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Qualitative Stability Analysis of Oscillations in Carleman Volterra Integral Equations

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Abstract. This paper investigates the stability properties of Carleman Volterra integral equations with oscillatory kernels characterized by sine and cosine functions, structures frequently encountered in modeling physical phenomena such as wave propagation, resonance, and signal transmission. Drawing upon the stability concepts introduced by Ulam–Hyers and Ulam–Hyers–Rassias, we establish novel and explicit criteria ensuring Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and σ–semi–Ulam–Hyers stability for the considered class of integral equations. The analysis is carried out using fixed–point theory within the framework of a generalized Bielecki metric, which enables the treatment of both finite and infinite domains. The main findings demonstrate that, under suitable conditions on the oscillatory kernels, the proposed integral equations admit stable solutions in the sense of the aforementioned stability concepts. In addition, the outcomes validate the robustness of the solutions to perturbation as well as offer verifiable criteria for establishing stability behavior. For a demonstration of the practicality and efficiency of the theoretical outcomes, some examples have been illustrated using graphical simulations. This information will lead to a more profound insight into the stability behavior of integral equations with oscillatory kernels.

2020 Mathematics Subject Classifications: 26D10, 45G10, 45M10, 47H10.

Key Words and Phrases: Oscillatory Carleman Volterra integral equations, Ulam–Hyers stability, Ulam–Hyers–Rassias stability, Banach fixed–point theory, Bielecki metric.

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1. Introduction and the Problem Formulation

In 1940, S. M. Ulam [1] formulated one of the key problems of mathematical analysis that has later become known as Ulam–Hyers stability. The core question regarded conditions under which an approximate solution to a certain functional equation may be considered as being close to the exact solution of the same equation and whether the latter existed. In 1941, D. H. Hyers [2] gave a partial response to Ulam's query in terms of Banach spaces for the case of an additive Cauchy functional equation, that is, $g(\zeta + \tau) = g(\zeta) + g(\tau)$, and showed the existence of the solution. That became the initial point of what was termed as Ulam–Hyers stability [3].

Further research by T. M. Rassias [4], who extended the results of Hyers for unbounded Cauchy differences, resulted in the concept of Ulam–Hyers–Rassias stability, which was expanded in further study [5]. Another important contribution in this field was provided by other researchers, such as Aoki [6] and Gajda [7], who conducted studies on different norms, functional equations, and their approximations. Thus, a significant advancement in the area of functional equation stability was achieved, providing a rich and profound theoretical framework with multiple mathematical applications [8].

The concepts of stability in the context of Ulam–Hyers and Ulam–Hyers–Rassias are currently considered an essential part of mathematical analysis, especially in fields such as functional analysis, operator theory, and numerical analysis [9]. The ongoing research in this field enables researchers to model complex systems accurately and solve them with greater precision [10]. It is worth noting that developments in this area allow researchers to develop reliable numerical approaches as well as find their implementations in applied science including engineering and physics. More details about this topic can be found in [11, 12].

During the last decade, there were significant results regarding the use of Ulam–Hyers type stability of solutions of differential, integro-differential, and integral equations with fractional derivatives. In particular, the paper of Irshad et al. [13] analyzed the Ulam–Hyers stability of nonlinear Schrödinger equations with fractional derivatives and discussed applications of the latter. Shah and Irshad [14] introduced the concept of Ulam–Hyers–Mittag–Leffler stability for nonlinear reaction-diffusion fractional equations with delays.

The Ulam–Hyers–Rassias stability analysis for nonlinear convolution integral equations was discussed by Irshad et al. [15], while Ulam–Hyers stability on the classical Bernoulli differential equation has been carried out by Shah and Irshad [16]. Other researchers studied impulsive and oscillatory problems, where the former included impulse in Fredholm [17] and Volterra integral equations [18]; the latter included hybrid systems using Gronwall–like inequalities [19, 20]. Impulsive Hammerstein integral equations were studied by Shah et al. [21], who found that the fixed–point technique was quite effective. Recently, Shah and Irshad [22] proposed a fixed–point approach for obtaining stability estimates of delay fractional integro–differential equations.

Other relevant work includes (k, ψ)-fractional differential equation [23] and Volterra-Fredholm systems which have been solved using successive approximations [24]. Basic principles like the Banach and Krasnosel'skiĭ fixed point theorem form the foundation for many of these papers. All of these demonstrate the close connection that exists between stability theory and functional analysis [25]. The importance of fractional and integral equations in modeling real life dynamic processes is emphasized again and again and further development is helping the way for future developments [26]. See [27] for more work on the subject of Ulam–Hyers and Ulam–Hyers–Rassias stability.

In light of the above concepts, the present paper deals with Ulam–Hyers stability, Ulam–Hyers–Rassias stability and a new concept of σ-semi-Ulam–Hyers stability for the oscillatory Carleman Volterra integral equation:

$$\begin{aligned} g(\zeta) = \varpi(\zeta) &+ \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau \\ &+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta [\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)]\, g(\tau)\, d\tau, \end{aligned} \tag{1.1}$$

with $\zeta \in [a, b]$, where a and b are fixed real numbers, $g \in C([a, b])$, $\nu, \varsigma, \gamma, \varrho \in \mathbb{R}$, and $\varpi : [a, b] \to \mathbb{R}$ is a continuous function.

Carleman Volterra equations, Volterra equations, and Fredholm integral equations constitute fundamental mathematical objects that have vast applications in many areas of physics and allied sciences [28]. Integral equations serve as a highly effective tool for studying problems involving memory effects, nonlocal interactions, and heredity properties. This is particularly relevant when we deal with physical systems characterized by retardation or time-dependent potentials. Electrodynamics and wave propagation processes are modeled using integral equations in the context of resonance effects and feedback control. The analysis of non-equilibrium and phase transition phenomena and their description in terms of integral equations is important in statistical physics and thermodynamics. Integral equations also prove indispensable in such fields as astrophysics and plasma dynamics as well as in the study of viscoelastic processes with a stress depending on past deformation histories. In addition, they find application in such diverse areas as population dynamics and epidemiology taking into account the aspects of migration and delay [29]. They are also applied in the analysis of chemical kinetics and finance. See [30] for more details.

The key novelties and innovations of the present manuscript can be highlighted as follows:

- ◇ The present investigation proposes a novel stability theory for Carleman Volterra integral equations with oscillating kernels based on sine and cosine functions, which naturally appear in practical applications such as wave and signal propagation, although not sufficiently considered in existing studies [19].

- ◇ Contrary to previous investigations, which have mainly focused on exponential-type kernels and standard Gronwall-type inequalities [18, 19], the suggested methodology utilizes trigonometric kernels for obtaining stability criteria for oscillating systems using the fixed-point theorem in a general Bielecki metric space.

- ◇ In particular, this paper presents new improved analytical estimations that can precisely reflect how the oscillating properties of the kernel function affect stability and, therefore, enhance the estimation accuracy compared to the known ones.
- ◇ Utilization of the generalized Bielecki metric within the scope of fixed–point theorem contributes to methodological progress compared to other methodologies like successive approximations and the weighted space method [24], since the latter one is not always efficient when dealing with oscillatory integral equations defined on both finite and infinite intervals.
- ◇ Through concentrating on the special Carleman type of integral equation, this paper attempts to investigate its special challenges that require special approach in analysis as opposed to other more general cases discussed in previous works [31].
- ◇ Practical application and verification of theoretical results are demonstrated via examples and graphics, stressing the importance and relevance of stability criteria, which were neglected in purely theoretical researches before [18].
- ◇ Overall, this research paper makes a significant contribution to the theory of oscillatory integral equations with applications in the field of mathematical physics and mechanics.

The remaining parts of the paper are structured as follows: In Section 2, the necessary fundamental concepts and preliminary results are presented for analyzing the oscillatory Carleman Volterra integral Equation (1.1). Stability results in the finite interval case of different types such as Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and σ–semi–Ulam–Hyers stability are given in Section 3. Section 4 contains stability results in the infinite interval case of the oscillatory Carleman Volterra integral Equation (1.1) in terms of Ulam–Hyers–Rassias stability and σ–semi–Ulam–Hyers stability. Some numerical examples to show the validity of the obtained theoretical results are provided in Section 5. The solution of the equation is analyzed graphically by presenting a three-dimensional surface plot and a two-dimensional contour plot of the solution in Section 6. The application of the theory of stability to the oscillatory Carleman Volterra integral Equation (1.1) is discussed in Section 7. Finally, some conclusions and further directions of research are drawn in Section 8.

2. Basic Concepts and Some Preliminary Results

In this section, we recall the definition of a generalized complete metric space and the Banach fixed–point theorem, which form the foundation of our stability analysis for the oscillatory Carleman Volterra integral Equation (1.1). We also present the definitions of stability including Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and the novel σ–semi–Ulam–Hyers stability.

