& (c_1 + I)\bar{\omega}_{s_1}(\omega_g(a_1 + I))f(s_1, h_1)s_1 h_1 + (c_2 + I)\bar{\omega}_{s_2}(\omega_g(a_2 + I))f(s_2, h_2)s_2 h_2 \\ + \;& \cdots + (c_n + I)\bar{\omega}_{s_n}(\omega_g(a_m + I))f(s_n, h_m)s_n h_m. \end{aligned}$$

Thus we have $c_i \omega_{s_i}(\omega_g(a_j))f(s_i, h_j)(s_i h_j) \subseteq I$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq m$ since R/I is strongly CM-reversible.

By induction on both n and m, considering every $g \in M$, and for $1 \leq i \leq n$ and $1 \leq j \leq m$. If we take $n = 1$. Then

$$(c_1 s_1)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0.$$

Thus, $(c_1 s_1)(a_1 h_1) + (c_1 s_1)g(a_2 h_2) + \cdots + (c_1 s_1)(a_m h_m) = c_1 \omega_{s_1}(\omega_g(a_1))f(s_1, h_1)(s_1 h_1) + c_1 \omega_{s_1}(\omega_g(a_2))f(s_1, h_2)(s_1 h_2) + \cdots + c_1 \omega_{s_1}(\omega_g(a_m))f(s_1, h_m)(s_1 h_m) = 0$ for any $g \in M$. By Lemma 1, M is cancellative we have $s_1 h_i \neq s_1 h_j$ for any i and j with $1 \leq i \neq j \leq m$. Then $c_1 \omega_{s_1}(\omega_g(a_j))f(s_1, h_j)(s_1 h_j) = 0, j = 1, 2, \ldots, m$. Thus, $c_1 \omega_{s_1}(\omega_g(a_j)) = 0$ for any j. If $m = 1$, then, the proof is similar.

Now suppose that $n \geq 2$ and $m \geq 2$. Since M is a $u.p.$-monoid, there exists i, j with $1 \leq$

$i \leq n$ and $1 \leq j \leq m$ such that $s_i g h_j$ is uniquely presented by considering two subsets $K = \{s_1 g, s_2 g, \ldots, s_n g\}$ and $H = \{h_1, h_2, \ldots, h_m\}$ of the monoid M. Without loss of generality, we may assume that $i = 1$ and $j = 1$. We can deduce that $c_1 \omega_{s_1}(\omega_g(a_1)) f(s_1 g, h_1) s_1(g h_1) = 0$, which implies that $c_1 \omega_{s_1}(\omega_g(a_1)) = 0$. Since ω_g and ω_{s_1} are automorphisms of R, we have $c_1 \omega_{s_1}(a_1) = 0$. Let $b = c_k a_q$, where $1 \leq k \leq n, 1 \leq q \leq m$. Then $b \in I$. Since $(a_1 b c_1)^2 = 0$ and I is reduced, we have $a_1 b c_1 = 0$. Thus,

$$\begin{aligned}&(a_1 b c_2 s_2 + a_1 b c_3 s_3 + \cdots + a_1 b c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m)\\ &= (a_1 b)(c_2 s_2 + c_3 s_3 + \cdots + c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0.\end{aligned}$$

By induction, we have $a_1 b c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for $2 \leq i \leq n$ and $1 \leq j \leq m$. Thus, $(a_1 b c_i)^2 = 0$. Since I is reduced and ω is automorphism, it follows that $a_1 b c_i \omega_{s_i}(a_j) = 0$. Note that $b c_i \omega_{s_i}(a_1) \subseteq I$. Thus $b c_i \omega_{s_i}(a_1) = 0$ for any i. Now we have

$$\begin{aligned}&(b c_1 s_1 + b c_2 s_2 + \cdots + b c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m)\\ &= (b)(c_1 s_1 + c_2 s_2 + \cdots + c_n s_n)(a_1 h_1 + a_2 h_2 + \cdots + a_m h_m) = 0.\end{aligned}$$

By applying the induction hypothesis, it follows that $b c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all $1 \leq i \leq n$ and $2 \leq j \leq m$. Thus, we have $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all i, j. Particularly, we have $b c_k \omega_{s_k}(\omega_g(a_q)) = 0$ and so $b^2 = 0$. Thus $b = 0$. This shows that $c_k \omega_{s_k}(\omega_g(a_q)) = 0$ for any $1 \leq k \leq n$ and $1 \leq q \leq m$. Consequently, we can see that $a_j \omega_{h_j}(\omega_g(c_i)) = 0$ for all $g \in M$, $1 \leq j \leq m$, and $1 \leq i \leq n$. Therefore, R is strongly CM-reversible. □

The notion of complete (M, ω)-compatibility is important in the following result [12].