Definition 1. *Let Z be a nonempty set and let $d : Z \times Z \to [0, +\infty]$ be a mapping. The function d is called a generalized metric on Z if it satisfies the following conditions:*
(i) $d(\zeta, \tau) = 0$ *if and only if* $\zeta = \tau$;
(ii) $d(\zeta, \tau) = d(\tau, \zeta)$ *for all* $\zeta, \tau \in Z$;
(iii) $d(\zeta, \rho) \le d(\zeta, \tau) + d(\tau, \rho)$ *for all* $\zeta, \tau, \rho \in Z$.

Theorem 1 ([32]). *Let (Z, d) be a generalized complete metric space, and let $\Theta : Z \to Z$ be a strictly contractive mapping, that is,*

$$d(\Theta\zeta, \Theta\tau) \le L\, d(\zeta, \tau), \quad \textit{for all } \zeta, \tau \in Z,$$

for some Lipschitz constant $0 \le L < 1$. Suppose there exists a nonnegative integer k such that $d(\Theta^{k+1}\zeta, \Theta^k\zeta) < \infty$ for some $\zeta \in Z$. Then the following statements hold:
(i) *the sequence $(\Theta^n\zeta)_{n\in\mathbb{N}}$ converges to a fixed–point ζ^* of Θ;*
(ii) *ζ^* is the unique fixed–point of Θ in the set $Z^* = \{\tau \in Z : d(\Theta^k\zeta, \tau) < \infty\}$;*
(iii) *for any $\tau \in Z^*$, the following estimate holds:*

$$d(\tau, \zeta^*) \le \frac{1}{1 - L}\, d(\Theta\tau, \tau). \tag{2.1}$$

In numerous studies addressing the stability of functional, integral, and integro–differential equations, fixed–point theorems [33] are frequently employed in conjunction with generalized metrics, even in scenarios where such applications may not be fully appropriate; see, for instance, [34, 35].

We now proceed to formally define the aforementioned stability concepts for the oscillatory Carleman Volterra integral Equation (1.1).

Definition 2. *Let g be a continuous function on $[a, b]$ such that*

$$\begin{aligned}
&\Bigg| g(\zeta) - \varpi(\zeta) - \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \big[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\big]\, g(\tau)\, d\tau \\
&\quad - \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \big[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\big]\, g(\tau)\, d\tau \Bigg| \le \sigma(\zeta), \quad \zeta \in [a, b],
\end{aligned}$$

where σ is a nonnegative function. If there exists a solution g_0 of the oscillatory Carleman Volterra integral equation and a constant $M > 0$, independent of both g and g_0, such that

$$\big|g(\zeta) - g_0(\zeta)\big| \le M\, \sigma(\zeta), \quad \textit{for all } \zeta \in [a, b],$$

then the oscillatory Carleman Volterra integral Equation (1.1) is said to exhibit Ulam–Hyers–Rassias stability.

Definition 3. *Let g be a continuous function on $[a, b]$ such that*

$$\Bigg| g(\zeta) - \varpi(\zeta) - \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \big[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\big]\, g(\tau)\, d\tau$$

$$-\frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\left[\varsigma\cos\left(\zeta+\tau\right)+\gamma\sin\left(\zeta+\tau\right)\right]g(\tau)\,d\tau\Bigg|\leq\theta,\quad \zeta\in[a,b],$$

where $\theta \geq 0$ is a constant. If there exists a solution g_0 of the oscillatory Carleman Volterra integral equation and a constant $M > 0$, independent of both g and g_0, such that

$$\left|g(\zeta)-g_0(\zeta)\right|\leq M\,\theta,\quad \textit{for all } \zeta\in[a,b],$$

then the oscillatory Carleman Volterra integral Equation (1.1) *is said to possess Ulam–Hyers stability.*

Next, we define a novel type of stability, recently introduced by Castro and Simões [36], which serves as an intermediate concept between Ulam–Hyers stability and Ulam–Hyers–Rassias stability discussed above.

Definition 4. *Let σ be a nondecreasing function defined on $[a,b]$. For each continuous function g satisfying*

$$\begin{aligned}\Bigg|g(\zeta)-\varpi(\zeta)-\frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta}\left[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\right]g(\tau)\,d\tau\\-\frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\left[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\right]g(\tau)\,d\tau\Bigg|\leq\theta,\quad \zeta\in[a,b],\end{aligned}\tag{2.2}$$

where $\theta \geq 0$, if there exists a solution g_0 of the oscillatory Carleman Volterra integral equation and a constant $M > 0$, independent of g and g_0, such that

$$|g(\zeta)-g_0(\zeta)|\leq M\,\sigma(\zeta),\quad \zeta\in[a,b],$$

then we say that the oscillatory Carleman Volterra integral Equation (1.1) *has the σ–semi–Ulam–Hyers stability.*

3. Stabilities in the Finite Interval Case

In this section, we investigate the stability characteristics of the oscillatory Carleman Volterra integral Equation (1.1) over the finite interval $\zeta \in [a,b]$, where a and b are fixed real constants.

We work within the space of continuous functions $C([a,b])$, endowed with a generalized form of the Bielecki metric, defined as

$$d(g,h)=\sup_{\zeta\in[a,b]}\frac{|g(\zeta)-h(\zeta)|}{\sigma(\zeta)},\tag{3.1}$$

where $\sigma : [a,b] \to (0,\infty)$ is a continuous, nondecreasing function. When $\sigma(\zeta) = e^{q(\zeta-a)}$ for some $q > 0$, the metric (3.1) coincides with the classical Bielecki metric. The adoption of this generalized form allows for greater flexibility in our stability analysis.

It is worth noting that the space $\big(C([a,b]),d\big)$ constitutes a complete metric space with respect to the generalized metric d (see, e.g., [37], [38]).

Under this framework, we establish the Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and the recently introduced σ–semi–Ulam–Hyers stability of Equation (1.1) on the interval $[a,b]$.

3.1. Ulam–Hyers–Rassias Stability

In this subsection, we prove the Ulam–Hyers–Rassias stability of the oscillatory Carleman Volterra integral Equation (1.1) in the finite interval case.

Theorem 2. *Let $\sigma:[a,b]\to(0,\infty)$ be a nondecreasing continuous function. Assume further that there exists $\omega\in\mathbb{R}$ such that*

$$\int_a^\zeta \sigma(\wp)\,d\wp \le \omega\,\sigma(\zeta), \tag{3.2}$$

for all $\zeta\in[a,b]$, and let $\varpi:[a,b]\to\mathbb{R}$ be a continuous function.

Suppose $g\in C([a,b])$ satisfies

$$\begin{aligned}\Bigg|g(\zeta)-\varpi(\zeta)-\frac{\nu}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau\\ -\frac{\varrho}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau\Bigg|\le\sigma(\zeta),\quad \zeta\in[a,b],\end{aligned}\tag{3.3}$$

and that $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho)<1$. Then, there exists a unique function $g_0\in C([a,b])$ that solves Equation (1.1), that is,

$$\begin{aligned}g_0(\zeta)=\varpi(\zeta)&+\frac{\nu}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g_0(\tau)\,d\tau\\ &+\frac{\varrho}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g_0(\tau)\,d\tau\end{aligned}\tag{3.4}$$

and satisfies the inequality

$$|g(\zeta)-g_0(\zeta)|\le\frac{\sqrt{2\pi}}{\sqrt{2\pi}-\omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}}\sigma(\zeta)\tag{3.5}$$

for all $\zeta\in[a,b]$. This implies that the oscillatory Carleman Volterra integral Equation (1.1) *is Ulam–Hyers–Rassias stable.*

Proof. We consider the operator $\Theta:C([a,b])\to C([a,b])$, defined by

$$(\Theta g)(\zeta)=\varpi(\zeta)+\frac{\nu}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau$$

$$+\frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau,$$

for all $\zeta\in[a,b]$ and $g\in C([a,b])$.

Observe that for any continuous function g, the image Θg remains continuous. In fact,

$$\begin{aligned}
\big|(\Theta g)(\zeta)-(\Theta g)(\zeta_0)\big| &= \Bigg|\varpi(\zeta)+\frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau \\
&\quad+\frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau-\varpi(\zeta_0) \\
&\quad-\frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta_0}\big[\varsigma\cos(\zeta_0-\tau)+\gamma\sin(\zeta_0-\tau)\big]\,g(\tau)\,d\tau \\
&\quad-\frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta_0}\big[\varsigma\cos(\zeta_0+\tau)+\gamma\sin(\zeta_0+\tau)\big]\,g(\tau)\,d\tau\Bigg| \\
&\leq \big|\varpi(\zeta)-\varpi(\zeta_0)\big| \\
&\quad+\frac{\nu}{\sqrt{2\pi}}\Bigg|\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau \\
&\quad-\int_a^{\zeta_0}\big[\varsigma\cos(\zeta_0-\tau)+\gamma\sin(\zeta_0-\tau)\big]\,g(\tau)\,d\tau\Bigg| \\
&\quad+\frac{\varrho}{\sqrt{2\pi}}\Bigg|\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau \\
&\quad-\int_a^{\zeta_0}\big[\varsigma\cos(\zeta_0+\tau)+\gamma\sin(\zeta_0+\tau)\big]\,g(\tau)\,d\tau\Bigg| \\
&= \big|\varpi(\zeta)-\varpi(\zeta_0)\big| \\
&\quad+\frac{\nu}{\sqrt{2\pi}}\Bigg|\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau \\
&\quad-\int_a^{\zeta}\big[\varsigma\cos(\zeta_0-\tau)+\gamma\sin(\zeta_0-\tau)\big]\,g(\tau)\,d\tau \\
&\quad+\int_a^{\zeta}\big[\varsigma\cos(\zeta_0-\tau)+\gamma\sin(\zeta_0-\tau)\big]\,g(\tau)\,d\tau \\
&\quad-\int_a^{\zeta_0}\big[\varsigma\cos(\zeta_0-\tau)+\gamma\sin(\zeta_0-\tau)\big]\,g(\tau)\,d\tau\Bigg| \\
&\quad+\frac{\varrho}{\sqrt{2\pi}}\Bigg|\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&- \int_a^{\zeta} \big[\varsigma \cos(\zeta_0 + \tau) + \gamma \sin(\zeta_0 + \tau)\big]\, g(\tau)\, d\tau \\
&+ \int_a^{\zeta} \big[\varsigma \cos(\zeta_0 + \tau) + \gamma \sin(\zeta_0 + \tau)\big]\, g(\tau)\, d\tau \\
&- \int_a^{\zeta_0} \big[\varsigma \cos(\zeta_0 + \tau) + \gamma \sin(\zeta_0 + \tau)\big]\, g(\tau)\, d\tau \Bigg| \\
\leq\ & \big|\varpi(\zeta) - \varpi(\zeta_0)\big| + \frac{\nu}{\sqrt{2\pi}} \Bigg| \int_a^{\zeta} \big[\varsigma \cos(\zeta - \tau) \\
&+ \gamma \sin(\zeta - \tau) - \varsigma \cos(\zeta_0 - \tau) - \gamma \sin(\zeta_0 - \tau)\big]\, g(\tau)\, d\tau \Bigg| \\
&+ \frac{\nu}{\sqrt{2\pi}} \Bigg| \int_{\zeta_0}^{\zeta} \big[\varsigma \cos(\zeta_0 - \tau) + \gamma \sin(\zeta_0 - \tau)\big]\, g(\tau)\, d\tau \Bigg| \\
&+ \frac{\varrho}{\sqrt{2\pi}} \Bigg| \int_a^{\zeta} \big[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau) \\
&- \varsigma \cos(\zeta_0 + \tau) - \gamma \sin(\zeta_0 + \tau)\big]\, g(\tau)\, d\tau \Bigg| \\
&+ \frac{\varrho}{\sqrt{2\pi}} \Bigg| \int_{\zeta_0}^{\zeta} \big[\varsigma \cos(\zeta_0 + \tau) + \gamma \sin(\zeta_0 + \tau)\big]\, g(\tau)\, d\tau \Bigg| \to 0
\end{aligned}
$$

when $\zeta \to \zeta_0$.

We will show that, under the given conditions, the operator Θ is a strict contraction with respect to the metric (3.1).

$$
\begin{aligned}
d(\Theta g, \Theta h) &= \sup_{\zeta \in [a,b]} \frac{\big|(\Theta g)(\zeta) - (\Theta h)(\zeta)\big|}{\sigma(\zeta)} \\
&= \sup_{\zeta \in [a,b]} \frac{1}{\sigma(\zeta)} \Bigg| \frac{\nu}{\sqrt{2\pi}} \int_a^{\zeta} \big[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\big]\, g(\tau)\, d\tau \\
&\quad + \frac{\varrho}{\sqrt{2\pi}} \int_a^{\zeta} \big[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\big]\, g(\tau)\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\nu}{\sqrt{2\pi}} \int_a^{\zeta} \big[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\big]\, h(\tau)\, d\tau \\
&- \frac{\varrho}{\sqrt{2\pi}} \int_a^{\zeta} \big[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\big]\, h(\tau)\, d\tau \Bigg| \\
\leq\ & \frac{\nu}{\sqrt{2\pi}} \sup_{\zeta \in [a,b]} \frac{1}{\sigma(\zeta)} \Bigg| \int_a^{\zeta} [\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)][g(\tau) - h(\tau)]\, d\tau \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{\varrho}{\sqrt{2\pi}} \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \left| \int_a^\zeta [\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)][g(\tau)-h(\tau)]\, d\tau \right| \\
\leq\ & \frac{\nu}{\sqrt{2\pi}} \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)|\, \sigma(\tau) \frac{|g(\tau)-h(\tau)|}{\sigma(\tau)}\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)|\, \sigma(\tau) \frac{|g(\tau)-h(\tau)|}{\sigma(\tau)}\, d\tau \\
\leq\ & \frac{\nu}{\sqrt{2\pi}} \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \sup_{\tau\in[a,b]} \frac{|g(\tau)-h(\tau)|}{\sigma(\tau)} \int_a^\zeta |\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)|\, \sigma(\tau)\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \sup_{\tau\in[a,b]} \frac{|g(\tau)-h(\tau)|}{\sigma(\tau)} \int_a^\zeta |\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)|\, \sigma(\tau)\, d\tau \\
\leq\ & \frac{\nu}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)|\, \sigma(\tau)\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)|\, \sigma(\tau)\, d\tau.
\end{aligned}
$$

By the Cauchy–Schwartz inequality and condition (3.2), we have

$$
\begin{aligned}
& \frac{\nu}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)|\, \sigma(\tau)\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta |\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)|\, \sigma(\tau)\, d\tau \\
\leq\ & \frac{\nu}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sqrt{[\varsigma^2+\gamma^2][\cos^2(\zeta-\tau) + \sin^2(\zeta-\tau)]}\, \sigma(\tau)\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sqrt{[\varsigma^2+\gamma^2][\cos^2(\zeta+\tau) + \sin^2(\zeta+\tau)]}\, \sigma(\tau)\, d\tau \\
=\ & \frac{\nu}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sqrt{\varsigma^2+\gamma^2}\, \sigma(\tau)\, d\tau \\
& + \frac{\varrho}{\sqrt{2\pi}} d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sqrt{\varsigma^2+\gamma^2}\, \sigma(\tau)\, d\tau \\
=\ & \sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}} \nu d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sigma(\tau)\, d\tau \\
& + \sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}} \varrho d(g,h) \sup_{\zeta\in[a,b]} \frac{1}{\sigma(\zeta)} \int_a^\zeta \sigma(\tau)\, d\tau \\
\leq\ & \sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}} \nu d(g,h) \sup_{\zeta\in[a,b]} \frac{\omega\sigma(\zeta)}{\sigma(\zeta)}
\end{aligned}
$$

$$+\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\varrho d(g,h)\sup_{\zeta\in[a,b]}\frac{\omega\sigma(\zeta)}{\sigma(\zeta)}$$

$$=\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\omega(\nu+\varrho)d(g,h).$$

Since $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho)<1$, it follows that the operator Θ is strictly contractive. Therefore, we can apply the Banach fixed–point theorem mentioned above, which guarantees the Ulam–Hyers–Rassias stability of the oscillatory Carleman Volterra integral equation.

Moreover, inequality (3.5) follows directly from (2.1) and (3.3). Indeed, from (3.3) we obtain

$$|g(\zeta)-(\Theta g)(\zeta)|\le\sigma(\zeta),\quad \zeta\in[a,b]. \tag{3.6}$$

Applying the Banach fixed–point theorem once again, and using (2.1), we have

$$d(g,g_0)\le\frac{1}{1-\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho)}d(\Theta g,g). \tag{3.7}$$

By the definition of the metric d and inequality (3.6), it follows that

$$\sup_{\zeta\in[a,b]}\frac{|g(\zeta)-g_0(\zeta)|}{\sigma(\zeta)}\le\frac{\sqrt{2\pi}}{\sqrt{2\pi}-\omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}}, \tag{3.8}$$

which implies that (3.5) holds.