Corollary 1. *Assume that R is a ring that is completely (M, ω)-compatible, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Let I be an ideal of R such that I is reduced ring and R/I is CM-Armendariz. Then R is strongly CM-reversible.*

Proof. Since every CM-Armendariz rings is strongly CM-reversible, the result follows directly from Theorem 2. □

Proposition 1. *Assume that R is a ring that is both (M, ω)-compatible and CM-Armendariz, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then R is reversible if and only if $R * M$ is reversible.*

Proof. Proving a necessary condition is sufficient. Let $\mu = \Sigma_{i=1}^{n} c_i s_i, \psi = \Sigma_{j=1}^{m} a_j h_j \in R * M$ satisfying $\mu\psi = 0$. Since R is CM-Armendariz, we have $c_i \omega_{s_i}(\omega_g(a_j)) f(s_i, h_j)(s_i h_j) = 0$ for all i, j. This implies that $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all i, j since R is M-compatible. Because R is a reversible ring, $a_j c_i = 0$. Then, $a_j \omega_{h_j}(\omega_g(c_i)) = 0$ for all i, j, and for any $g \in M$. Thus, we have

$$\psi\mu = \Sigma_{j=1}^{m} \Sigma_{i=1}^{n} a_j \omega_{h_j}(\omega_g(c_i)) f(h_j, s_i)(h_j s_i) = 0.$$

Therefore, $R * M$ is reversible. □

If N is an ideal of the monoid M with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, then the restrictions $f|_{N\times N} : N \times N \to U(R)$ and $\omega|_N : N \to Aut(R)$ are induced twisting and action.

Proposition 2. *Let R be an M-compatible ring and M be a commutative, cancellative monoid and N be an ideal of M with a center element λ. If R is strongly CN-reversible, then R is strongly CM-reversible.*

Proof. Let $\mu = \sum_{i=1}^{n} c_i s_i, \psi = \sum_{j=1}^{m} a_j h_j \in R * M$ satisfying $\mu\psi = 0$. Since $\gamma \in N$ is a center element, this implies that $\gamma s_1, \gamma s_2, \ldots, \gamma s_n, h_1\gamma, h_2\gamma, \ldots, h_m\gamma \in N$, such that $\gamma l_i \neq \gamma l_j$ and $h_i\gamma \neq h_j\gamma$ for all $i \neq j$. Then, we have

$$\mu_1\psi_1 = \sum_{i=1}^{n}\sum_{j=1}^{m}(c_i\omega_{s_i}(\omega_\gamma(a_j)))f(s_i\gamma, h_j)(\gamma^2 s_i h_j) = 0.$$

Since μ and ψ are nonzero in $R * M$, so μ_1 and ψ_1 are nonzero elements in $(R * M)[N]$. Moreover, from $\mu\psi = 0$ and ω compatible automorphism, γ a center element of N one can easily obtain that $\mu_1\psi_1 = 0$. Since R is strongly CN-reversible. Then, $c_i\omega_{s_i}(\omega_\lambda(a_j))f(s_i, h_j)(s_i h_j) = 0$. So $c_i\omega_{s_i}(\omega_\gamma(a_j)) = 0$. By a compatible automorphism, we have $a_j\omega_{h_j}(\omega_\gamma(c_i)) = 0$. Therefore, R is strongly CM-reversible. □

Corollary 2. *[9] Let M be a cancellative monoid and N an ideal of M. If R is strongly N-reversible, then R is strongly M-reversible.*

3. Some results on ring extensions of Crossed products type

In this section we examining several kinds of ring extensions which have roles in ring theory, being concerned with cross product of reversible rings. A ring R is considered to be a right (or left) Ore if, for any $x, y \in R$ is non zero-divisor, there exist $x_1, y_1 \in R$ such that y_1 is non zero-divisor, and $xy_1 = yx_1$ (or $y_1x = x_1y$, respectively). According to Theorem 6.2 [20], R possesses a right quotient ring if and only if the collection of regular elements in R constitutes a right Ore set.

Assume that the classical right quotient ring Q of R exists. Then, for any automorphism ω of R, we can define an induced map $\bar{\omega} : Q \to Q$ as follows: for any element $xy^{-1} \in Q$, where $x, y \in R$ and y is regular, $\bar{\omega}(xy^{-1}) = \omega(x)\omega(y)^{-1}$, if R is a ring and ω is a monomorphism, then by Lemma 2.11 [10], we obtain the following results for strongly CM-reversible.