3.2. σ–Semi–Ulam–Hyers and Ulam–Hyers Stabilities

In this subsection, we prove the novel concept of σ–semi–Ulam–Hyers stability, as well as the Ulam–Hyers stability, for the oscillatory Carleman Volterra integral Equation (1.1) in the finite interval case.

Theorem 3. *Let $\sigma:[a,b]\to(0,\infty)$ be a nondecreasing and continuous function. Additionally, suppose there exists a constant $\omega\in\mathbb{R}$ such that*

$$\int_a^\zeta\sigma(\wp)\,d\wp\le\omega\,\sigma(\zeta), \tag{3.9}$$

for all $\zeta\in[a,b]$, and let $\varpi:[a,b]\to\mathbb{R}$ be a continuous function.

Assume that $g\in C([a,b])$ satisfies

$$\left|g(\zeta)-\varpi(\zeta)-\frac{\nu}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta-\tau)+\gamma\sin(\zeta-\tau)\big]\,g(\tau)\,d\tau\right. \tag{3.10}$$
$$\left.-\frac{\varrho}{\sqrt{2\pi}}\int_a^\zeta\big[\varsigma\cos(\zeta+\tau)+\gamma\sin(\zeta+\tau)\big]\,g(\tau)\,d\tau\right|\le\theta,\quad \zeta\in[a,b],$$

where $\theta \geq 0$, and the condition $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho) < 1$ holds. Then, there exists a unique function $g_0 \in C([a,b])$ which is a solution to Equation (1.1)*, that is,*

$$g_0(\zeta) = \varpi(\zeta) + \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)\right] g_0(\tau)\, d\tau \tag{3.11}$$
$$+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)\right] g_0(\tau)\, d\tau,$$

and satisfies the estimate

$$|g(\zeta) - g_0(\zeta)| \leq \frac{\sqrt{2\pi}\,\theta}{\left(\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}\right)\sigma(a)}\,\sigma(\zeta), \tag{3.12}$$

for all $\zeta \in [a,b]$, meaning that the oscillatory Carleman Volterra integral Equation (1.1) *is σ–semi–Ulam–Hyers stable.*

Proof. Following the same reasoning as previously discussed, the operator Θ is a strict contraction with respect to the metric defined in (3.1), due to the inequality $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho) < 1$. Hence, by the Banach fixed–point theorem, the oscillatory Carleman Volterra integral Equation (1.1) is guaranteed to have σ–semi–Ulam–Hyers stability.

Moreover, considering (3.10) along with the definition of Θ, it follows that

$$|g(\zeta) - (\Theta g)(\zeta)| \leq \theta, \quad \zeta \in [a,b]. \tag{3.13}$$

From (2.1), the definition of the metric d, and inequality (3.13), we deduce that

$$\sup_{\zeta\in[a,b]} \frac{|g(\zeta) - g_0(\zeta)|}{\sigma(\zeta)} \leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}} \sup_{\zeta\in[a,b]} \frac{\theta}{\sigma(\zeta)}, \tag{3.14}$$

which immediately implies, by the definition of σ, the validity of inequality (3.12).

Corollary 1. *Let $\sigma : [a,b] \to (0,\infty)$ be a continuous and nondecreasing function. Furthermore, assume there exists $\omega \in \mathbb{R}$ such that*

$$\int_a^\zeta \sigma(\wp)\, d\wp \leq \omega\, \sigma(\zeta), \tag{3.15}$$

for all $\zeta \in [a,b]$, and let $\varpi : [a,b] \to \mathbb{R}$ be a continuous function.

Suppose $g \in C([a,b])$ satisfies

$$\left| g(\zeta) - \varpi(\zeta) - \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta-\tau) + \gamma \sin(\zeta-\tau)\right] g(\tau)\, d\tau \right. \tag{3.16}$$
$$\left. - \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta+\tau) + \gamma \sin(\zeta+\tau)\right] g(\tau)\, d\tau \right| \leq \theta, \quad \zeta \in [a,b],$$

where $\theta \geq 0$ *and* $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu+\varrho) < 1$. *Then there exists a unique function* $g_0 \in C([a,b])$, *which solves Equation* (1.1), *such that*

$$g_0(\zeta) = \varpi(\zeta) + \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma\cos(\zeta-\tau) + \gamma\sin(\zeta-\tau)\right] g_0(\tau)\,d\tau \qquad (3.17)$$
$$+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma\cos(\zeta+\tau) + \gamma\sin(\zeta+\tau)\right] g_0(\tau)\,d\tau,$$

and it satisfies the inequality

$$|g(\zeta) - g_0(\zeta)| \leq \frac{\sqrt{2\pi}\,\sigma(b)}{\left(\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}\right)\sigma(a)}\,\theta, \qquad (3.18)$$

for all $\zeta \in [a,b]$, *which implies that the oscillatory Carleman Volterra integral Equation* (1.1) *possesses Ulam–Hyers stability.*

4. Stabilities in the Infinite Interval Case

In this section, we investigate the Ulam–Hyers–Rassias stability and the σ–semi–Ulam–Hyers stability of the oscillatory Carleman Volterra integral Equation (1.1) on the infinite interval $[a,\infty)$, where $a \in \mathbb{R}$ is fixed. This extends the earlier results obtained on the finite interval $[a,b]$, with $a,b \in \mathbb{R}$. Analogous results can be formulated for the intervals $(-\infty,a]$ and $(-\infty,\infty)$ with suitable adjustments. Hence, we shall focus on the oscillatory Carleman Volterra integral equation given by

$$g(\zeta) = \varpi(\zeta) + \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma\cos(\zeta-\tau) + \gamma\sin(\zeta-\tau)\right] g(\tau)\,d\tau \qquad (4.1)$$
$$+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma\cos(\zeta+\tau) + \gamma\sin(\zeta+\tau)\right] g(\tau)\,d\tau,$$

for $\zeta \in [a,\infty)$, where a is a fixed real number, $g \in C([a,\infty))$, $\nu,\varsigma,\gamma,\varrho \in \mathbb{R}$, and $\varpi : [a,\infty) \to \mathbb{R}$ is a continuous function.

Motivated by the previously derived results on finite intervals, our method of analysis on the infinite interval will rely on a recurrence technique.

To this, consider a fixed nondecreasing continuous function $\sigma : [a,\infty) \to (\epsilon,\rho)$, where $\epsilon,\rho > 0$, and define the space $C_b([a,\infty))$ of bounded continuous functions, endowed with the metric

$$d_b(g,h) = \sup_{\zeta\in[a,\infty)} \frac{|g(\zeta)-h(\zeta)|}{\sigma(\zeta)}. \qquad (4.2)$$

4.1. Ulam–Hyers–Rassias Stability

In this subsection, we prove the Ulam–Hyers–Rassias stability of the oscillatory Carleman Volterra integral Equation (4.1) in the infinite interval case.

Theorem 4. *Let $\sigma : [a, \infty) \to (0, \infty)$ be a continuous, nondecreasing function, and suppose there exists a real number $\omega \in \mathbb{R}$ such that*

$$\int_a^\zeta \sigma(\wp)d\wp \le \omega\,\sigma(\zeta), \tag{4.3}$$

for every $\zeta \in [a, \infty)$ and let $\varpi : [a, \infty) \to \mathbb{R}$ be a continuous function.

Additionally, assume that the expressions

$$\int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau$$

and

$$\int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g(\tau)\, d\tau$$

are bounded and continuous for any bounded continuous function g.