Lemma 3. *Let R be a ring, M be a u.p.-monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$ a monoid homomorphism. If R is strongly CM-reversible, then $R * M$ is semicommutativity.*

Proof. Let $\mu = c_1s_1 + c_2s_2 + \cdots + c_ns_n, \psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m \in R*M$ satisfying $\mu\psi = 0$ implies that $c_i\omega_{s_i}(\omega_g(a_j))f(s_i, h_j))(s_ih_j) = 0$ for any $g \in M$. So, by Lemma 1,

M is a cancellative, this means $s_1h_i \neq s_1h_j$ for any i and j with $1 \leq i \neq j \leq m$. Thus, $c_i\omega_{s_i}(\omega_g(a_j)) = 0$. Since R is strongly CM-reversible. For any $\varphi = b_1t_1 + b_2t_2 + \cdots + b_dt_d \in R * M$ for each $1 \leq k \leq d$. Hence, $c_i\omega_{s_i}(\omega_g(Ra_j)) = 0$ by compatibility. Thus, $\mu\varphi\psi = 0$. Therefore, $R * M$ is semicommutative. □

Theorem 3. *Assume that R is an (M,ω)-compatible ring and M is a cancellative monoid with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. Considering R as a right Ore ring with the classical right quotient ring Q, the R is strongly CM-reversible if and only if Q is strongly CM-reversible.*

Proof. It is enough showing necessary. Assume that R is strongly CM-reversible. Let $\mu = \Sigma_{i=1}^{m}\alpha_i s_i, \psi = \Sigma_{k=1}^{p}\gamma_k h_k$ be an elements in $Q*M$ satisfying $\mu\psi = 0$. Using the property of right Ore rings a cording to Proposition 2.1.16 [16], we can represent all coefficients α_i and γ_k using common right denominators. There exist regular elements $u, w \in R$ and elements $a_i, c_k \in R$ such that $\alpha_i = a_iu^{-1}$ and $\gamma_k = c_kw^{-1}$. Thus,

$$\mu = \left(\sum_{i=1}^{m} a_i s_i\right) u^{-1} \quad \text{and} \quad \psi = \left(\sum_{k=1}^{p} c_k h_k\right) w^{-1}.$$

Let $\mu_1 = \sum_{i=1}^{m} a_i s_i$ and $\psi_1 = \sum_{k=1}^{p} c_k h_k \in R * M$. The condition

$$\mu\psi = 0 \implies (\mu_1 u^{-1})(\psi_1 w^{-1}) = 0.$$

By the Ore condition, there exist regular element $t \in R$ such that $u^{-1}\psi_1$ can be rewritten as $\psi_2 t^{-1}$ for some $\psi_2 = \sum c_k h_k \in R * M$. Since R is strongly CM-reversible, ω is an automorphism of R and M is a cancellative monoid by Lemma 1, $s_ih_1 \neq s_jh_1$ for $s_i \neq s_j$ it follows that by substituting,

$$\begin{aligned}
\mu\psi &= \Sigma_{i=1}^{m}\Sigma_{k=1}^{p}(a_iu^{-1})\omega_{s_i}(\omega_g(c_kw^{-1}))f(s_i,h_k)(s_ih_k) \\
&= \Sigma_{i=1}^{m}\Sigma_{k=1}^{p}a_i\omega_{s_i}(\omega_g(c_k))f(s_i,h_k)(s_ih_k)(\omega_{s_i}(t^{-1})) \\
&= \Sigma_{i=1}^{m}\Sigma_{k=1}^{p}a_i\omega_{s_i}(\omega_g(c_k)) \\
&= \mu_1\psi_2 = 0.
\end{aligned}$$

Therefore, $\mu_1\psi_2 = 0$ implies that the coefficients vanish coefficient-wise, $a_i\omega_{s_i}(\omega_g(c_k)) = 0$ for all i, k and $g \in M$. Thus, $\mu_1\psi_2 = 0$ implies that $\psi_2\mu_1 = 0$. Using the (M,ω)-compatibility and the relation $u\psi_2 = \psi_1 t$, we translate the vanishing property back to the original elements. Since the annihilator structure is preserved in Q,

$$\psi\mu = (\psi_1 w^{-1})(\mu_1 u^{-1}) = 0.$$

Consequently, the individual coefficients in Q must satisfy $c_k\omega_{h_k}(\omega_g(a_i)) = 0$, therefore, proving that Q is strongly CM-reversible. □

Corollary 3. *[9, Theorem 2]. If M be a monoid and R be a right Ore ring with classical right quotient ring Q of a ring R, then the ring R is strongly M-reversible if and only if its quotient ring Q is also strongly M-reversible.*

Proposition 3. *Let R be an (M,ω)-compatible ring and M a monoid, with a twisting map $f : M \times M \to U(R)$ and an action map $\omega : M \to Aut(R)$. Let e be a central idempotent of R such that $\omega_s(e) = e$ for all $s \in M$. Then R is strongly CM-reversible if and only if eR and $(1-e)R$ are strongly CM-reversible.*

Proof. ($\Longrightarrow$) Let R be strongly CM-reversible and $S \subseteq R$ be a subring. Since e is a central idempotent and $\omega_s(e) = e$, and $\omega_s(1-e) = 1-e$, S is a subring of R invariant under the action ω (i.e., $\omega_s(S) \subseteq S$). Let $\mu = \sum_{i=1}^{n} c_i s_i, \psi = \sum_{j=1}^{m} a_j h_j \in S * M$. If $\mu\psi = 0$ in $S * M$, then $\mu\psi = 0$ in $R * M$ because $S * M \subseteq R * M$. By the strong CM-reversibility of R, we have $c_i \omega_{s_i}(\omega_g(a_j)) = 0$ for all i, j and $g \in M$. Since the coefficients c_i, a_j already belong to S, and ω preserves S, these relations hold within S. Thus, eR and $(1-e)R$ are strongly CM-reversible.

($\Longleftarrow$) Assume eR and $(1-e)R$ are strongly CM-reversible. Let $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j$ are elements in $R * M$ satisfying that $\mu\psi = 0$. Using the central idempotent e, we decompose the elements, $\mu = e\mu + (1-e)\mu$ and $\psi = e\psi + (1-e)\psi$. Since $\omega_s(e) = e$ and e is central, the twisting map $f(s,h)$ and action ω respect the decomposition $R \cong eR \oplus (1-e)R$. Thus, $0 = e(\mu\psi) = (e\mu)(e\psi) \in (eR) * M$ and $0 = (1-e)(\mu\psi) = ((1-e)\mu)((1-e)\psi) \in ((1-e)R) * M$. Applying the strong CM-reversibility of eR and $(1-e)R$ to the coefficients for the eR component, $(ec_i)\omega_{s_i}(\omega_g(ea_j)) = 0$ for all i, j and $g \in M$, similarly for the $(1-e)R$ component, $((1-e)c_i)\omega_{s_i}(\omega_g((1-e)a_j)) = 0$ for all i, j and $g \in M$. Taking the direct sum of these relations, $e[c_i\omega_{s_i}(\omega_g(a_j))] + (1-e)[c_i\omega_{s_i}(\omega_g(a_j))] = 0$ which simplifies to $c_i\omega_{s_i}(\omega_g(a_j)) = 0$ in R. Similarly, if $\psi\mu = 0$, the same applies to the components $e(\psi\mu)$ and $(1-e)(\psi\mu)$ to conclude that $a_j\omega_{h_j}(\omega_g(c_i)) = 0$ for all i, j and $g \in M$. Therefore, R is strongly CM-reversible. $\square$

Proposition 4. *Let R be a CM-Armendariz ring and M a strictly ordered monoid, equipped with a twisting map $f : M \times M \to U(R)$ and an action $\omega : M \to Aut(R)$. Let $e \in R$ be a non-zero idempotent such that $\omega_g(e) = e$ for all $g \in M$. Then, the corner subring eRe is strongly CM-reversible.*

Proof. The proof is a variant of the proof given in Proposition 2.9 [21]. Let $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j \in (eRe) * M$ satisfying $\mu\psi = 0$. Since M is a strictly totally ordered monoid, we can assume that $s_i \preceq s_j$ and $h_i \preceq h_j$ whenever $i < j$. Since R is CM-Armendariz, then so is eRe. Thus, we have $c_i\omega_{s_i}(\omega_g(a_j))f(s_i,h_j)(s_ih_j) = 0$ for all i, j. This implies that $c_i\omega_{s_i}(\omega_g(a_j)) = 0$ for all i, j since R is M-compatible and ω is an automorphism. Therefore, by Proposition 3, eRe is strongly CM-reversible. $\square$

Proposition 5. *Let R be a ring and M be a monoid, $\omega : M \to Aut(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$. A ring R is (M,ω)-rigid if and only if R is strongly CM-reversible and semiprime.*

Proof. Note that any (M,ω)-rigid ring is reduced by [16]. Assume that $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j \in R*M$ satisfying $\mu\psi = 0$. Then, $c_i\omega_{s_i}(a_j)\omega_{s_i}(c_i\omega_{s_i}(a_j))f(s_i,h_j)(s_ih_j)$ $= c_i\omega_{s_i}(a_jc_i)\omega_{s_i}(a_j) = 0$. Thus, $c_i\omega_{s_i}(a_j) = 0$. Therefore, R is strongly CM-reversible.