If $g \in C_b([a, \infty))$ satisfies the inequality

$$\begin{aligned} &\Bigg| g(\zeta) - \varpi(\zeta) - \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau \\ &- \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g(\tau)\, d\tau \Bigg| \le \sigma(\zeta), \quad \zeta \in [a, \infty), \end{aligned} \tag{4.4}$$

and the condition $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu + \varrho) < 1$ holds, then there exists a unique function $g_0 \in C_b([a, \infty))$ that solves Equation (4.1), that is,

$$\begin{aligned} g_0(\zeta) = \varpi(\zeta) &+ \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g_0(\tau)\, d\tau \\ &+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g_0(\tau)\, d\tau \end{aligned} \tag{4.5}$$

such that

$$|g(\zeta) - g_0(\zeta)| \le \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega(\nu + \varrho)\sqrt{\varsigma^2 + \gamma^2}}\sigma(\zeta) \tag{4.6}$$

for all $\zeta \in [a, \infty)$. This implies that the oscillatory Carleman Volterra integral Equation (4.1) *is Ulam–Hyers–Rassias stable.*

Proof. For each $m \in \mathbb{N}$, define the interval $I_m = [a, a + m]$. According to Theorem 1, there exists a unique continuous function $g_{0,m} \in C(I_m)$ such that

$$\begin{aligned} g_{0,m}(\zeta) = \varpi(\zeta) &+ \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g_{0,m}(\tau)\, d\tau \\ &+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g_{0,m}(\tau)\, d\tau \end{aligned} \tag{4.7}$$

and

$$|g(\zeta) - g_{0,m}(\zeta)| \leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}}\,\sigma(\zeta) \tag{4.8}$$

for all $\zeta \in I_m$. The uniqueness of $g_{0,m}$ implies that if $\zeta \in I_m$, then

$$g_{0,m}(\zeta) = g_{0,m+1}(\zeta) = g_{0,m+2}(\zeta) = \cdots . \tag{4.9}$$

For any $\zeta \in [a,\infty)$, let us define $m(\zeta) \in \mathbb{N}$ as $m(\zeta) = \min\{m \in \mathbb{N} : \zeta \in I_m\}$. We also define a function $g_0 : [a,\infty) \to \mathbb{R}$ by

$$g_0(\zeta) = g_{0,m(\zeta)}(\zeta). \tag{4.10}$$

For any $\zeta_1 \in [a,\infty)$, let $m_1 = m(\zeta_1)$. Then $\zeta_1 \in IntI_{m_1+1}$ and there exists an $\epsilon > 0$ such that $g_0(\zeta) = g_{0,m_1+1}(\zeta)$ for all $\zeta \in (\zeta_1-\epsilon, \zeta_1+\epsilon)$, (where $IntI_{m_1+1}$ denotes the interior of the set I_{m_1+1}). By Theorem 1, g_{0,m_1+1} is continuous at ζ_1, and so it is g_0.

Now, we will prove that g_0 satisfies

$$\begin{aligned} g_0(\zeta) = \varpi(\zeta) &+ \frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau) + \gamma\sin(\zeta-\tau)\big]\,g_0(\tau)\,d\tau \\ &+ \frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau) + \gamma\sin(\zeta+\tau)\big]\,g_0(\tau)\,d\tau \end{aligned} \tag{4.11}$$

and

$$|g(\zeta) - g_0(\zeta)| \leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}}\,\sigma(\zeta). \tag{4.12}$$

For an arbitrary $\zeta \in [a,\infty)$, we choose $m(\zeta)$ such that $\zeta \in I_{m(\zeta)}$. By (4.7) and (4.10), we have

$$\begin{aligned} g_0(\zeta) &= g_{0,m(\zeta)}(\zeta) \\ &= \varpi(\zeta) + \frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau) + \gamma\sin(\zeta-\tau)\big]\,g_{0,m(\zeta)}(\tau)\,d\tau \\ &\quad + \frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau) + \gamma\sin(\zeta+\tau)\big]\,g_{0,m(\zeta)}(\tau)\,d\tau \\ &= \varpi(\zeta) + \frac{\nu}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta-\tau) + \gamma\sin(\zeta-\tau)\big]\,g_0(\tau)\,d\tau \\ &\quad + \frac{\varrho}{\sqrt{2\pi}}\int_a^{\zeta}\big[\varsigma\cos(\zeta+\tau) + \gamma\sin(\zeta+\tau)\big]\,g_0(\tau)\,d\tau. \end{aligned} \tag{4.13}$$

Note that $m(\wp) \leq m(\zeta)$, for any $\wp \in I_{m(\zeta)}$, and it follows from (4.9) that $g_0(\wp) = g_{0,m(\wp)}(\wp) = g_{0,m(\zeta)}(\wp)$, so the last equality in (4.13) holds.

To prove (4.6), by (4.10) and (4.8), we have that for all $\zeta \in [a,\infty)$,

$$|g(\zeta) - g_0(\zeta)| = |g(\zeta) - g_{0,m(\zeta)}(\zeta)| \leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega(\nu+\varrho)\sqrt{\varsigma^2+\gamma^2}}\,\sigma(\zeta). \tag{4.14}$$

Finally, we demonstrate the uniqueness of g_0. Suppose there exists another bounded continuous function g_1 that also satisfies (4.5) and (4.6) for all $\zeta \in [a, \infty)$. Due to the uniqueness of the solution over the interval $I_{m(\zeta)}$ for any $m(\zeta) \in \mathbb{N}$, it follows that $g_{0|I_{m(\zeta)}} = g_{0,m(\zeta)}$, and $g_{1|I_{m(\zeta)}}$ satisfies both (4.5) and (4.6) for all $\zeta \in I_{m(\zeta)}$. Therefore, we obtain $g_0(\zeta) = g_{0|I_{m(\zeta)}}(\zeta) = g_{1|I_{m(\zeta)}}(\zeta) = g_1(\zeta)$.

4.2. σ–Semi–Ulam–Hyers Stability

In this subsection, we prove the novel concept of σ–semi–Ulam–Hyers stability for the oscillatory Carleman Volterra integral Equation (4.1) in the infinite interval case.

Theorem 5. *Let $\sigma : [a, \infty) \to (0, \infty)$ be a nondecreasing continuous function and suppose that there is $\omega \in \mathbb{R}$ such that*

$$\int_a^\zeta \sigma(\wp) d\wp \le \omega\, \sigma(\zeta), \tag{4.15}$$

for all $\zeta \in [a, \infty)$ and $\varpi : [a, \infty) \to \mathbb{R}$ is a continuous function.

In addition, suppose that

$$\int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau$$

and

$$\int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g(\tau)\, d\tau$$

are bounded continuous functions for any bounded continuous function g.

If $g \in C_b([a, \infty))$ is such that

$$\begin{aligned} \Bigg| g(\zeta) - \varpi(\zeta) - \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau \\ - \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g(\tau)\, d\tau \Bigg| \le \theta, \quad \zeta \in [a, \infty), \end{aligned} \tag{4.16}$$

where $\theta \ge 0$ and $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\,\omega(\nu + \varrho) < 1$, then there is a unique function $g_0 \in C_b([a, \infty))$, solution of Equation (4.1), that is,

$$\begin{aligned} g_0(\zeta) = \varpi(\zeta) &+ \frac{\nu}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g_0(\tau)\, d\tau \\ &+ \frac{\varrho}{\sqrt{2\pi}} \int_a^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g_0(\tau)\, d\tau \end{aligned} \tag{4.17}$$

such that

$$|g(\zeta) - g_0(\zeta)| \le \frac{\sqrt{2\pi}\, \theta}{\left(\sqrt{2\pi} - \omega(\nu + \varrho)\sqrt{\varsigma^2 + \gamma^2}\right) \sigma(a)}\, \sigma(\zeta) \tag{4.18}$$

for all $\zeta \in [a, \infty)$, which means that the oscillatory Carleman Volterra integral Equation (4.1) *is σ–semi–Ulam–Hyers stable.*

Proof. By the same procedure as above and Theorem 3, the proof is visible. For that reason, we leave it out here.

5. Examples

We shall give some examples to show that the above results conditions can be fulfilled. We will first discuss some issues regarding our equation's solution.

If $g \in C([0,1])$ satisfies $g(\zeta) \neq 0$ for all $\zeta \in [0,1]$, $g(0) = 1$, and $\varpi(\zeta) = \zeta + 1$, then the solution of

$$\begin{aligned} g(\zeta) = \zeta + 1 + \frac{\nu}{\sqrt{2\pi}} \int_0^\zeta \left[\varsigma \cos(\zeta - \tau) + \gamma \sin(\zeta - \tau)\right] g(\tau)\, d\tau \\ + \frac{\varrho}{\sqrt{2\pi}} \int_0^\zeta \left[\varsigma \cos(\zeta + \tau) + \gamma \sin(\zeta + \tau)\right] g(\tau)\, d\tau, \end{aligned} \tag{5.1}$$

is

$$\begin{aligned} g_0(\zeta) = \exp\left[\frac{\varsigma\nu}{2} C\left(\sqrt{\frac{2}{\pi}}\right) + \frac{\gamma\nu}{2} S\left(\sqrt{\frac{2}{\pi}}\right)\right] \\ + \exp\left[\frac{\varsigma\varrho}{2} C\left(\sqrt{\frac{2}{\pi}}\right) + \frac{\gamma\varrho}{2} S\left(\sqrt{\frac{2}{\pi}}\right)\right] \end{aligned} \tag{5.2}$$

with the Fresnel C integral and the Fresnel S integral, respectively, denoted by $C(.)$ and $S(.)$.

Example 1. *Let us consider the nondecreasing continuous function $\sigma \in C([0,1])$ defined by $\sigma(\zeta) = \exp(0.5\zeta)$. We have that*

$$\int_0^\zeta \sigma(\wp) d\wp \leq \omega\, \sigma(\zeta)$$

for all $\zeta \in [0,1]$ with $\omega \geq 0.978$.