Conversely, assume $c_i\omega_{s_i}(c_i) = 0$ for $c \in R$. For any $r \in R, 0 = c_i\omega_{s_i}(c_i)\omega_{s_i}(r) = c_i\omega_{s_i}(rc_i)$. Since R is strongly CM-reversible, $\omega_{s_i}(c_irc_i) = 0$ for any $r \in R$. Thus, $c_i = 0$ since ω is a homomorphism of a semiprime ring R. Therefore, R is (M, ω)-rigid. □

Corollary 4. *Let $(M, \leq)$ a strictly ordered a.n.u.p.-monoid, $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$ and R be a locally finite ring. Assume that R is (M, ω)-compatible. Then the following statements are equivalent:*
(1) R is semiprime and CM-Armendariz;
(2) R is (M, ω)-rigid;
(3) $R * M$ is reduced.

Proof. (1) ⇔ (2). If R is CM-Armendariz, then R is strongly CM-reversible. Thus the result follows from Proposition 5. (2) ⇔ (3). Since R is (M, ω)-compatible, so R is (M, ω)-rigid. Thus the result follows from Lemma 2. □

Theorem 4. *Let R be a semiprime, M be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$. Then $R * M$ is reversible if and only if $R * M$ is semicommutative and ω_s is injective for each $s \in M$.*

Proof. Let $\mu = \sum_{i=1}^{n} c_is_i, \psi = \sum_{j=1}^{m} a_jh_j \in R * M$ satisfying $\mu\psi = 0$. If $R * M$ is reversible, then we have $c_i\omega_{s_i}(a_j)f(s_i, h_j)(s_ih_j) = c_i\omega_{s_i}(a_j) = 0$ then by Theorem 4.12 [22] we have ω_g is injective for each $g \in M$. So, $R * M$ is semicommutative by Lemma 3. Now, assume $g \in M$, $\omega_g \in End(R)$ and $b \in R$ such that $b\omega_g(b) = 0$. Since R is semiprime and semicommutative, then R is a reduced ring and so $\omega_g(b)b = 0$. Thus, by Lemma 4.4 [22], we have $\omega_g(b)\omega_g(b) = 0$. Then $b^2 = 0$ and so $b = 0$. Thus for each $g \in M, \omega_g$ is a rigid endomorphism. Then by Theorem 4.12 [22], $R * M$ is reversible and the result follows. □

The following proposition shows that a crossed products over semicommutative rings R are semicommutative when R is CM-Armendariz.

Proposition 6. *Let M be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$ and R be an CM-Armendariz. Then R is semicommutative if and only if $R * M$ is semicommutative.*

Proof. Suppose that R is semicommutative and $\mu = \sum_{i=1}^{n} c_is_i, \psi = \sum_{j=1}^{m} a_jh_j \in R * M$ are such that $\mu\psi = 0$. Since R is CM-Armendariz and semicommutative, we have $c_i\omega_{s_i}(a_j)f(s_i, h_j)(s_ih_j) = 0$. Now for any $\phi = \sum_{k=1}^{q} b_k\lambda_k \in R * M$ and any $g \in M$, we have $(\mu\phi\psi)(g) = \sum_{(c,b,a)\in X_g(\mu,\lambda,\psi)} c_i\omega_{s_i}(\omega_\lambda(Ra_j))f(s_i, h_j)(s_ih_j) = c_i\omega_{s_i}(\omega_\lambda(Ra_j)) = 0$. Thus, $\mu\phi\psi = 0$. This shows that $\mu(R * M)\psi = 0$. This means $R * M$ is semicommutative. Conversely, if $R * M$ is semicommutative, then R is semicommutative since subrings of semicommutative are also semicommutative. □

Proposition 7. *Let R be a ring that is both (M, ω)-compatible and CM-Armendariz, where M is a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then R is symmetric if and only if $R * M$ is symmetric.*

Proof. Since any subring of a symmetric ring is again a symmetric ring. It suffices to show that if R is a symmetric ring, then so is $R * M$.
Let $\mu = \sum_{i=1}^{n} c_i s_i, \psi = \sum_{j=1}^{m} a_j h_j, \phi = \sum_{k=1}^{q} b_k \lambda_k \in R * M$ are such that

$$\mu\phi\psi = \sum_{(c,b,a)\in X_{(s,\lambda,h)}(\mu,\phi,\psi)} c_i\omega_{s_i}(b_k)\omega_{s_i}(\omega_{\lambda_k}(a_j))f(s_i,\lambda_k,h_j)(s_i\lambda_k h_j) = c_i\omega_{s_i}(\omega_\lambda(Ra_j)) = 0.$$

Since R is both CM-Armendariz and (M,ω)-compatible, we have

$$c_i\omega_{s_i}(R\omega_{h_j}(b_k))f(s_i,\lambda_k,h_j)(s_i\lambda_k h_j) = c_i\omega_{s_i}(R\omega_{h_j}(b_k)) = 0, \forall\ s_i,\lambda_k,h_j \in M.$$

Thus, $\mu\psi\phi = 0$. Therefore, $R * M$ is a symmetric ring. □

A ring R is said to be Dedekind finite if $c\,a = 1$ implies $a\,c = 1$ for any $a, c \in R$.

Proposition 8. *Let R be a ring that is (M,ω)-compatible, where M is a monoid with twisting map $f : M \times M \to U(R)$ and action map $\omega : M \to Aut(R)$. If R is CM-Armendariz, then both R and the crossed product $R * M$ are Dedekind finite.*

Proof. To prove R is Dedekind finite. Let $c, a \in R$ such that $c\,a = 1$. Consider the elements $\mu = c\,s_1$ and $\psi = a\,s_1 \in R * M$. Then $\mu\psi = (c\,s_1)(a\,s_1) = c\,\omega_1(a)f(1,1)s_1$. Since $\omega_1 = id_R$ and $f(1,1) = 1$, we get $\mu\psi = c\,a s_1 = s_1$. Thus, $\mu\psi = s_1$. Because R is CM-Armendariz, from $\mu\psi = s_1$ we conclude that the constant coefficients satisfy $a\,c = 1$. Therefore, R is Dedekind finite.
Now, to prove $R*M$ is Dedekind finite. Suppose that $\mu = \sum_{i=1}^{n} c_i s_i$ and $\psi = \sum_{j=1}^{m} a_j h_j \in R*M$ are such that $\mu\psi = \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\omega_{s_i}(a_j)f(s_i,h_j)s_i h_j = 0$. Since R is CM-Armendariz, it follows that $c_i\omega_{s_i}(a_j) = 0$ for all i, j. Now consider the coefficient of s_1 in $\mu\psi$. Hence $c_1\,a_1 = 1$ where $s_1 = h_1 = 1$. Since R is Dedekind finite, we conclude that $a_1\,c_1 = 1$. Now, $(\psi\mu)(1) = a_1\,c_1 = 1$. Moreover, the coefficient of $\psi\mu$ is $a_j\omega_{h_j}(c_i)f(h_j,s_i)$. Using (M,ω)-compatibility together with the relations $c_i\,\omega_{s_i}(a_j) = 0$, we obtain $a_j\omega_{h_j}(c_i) = 0$. Hence, $\psi\mu = s_1 = 1$. Therefore, $R * M$ satisfies this property. □

Theorem 5. *Let R be a ring, M be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$ and R a ring without nonzero zero-divisors. Then R is Dedekind finite if and only if $R * M$ is Dedekind finite ring.*

Proof. It is evident that if $R * M$ is Dedekind finite, then R must also be Dedekind finite. Now suppose that R is Dedekind finite, and $\mu = c_i\,s_i, \psi = a_j\,h_j \in R * M$ are such that $\mu\psi = \sum_{i=1}^{n}\sum_{j=1}^{m} c_i\omega_{s_i}(a_j)f(s_i,h_j)s_i h_j$. We will show that $\psi\mu = c_1$. Note that $\mu \neq 0$ and $\psi \neq 0$. By Ribenboim [23], there exists a compatible strict total order $\leq'$ on M, which is finer than $\leq$ (that is, for all $s, h \in M, s \leq h$ implies $s \leq'$ h). Since $supp(\mu)$ and $supp(\psi)$ are non-empty artinian and narrow subsets of M, the sets $Min(supp(\mu))$ and $Min(supp(\psi))$ of minimal elements of $supp(\mu)$ and $supp(\psi)$, respectively, are finite and non-empty. Let $Min(supp(\mu)) = \{s_1, \ldots, s_n\}$ with $s_1 <' s_2 <' \ldots <' s_n$ and

$Min(supp(\psi)) = \{h_1, h_2, \ldots, h_m\}$ with $h_1 <' h_2 <' \ldots <' h_m$. Then

$$c_1(s_1 + h_1) = \mu\psi(s_1 + h_1) = \sum_{(c,a)\in X_{(s_1+h_1)}(\mu,\psi)} c_i\omega_{s_i}(a_j)f(s_i, h_j)(s_ih_j).$$