Let us consider for $\varsigma = 0.6$, $\gamma = 0$, $\nu = 0.3$, $\varrho = 0.9$, and the perturbation of the *solution* $g(\zeta) = \exp(0.3\zeta)$. *It follows that*

$$\begin{aligned}
&\Bigg| g(\zeta) - \zeta - 1 - \frac{0.3}{\sqrt{2\pi}} \int_0^\zeta 0.6\cos(\zeta - \tau)\, g(\tau)\, d\tau \\
&- \frac{0.9}{\sqrt{2\pi}} \int_0^\zeta 0.6\cos(\zeta + \tau)\, g(\tau)\, d\tau \Bigg| \\
&= \Bigg| \exp(0.3\zeta) - \zeta - 1 + \frac{0.2308}{\sqrt{2\pi}} \Big[4sin(\zeta) + 2.4\cos(\zeta) \\
&- \exp(0.3\zeta)\big[6\sin(2\zeta) + 0.6(1 + 3\cos(2\zeta))\big]\Big] \Bigg| \\
&\le \sigma(\zeta) = \exp(0.5\zeta), \quad \zeta \in [0,1],
\end{aligned} \tag{5.3}$$

see Figure 1.

We realize that for $\omega = 0.978$, *we have* $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\omega(\nu + \varrho) < 1$. *So we have*

$$\begin{aligned}
|g(\zeta) - g_0(\zeta)| &= \left| \exp(0.3\zeta) - \exp\left[\frac{9}{100}C\left(\sqrt{\frac{2}{\pi}}\right)\right] - \exp\left[\frac{27}{100}C\left(\sqrt{\frac{2}{\pi}}\right)\right] \right| \\
&\le \frac{\sqrt{2\pi}}{\sqrt{2\pi} - 0.978 \times 0.6 \times 1.2} \exp(0.5\zeta) \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega\sqrt{\varsigma^2 + \gamma^2}(\nu + \varrho)} \sigma(\zeta),
\end{aligned} \tag{5.4}$$

see Figure 1.

Therefore, the conditions of Theorem 1 are all fulfilled when these assumptions are taken into consideration. This indicates that the oscillatory Carleman Volterra integral Equation (5.1) *possesses the Ulam–Hyers–Rassias stability under the aforementioned conditions.*

Example 2. *Let us consider the nondecreasing continuous function* $\sigma \in C([0,1])$ *defined by* $\sigma(\zeta) = 0.3 + 0.8\exp(0.01\zeta)$. *We have that*

$$\int_0^\zeta \sigma(\wp)d\wp \le \omega\,\sigma(\zeta),$$

for all $\zeta \in [0,1]$ *with* $\omega \ge 0.961$.

Let us consider for $\varsigma = 0$, $\gamma = 0.2$, $\nu = 0.4$, $\varrho = 0.7$, *and the perturbation of the*

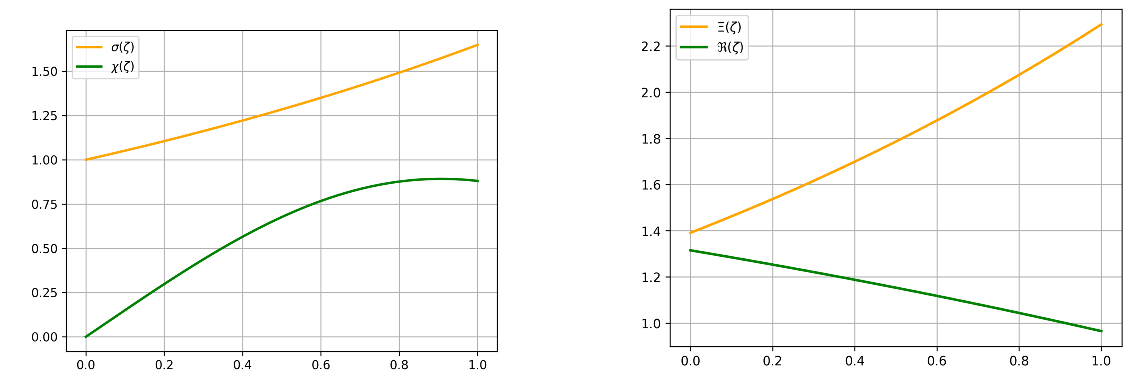

Figure 1: On the left, $\sigma(\zeta) = \exp(0.5\zeta)$ is represented by an orange line, while $\chi(\zeta) = \Big|\exp(0.3\zeta) - \zeta - 1 + \frac{0.2308}{\sqrt{2\pi}}\left[4\sin(\zeta) + 2.4\cos(\zeta) - \exp(0.3\zeta)\left[6\sin(2\zeta) + 0.6(1+3\cos(2\zeta))\right]\right]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in inequality (5.3). On the right, $\Xi(\zeta) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}-0.978\times 0.6\times 1.2}\exp(0.5\zeta) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}-\omega\sqrt{\varsigma^2+\gamma^2}(\nu+\varrho)}\sigma(\zeta)$ is represented by an orange line, and $\Re(\zeta) = \Big|\exp(0.3\zeta) - \exp\left[\frac{9}{100}C\left(\sqrt{\frac{2}{\pi}}\right)\right] - \exp\left[\frac{27}{100}C\left(\sqrt{\frac{2}{\pi}}\right)\right]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in (5.4).

solution $g(\zeta) = \exp(0.0183\zeta)$. *It follows that*

$$\begin{aligned}
&\Bigg| g(\zeta) - \zeta - 1 - \frac{0.4}{\sqrt{2\pi}}\int_0^{\zeta} 0.2\sin(\zeta-\tau)\, g(\tau)\, d\tau \\
&\quad - \frac{0.7}{\sqrt{2\pi}}\int_0^{\zeta} 0.2\sin(\zeta+\tau)\, g(\tau)\, d\tau \Bigg| \\
&= \Bigg| \exp(0.0183\zeta) - \zeta - 1 + \frac{1}{\sqrt{2\pi}}\big(0.0011\sin(\zeta) - 0.22\cos(\zeta)\big) \\
&\quad + \frac{\exp(0.0183\zeta)}{\sqrt{2\pi}}\Big[0.0802 - 0.0026\sin(2\zeta) + 0.14\cos(2\zeta)\Big] \Bigg| \\
&\leq \sigma(\zeta) = 0.3 + 0.8\exp(0.01\zeta), \quad \varsigma \in [0,1],
\end{aligned} \tag{5.5}$$

see Figure 2.

We realize that for $\omega = 0.961$, *we have* $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\omega(\nu+\varrho) < 1$. *So we have*

$$\begin{aligned}
|g(\zeta) - g_0(\zeta)| &= \left| \exp(0.0183\zeta) - \exp\left[\frac{1}{25}S\left(\sqrt{\frac{2}{\pi}}\right)\right] - \exp\left[\frac{7}{100}S\left(\sqrt{\frac{2}{\pi}}\right)\right]\right| \\
&\leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - 0.961\times 0.2\times 1.1}\big(0.3 + 0.8\exp(0.01\zeta)\big) \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega\sqrt{\varsigma^2+\gamma^2}(\nu+\varrho)}\sigma(\zeta),
\end{aligned} \tag{5.6}$$

see Figure 2.

Therefore, the conditions of Theorem 1 are all fulfilled when these assumptions are taken into consideration. This indicates that the oscillatory Carleman Volterra integral Equation (5.1) *possesses the Ulam–Hyers–Rassias stability under the aforementioned conditions.*