Since $\mu\psi = 1$ in $R*M$ this means $c_i\omega_{s_i}(a_j)f(s_i, h_j)(s_ih_j) = 1$. For $\mu\psi = 1$ to hold, it must be that there are non-zero contributions from c_i and $\omega_{s_i}(a_j)$. By assumption of Dedekind finiteness of R the fact $c_i\omega_{s_i}(a_j) = 0$ for some i, j implies that $c_i = 0$ or $\omega_{s_i}(a_j) = 0$. Thus, if $c\,a = 1$ for some $c, a \in R$, we conclude $a\,c = 1$. Therefore, $R * M$ is Dedekind finite. □

Corollary 5. *[24, Proposition 3.15]. Let R be a ring, $(M, \leq)$ be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism. If R is (M, ω)-Armendariz, then R and $R[[M, \omega]]$ are Dedekind finite.*

According to Nicholson [25], a ring R is called a clean (uniquely clean) ring if every element $r \in R$ can be written (uniquely) in the form $r = u + e$ where $u \in U(R)$ and $e^2 = e \in R$.

Lemma 4. *Liu [26] A homomorphic image of a clean ring is a clean ring.*

Lemma 5. *Let R be a ring and M be a strictly ordered monoid satisfying the condition $1 \leq s$ for every $s \in M$. Let $f : M \times M \to U(R)$ be a twisting map and $\omega : M \to Aut(R)$ be an action map. Suppose that $\mu \in R * M$. Then μ is a unit in the crossed product, denoted $\mu \in U(R * M)$ if and only if its constant coefficient is a unit, $\mu(1) \in U(R)$.*

Proof. Suppose $\mu \in U(R * M)$. There must exist an inverse $\psi \in R * M$ such that $\mu\psi = 1$ and $\psi\mu = 1$. Let $\mu = c_1s_1 + c_2s_2 + \cdots + c_ns_n$ and $\psi = a_1h_1 + a_2h_2 + \cdots + a_mh_m$. Since M is strictly ordered and $1 \leq s$ for all $s \in M$, we can order the supports such that $1 = s_1 < s_2 < \cdots < s_n$ and $1 = h_1 < h_2 < \cdots < h_m$. The coefficient of the identity element $1 \in M$ in the product $\mu\psi = c_1\omega_{s_1}(a_1)f(s_1, h_1) = 1$. Since $s_1 = 1$ and $h_1 = 1$, and ω_1 is the identity automorphism, this simplifies to $c_1a_1f(1, 1) = 1$. Because $f(1, 1)$ is a unit, c_1 has a right inverse in R. A symmetric argument using $\psi\mu = 1$ shows c_1 has a left inverse. Thus, $\mu(1) \in U(R)$.
Now assume $\mu(1) \in U(R)$. We want to show that $\mu \in U(R*M)$. We construct a right inverse $\psi = \sum a_jh_j$ such that $\mu\psi = 1$. Write $\mu = \mu(1)u_1 + \sum_{s>1} \mu(s)u_s$. Let $\phi = \mu(1)^{-1} \in R$. Define $\psi(1) = \phi$. It follows that the constant coefficient of the product $\mu\psi$ is given by $(\mu\psi)(1) = \mu(1)\psi(1) = 1$. For the identity $1 \in M$, we require $c_1a_1f(1, 1) = 1$. Since c_1 and $f(1, 1)$ are units, we uniquely define $a_1 = c_1^{-1}f(1, 1)^{-1}$. For any $\lambda > 1$ in M, the coefficient of λ in the product $\mu\psi$ must vanish. The coefficient of λ is

$$c_1\omega_1(a_\lambda)f(1, \lambda) + \sum_{s_ih_j=\lambda, s_i>1} c_i\omega_{s_i}(a_j)f(s_i, h_j) = 0.$$

In a strictly ordered monoid with $1 \leq s$, if $s_ih_j = \lambda$ and $s_i > 1$, then it must be that $h_j < \lambda$. By the induction hypothesis, all such a_j for $h_j < \lambda$ have already been uniquely determined. Therefore, we can solve for a_λ as follows:

$$a_\lambda = -\left[c_1^{-1}\left(\sum_{s_ih_j=\lambda, s_i>1} c_i\omega_{s_i}(a_j)f(s_i,h_j)\right)f(1,\lambda)^{-1}\right].$$

Since μ has finite support and M is strictly ordered, the coefficients a_j are obtained as unique solutions to the recursive system of equations. Thus, μ has a right inverse. A similar construction yields a left inverse, Therefore, $\mu \in U(R * M)$. □

Theorem 6. *Let R be a ring and M be a strictly ordered monoid satisfying the condition $1 \leq s$ for every $s \in M$. Let $f : M \times M \to U(R)$ be a twisting map and $\omega : M \to Aut(R)$ be an action map. Then the crossed product $R * M$ is a clean ring if and only if R is a clean ring.*