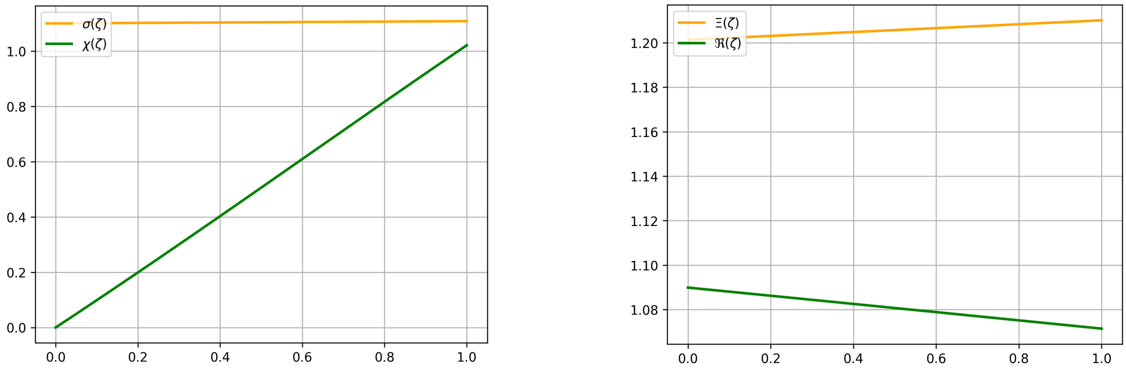

Figure 2: On the left, $\sigma(\zeta) = 0.3 + 0.8\exp(0.01\zeta)$ is represented by an orange line, while $\chi(\zeta) = \Big|\exp(0.0183\zeta) - \zeta - 1 + \frac{1}{\sqrt{2\pi}}\big(0.0011\sin(\zeta) - 0.22\cos(\zeta)\big) + \frac{\exp(0.0183\zeta)}{\sqrt{2\pi}}\big[0.0802 - 0.0026\sin(2\zeta) + 0.14\cos(2\zeta)\big]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in inequality (5.5). On the right, $\Xi(\zeta) = \frac{\sqrt{2\pi}}{\sqrt{2\pi} - 0.961\times 0.2\times 1.1}(0.3 + 0.8\exp(0.01\zeta)) = \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega\sqrt{\varsigma^2+\gamma^2}(\nu+\varrho)}\sigma(\zeta)$ is represented by an orange line, and $\Re(\zeta) = \Big|\exp(0.0183\zeta) - \exp\big[\frac{1}{25}S\big(\sqrt{\frac{2}{\pi}}\big)\big] - \exp\big[\frac{7}{100}S\big(\sqrt{\frac{2}{\pi}}\big)\big]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in (5.6).

Example 3. *Let us consider now the nondecreasing continuous polynomial function* $\sigma \in C([0,1])$ *defined by* $\sigma(\zeta) = 0.05\zeta + 2$. *We have that*

$$\int_0^{\zeta} \sigma(\wp)d\wp \leq \omega\,\sigma(\zeta),$$

for all $\zeta \in [0,1]$ *with* $\omega \geq 0.59$.

Let consider for $\varsigma = 0.8$, $\gamma = 0.8$, $\nu = 0.1$, $\varrho = 0.7$, *and the perturbation of the solution* $g(\zeta) = \exp(0.40\zeta)$. *It follows that*

$$
\begin{aligned}
&\left| g(\zeta) - \zeta - 1 - \frac{0.1}{\sqrt{2\pi}} \int_0^\zeta \left[0.8\cos(\zeta-\tau) + 0.8\sin(\zeta-\tau)\right] g(\tau)\, d\tau \right. \\
&\quad \left. - \frac{0.7}{\sqrt{2\pi}} \int_0^\zeta \left[0.8\cos(\zeta+\tau) + 0.8\sin(\zeta+\tau)\right] g(\tau)\, d\tau \right| \\
&= \left| \left[1 - \frac{0.0966}{\sqrt{2\pi}}\right] \exp(0.40\zeta) - \zeta - 1 + \frac{1}{\sqrt{2\pi}}\left[1.0138\cos(\zeta) \right.\right. \\
&\quad \left.\left. + 3.6517\sin(\zeta)\right] + \frac{\exp(0.40\zeta)}{\sqrt{2\pi}}\left[0.2897\cos(2\zeta) - 0.6759\sin(2\zeta)\right] \right| \\
&\leq \sigma(\zeta) = 0.05\zeta + 2, \quad \zeta \in [0,1],
\end{aligned}
\tag{5.7}
$$

see Figure 3.

We realize that for $\omega = 0.59$*, we have* $\sqrt{\frac{\varsigma^2+\gamma^2}{2\pi}}\omega(\nu+\varrho) < 1$*. So we have*

$$
\begin{aligned}
|g(\zeta) - g_0(\zeta)| &= \left| \exp(0.40\zeta) - \exp\left[\frac{1}{25}C\left(\sqrt{\frac{2}{\pi}}\right) + \frac{1}{25}S\left(\sqrt{\frac{2}{\pi}}\right)\right] \right. \\
&\quad \left. - \exp\left[\frac{7}{25}C\left(\sqrt{\frac{2}{\pi}}\right) + \frac{7}{25}S\left(\sqrt{\frac{2}{\pi}}\right)\right] \right| \\
&\leq \frac{\sqrt{2\pi}}{\sqrt{2\pi} - 0.59\sqrt{0.8^2+0.8^2}(0.8)}(0.05\zeta+2) \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega\sqrt{\varsigma^2+\gamma^2}(\nu+\varrho)}\sigma(\zeta),
\end{aligned}
\tag{5.8}
$$

see Figure 3.

Therefore, the conditions of Theorem 1 are all fulfilled when these assumptions are taken into consideration. This indicates that the oscillatory Carleman Volterra integral Equation (5.1) *possesses the Ulam–Hyers–Rassias stability under the aforementioned conditions.*

6. Graphical Interpretation and Analysis

In this section, we provide a graph depicting the solution function $g_0(\zeta)$ obtained from the Carleman-type integral equation as shown in Equation (5.1). The solution is provided in a closed-form expression involving Fresnel integrals $C(\cdot)$ and $S(\cdot)$ that describe the oscillations and accumulation processes induced by the parameters ν, ϱ, ς, and γ of the integral equation. The range of the independent variable ζ is confined to the interval $[0,1]$ under the initial condition $g(0) = 1$ while maintaining $g(\zeta) \neq 0$ on the interval $[0,1]$.

The graphical presentation above supports the mathematical form of the solution, particularly the effects caused by the sine and cosine terms in the integral kernel. The

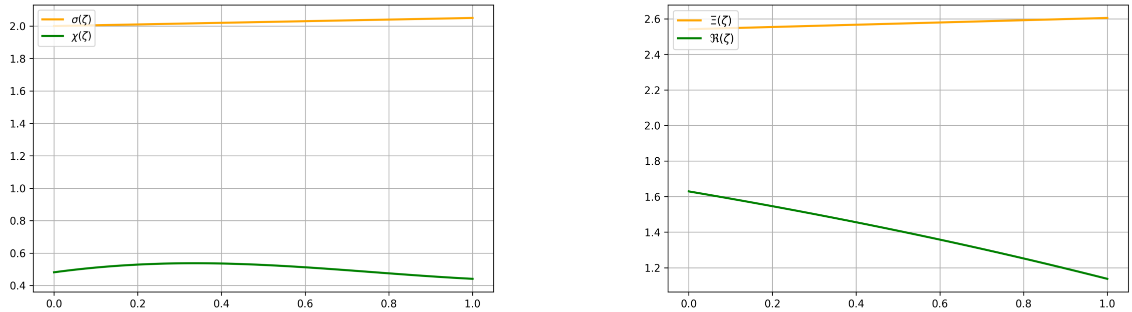

Figure 3: On the left, $\sigma(\zeta) = 0.05\zeta + 2$ is represented by an orange line, while $\chi(\zeta) = \Big|\big[1 - \frac{0.0966}{\sqrt{2\pi}}\big]\exp(0.40\zeta) - \zeta - 1 + \frac{1}{\sqrt{2\pi}}\big[1.0138\cos(\zeta) + 3.6517\sin(\zeta)\big] + \frac{\exp(0.40\zeta)}{\sqrt{2\pi}}\big[0.2897\cos(2\zeta) - 0.6759\sin(2\zeta)\big]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in inequality (5.7). On the right, $\Xi(\zeta) = \frac{\sqrt{2\pi}}{\sqrt{2\pi} - 0.59\sqrt{0.8^2 + 0.8^2}(0.8)}(0.05\zeta + 2) = \frac{\sqrt{2\pi}}{\sqrt{2\pi} - \omega\sqrt{\varsigma^2 + \gamma^2}(\nu + \varrho)}\sigma(\zeta)$ is represented by an orange line, and $\Re(\zeta) = \Big|\exp(0.40\zeta) - \exp\big[\frac{1}{25}C\big(\sqrt{\frac{2}{\pi}}\big) + \frac{1}{25}S\big(\sqrt{\frac{2}{\pi}}\big)\big] - \exp\big[\frac{7}{25}C\big(\sqrt{\frac{2}{\pi}}\big) + \frac{7}{25}S\big(\sqrt{\frac{2}{\pi}}\big)\big]\Big|$ is represented by a green line, enabling a comparison between the two functions involved in (5.8).

periodic variation induced by the joint effect of ν and ϱ using Fresnel integrals creates a subtle change in the curve shape of the solution function $g_0(\zeta)$. The highest value is obtained when ζ is close to 1.

Figure 4 shows the 3D plot of the exact solution $g_0(\zeta)$, which gives us the visual representation of how the solution varies as a function of ζ. In this plot, we see a continuous graph showing the variation of $g_0(\zeta)$ as ζ becomes larger. There are no sharp corners or sudden jumps in the graph, which signifies that the solution is stable and well-defined for the given parameter value. A gradient of colors is used in the plot, where dark colors represent low values and light colors represent high values of the solution.

The Figure 5 provides a visualization of the 2D contour plot of the solution $g_0(\zeta)$ for $\zeta \in [0, 1]$. Such a representation provides a bird's eye view of the magnitude of the function, through contours and a gradual color change. Contour lines connect points with the same magnitude value, allowing one to trace the variation in the solution magnitude within the region. A color scheme, placed to the right of the plot, helps in interpreting the solution magnitudes corresponding to different colors, varying from darker shades of blue representing smaller magnitudes to lighter shades of yellow indicating larger magnitudes.

The contour plot serves as an additional tool along with the 3D graph of the function, as it clearly illustrates the increasing trend in magnitude along the gradient flow. It reinforces the conclusion that the solution is characterized by a monotonic growth. This can be concluded from the appearance of concentric contour lines around the central part of the plot, implying that the magnitude changes continuously, being insensitive to small fluctuations of the parameter ζ.

All together, they prove the analytical conclusions through visualization, confirming that the obtained solution is indeed oscillating. Smooth transition of the solution surface and continuous increase in contours indicate absence of abrupt changes in solution values, implying Ulam–Hyers type stability of the solution. Furthermore, the introduced

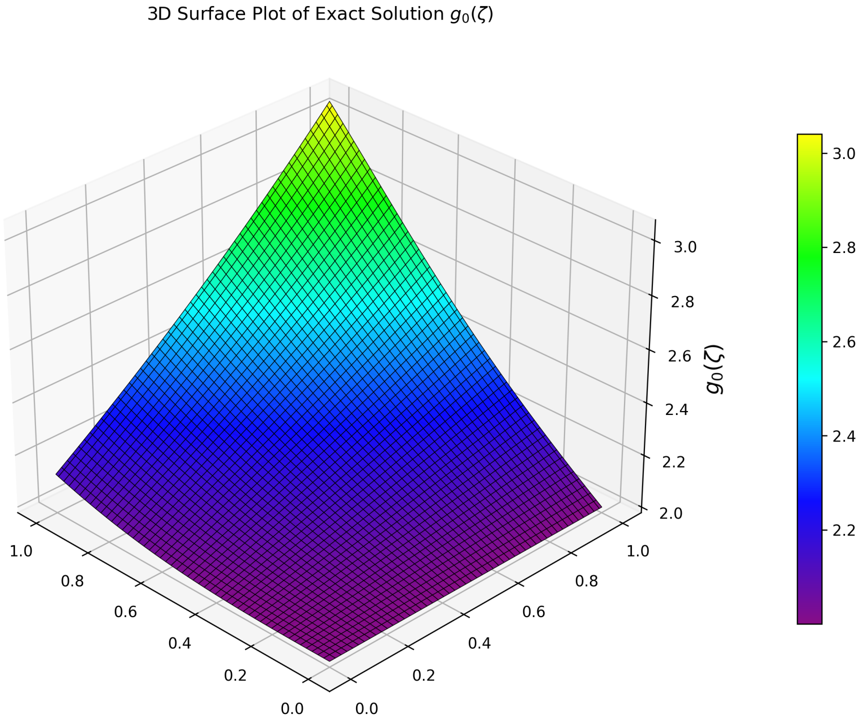

Figure 4: 3D surface plot of the exact solution $g_0(\zeta)$ for the given Carleman–type integral Equation (5.1), showing how the solution evolves with respect to the parameter ζ.

oscillatory nature of the solution due to sine and cosine functions in the integral kernel, expressed through Fresnel functions, is visually evident, demonstrating accumulation of the oscillations in the entire domain of integration. All these means that both the memory-dependent property and stable oscillatory nature of the solution have been successfully captured using visualization techniques.

7. Discussion on Real–Life Applications

The importance of these results in applications is discussed, with an emphasis on their physical significance in the context of the applied sciences and engineering domains. The oscillation of sine and cosine functions is characteristic of wave motion, resonance, and transmission of signals in physical systems, making them crucial elements in many physical applications. Maintaining the stability of solutions with respect to perturbations is vital to ensure that the predictions made by the model remain valid under realistic conditions where there may be some uncertainty and noise.

- **Wave Propagation and Acoustics:** These integral equations play an important role in modeling oscillatory waves in physical systems with memory and dissipation. Stability ensures that the predicted behavior is robust with respect to small perturbations in initial or environmental conditions.

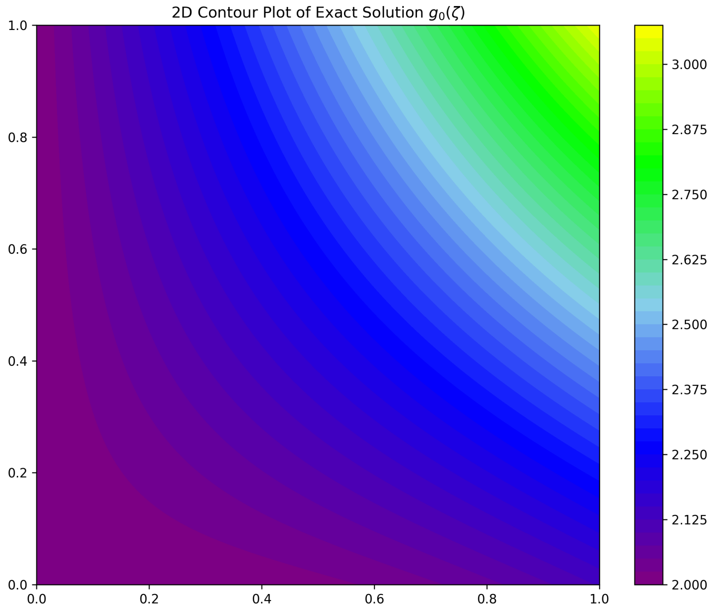

Figure 5: 2D contour plot of the exact solution $g_0(\zeta)$ to the Carleman–type integral Equation (5.1).

- **Signal Processing and Communication Systems:** Oscillatory kernels correspond to signal dynamics under the influence of noise and interference. The proven stability guarantees the reliability of methods for signal reconstruction, filtering, and identification, hence enhancing the quality of signal processing in telecommunication and other domains.

- **Resonance in Mechanical and Electrical Systems:** Resonance is a common property in many physical devices, where resonance is associated with oscillatory responses. Stable resonances ensure that small perturbations in parameters do not affect their prediction, thus facilitating safe design and control of such systems.

- **Mathematical and Theoretical Physics:** The obtained results can be straightforwardly generalized to include oscillatory integral equations appearing in different problems of mathematical physics, including electromagnetism and quantum mechanics.

- **Numerical Analysis and Computational Physics:** Stability in Ulam's sense allows us to rely on numerical computations for finding approximations to solutions of oscillatory integral equations. Namely, numerical algorithms provide reliable results even under possible errors in computations and discretization of initial equations.

These conclusions confirm the effectiveness of using oscillatory kernels in Carleman-Volterra integral equations to describe physical processes. The stability theorem proven here is an important step towards obtaining practical applications of the theory in physics and engineering problems.

8. Conclusions and Future Directions

Significant achievements have been reached in terms of stability analysis of Carleman Volterra integral equations with oscillatory kernels in the form of trigonometric functions. Using novel fixed point methods in the general setting of a generalized Bielecki metric, we have obtained sufficient criteria that assure Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and σ-semi-Ulam–Hyers stability for the given class of integral equations over finite and infinite intervals. These achievements constitute an effective stability framework for integral equations with oscillatory properties.

As shown in the paper, the suggested Carleman Volterra integral equations have stable solutions under specific conditions imposed on oscillatory kernels. In addition, the proven sufficient criteria ensure stability of solutions with respect to perturbations, which proves the validity and properness of these models. Consequently, this study is valuable from a theoretical standpoint since it sheds light on the stability of integral equations that model wave propagation and signal transmission.

In addition, the theoretical findings are accompanied by illustrative examples and computer simulation results, which show the validity of the achieved stability conclusions and their usefulness from both practical and computational perspectives.

As far as future research is concerned, several promising directions can be explored. In particular, the current stability scheme can be generalized to more sophisticated kernel functions such as nonlinear, non-oscillatory, or time-varying types, thereby widening the applicability domain. On the other hand, one may consider the inclusion of some stochastic factors, delay components, or higher-dimensional settings.

To sum up, this paper makes an essential contribution to the theoretical investigation of the stability property of the solutions of Carleman Volterra integral equations with oscillatory kernels and opens up new possibilities for future developments in this area.

Ethics Declarations

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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