Proof. We use the method in the proof of Theorem 5.3 [26]. Let R be a clean ring and $\mu \in R * M$. Write $\mu(1) = u + e$ where $u \in U(R)$ and $e^2 = e \in R$. Define $\psi \in R * M$ via

$$\psi(s) = \begin{cases} u, & s = 1 \\ \mu(s), & 1 < s. \end{cases}$$

We obtain a map $\psi : M \to R$ with $suup(\psi) = supp(\mu) \cup \{1\}$. Since $supp(\mu)$ is artinian and narrow, $\psi \in R * M$. Then, by Lemma 5, $\psi \in U(R * M)$. Clearly, $\mu = \psi + c_e$ and $c_e^2 = c_e$. Thus, $R * M$ is a clean ring.

Conversely, suppose that $R * M$ is a clean ring, we show that R is a clean ring. Let $I = \{\mu \in R * M \mid \mu(1) = 1\}$. For any $\mu \in I$ and any $\psi \in R * M$, since M satisfying the condition that $1 \leq s$ for every $s \in M$, so we have $\mu\psi(1) = 0$ which implies that $\mu\psi \in I$. Similarly, $\psi\mu \in I$. Thus, I is an ideal of $R*M$. Define a mapping $\phi : R \to R*M/I$ for any $\vartheta = bt \in R * M$ by setting $\phi(b) = \vartheta + I$ for every $b \in R$ and $t \in M$. It is straightforward to verify that ϕ is a ring isomorphism. Therefore, by Lemma 4. R is a clean ring. □

Theorem 7. Let R be a ring, M be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Assume that R is CM-Armendariz. Then $R * M$ is a uniquely clean ring if and only if R is a uniquely clean ring.

Proof. We use the method in the proof of Theorem 5.6 [26]. Suppose that R is a uniquely clean ring. Then by Theorem 6, $R*M$ is a clean ring. Assume that $\mu = \psi_1 + h_1 = \psi_2 + h_2$ where $\psi_1, \psi_2 \in U(R * M)$ and $h_1, h_2 \in R * M$ are idempotents. Since R is CM-Armendariz, by Proposition 4.10 [22], $h_1(1)$ and $h_2(1)$ are idempotent elements of R and $h_1 = c_{h_1}(1), h_2 = c_{h_2}(1)$. Now from $\psi_1 - \psi_2 = c_{h_2}(1) - c_{h_1}(1)$ it follows that $\psi_1(s) = \psi_2(s)$ for any $1 < s$. Also $\mu(1) = \psi_1(1)+h_1(1) = \psi_2(1)+h_2(1)$. By Lemma 5, $\psi_1(1), \psi_2(1) \in U(R)$. Since R is a uniquely clean ring, thus $\psi_1(1) = \psi_2(1)$ and $h_1(1) = h_2(1)$. Hence $\psi_1 = \psi_2$ and $h_1 = h_2$. This proves that $R * M$ is a uniquely clean. Conversely, suppose that $R * M$

is a uniquely clean ring. Similar to the proof of Theorem 6, we can show that the ring R is isomorphic to a factor ring of $R * M$. By Theorem 22 [27] every factor ring of a uniquely clean ring is also uniquely clean. Thus, the result follows.

Lemma 6. *[25, Proposition 1.8(2) and Theorem 2.1]. Clean rings are always exchange rings. The converse holds true if R is an Abelian ring.*

Proposition 9. Let R be an abelian ring and M be a strictly ordered monoid and $\omega : M \to End(R)$ a monoid homomorphism with a twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then $R * M$ is an exchange ring if and only if R is an exchange ring.

Proof. If R is an exchange ring, then, from Lemma 6 and Theorem 6, it follows that $R * M$ is a clean ring, and consequently, it is an exchange ring. Conversely, if $R * M$ is an exchange ring, then, similar to the proof of Theorem 6, and by applying Lemma 4 and Lemma 6, there exists a ring isomorphism $R \cong R * M/I$. Since every homomorphic image of an exchange ring is also an exchange ring, it follows that R is an exchange ring.

4. Conclusion

In this study, the author investigates the properties of crossed products of a ring R with a strictly ordered monoid that has a twisting map and an action on R, focusing on the CM-Armendariz condition from polynomial rings to these crossed products. The paper explores various classes of nonreduced CM-Armendariz rings and establishes that certain properties, such as being semicommutative, symmetric, Dedekind finite, clean, uniquely clean, or exchange, hold for the crossed product $R * M$. This framework opens avenues for future research in several directions can be transferred from R to $R * M$. Additionally, there are applications in noncommutative algebra that explore the implications of these findings in noncommutative settings, particularly in the context of module theory.

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