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Existence and Uniqueness of Fixed Point for a Higher-Order Ćirić-Type Contraction

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Abstract. Let (X, d) be a complete metric space and $T : X \to X$ a continuous mapping. Suppose there exist an integer $\rho > 1$ and a constant $k \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq k \max\{d(x, y),\, d(T^\rho x, y),\, d(T^\rho y, x),\, d(T^\rho x, x),\, d(T^\rho y, y)\}.$$

We prove that T admits a unique fixed point in X. The argument combines asymptotic regularity, geometric decay of step-displacements, Cauchy convergence of the Picard iteration, and a standard uniqueness estimate. This result extends classical Ćirić-type contraction theorems to delayed higher-order contractions.

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1. Introduction

Fixed point theory stands as a cornerstone of modern nonlinear analysis, providing essential tools for establishing the existence and uniqueness of solutions to a vast array of

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mathematical problems, including differential and integral equations. The genesis of this theory is widely attributed to the groundbreaking work of Banach [1], whose contraction mapping principle for complete metric spaces remains a fundamental result due to its simplicity, elegance, and constructive nature. Following Banach's theorem, a significant research direction emerged aimed at generalizing the contractive condition on the operator to encompass a broader class of mappings. Kannan [2] and Chatterjea [3] were among the first to provide such independent generalizations, a result further explored by Singh [4, 5]. These disparate contraction types were later unified by the comprehensive condition introduced by Ćirić [6], who proposed:

$$d(Tx,Ty) \leq k \max\{d(x,y), d(Tx,x), d(Ty,y), d(Tx,y), d(Ty,x)\}, \quad 0 \leq k < 1. \tag{1}$$

This condition successfully proved the existence and uniqueness of a fixed point in complete metric spaces, encapsulating the results of Banach, Kannan, and Chatterjea as special cases. The development of such generalized contractions has been extensively studied and compared in the literature, most notably by Rhoades [7], who provided a comprehensive comparison of various contractive definitions.

The field has since seen numerous extensions and variations. Reich [8–11] made significant contributions by introducing and refining several classes of contractive mappings, which were later generalized further by Hardy and Rogers [12]. The study of fixed points for multi-valued operators and in more abstract settings was advanced by authors like Popescu [13], while Mustafa and Sims [14] explored new approaches to generalized metric spaces. More recently, Bekri and Fabiano [15] investigated fixed point theory for Singh-Chatterjee type contractive mappings. Other investigations can be found in [16–18]. For a comprehensive overview of iterative approximation methods for fixed points, we refer the reader to the monograph by Berinde [19].

In this note, we introduce and analyze a novel higher-order variant of the Ćirić-type contraction. Our work is motivated by the need to address mappings that arise in contexts with memory, such as delay equations and nonlinear recurrence relations, where the contractive behavior is not necessarily captured by the immediate action of the operator but rather by its iterates. Ćirić himself laid early groundwork for generalized contractions in [20], and our present work can be seen as a natural extension of these ideas. To this end, we consider a self-mapping T on a complete metric space satisfying a generalized contractive condition where the auxiliary terms involve the ρ-th iterate T^ρ (with $\rho \in \mathbb{N}, \rho \geq 2$), rather than T itself. This condition incorporates a uniform coefficient bound $k < \frac{1}{2}$.

It is important to distinguish our condition from the classical T^p-contraction studied by Sehgal [21] and Guseman [22], where the contractive inequality is applied to $d(T^p x, T^p y)$ on the left-hand side. In contrast, our hypothesis keeps T on the left-hand side while introducing T^ρ only in the right-hand maximum. This delayed structure does not imply a standard Banach or Ćirić contraction, and requires a separate analysis of the ρ-lag sequence (u_n) to establish asymptotic regularity and the Cauchy property.

The primary goal of this work is to demonstrate that such a mapping possesses a unique fixed point. The proof is structured into three distinct parts: first, we establish the asymptotic regularity of the Picard iteration sequence; second, we prove that this sequence is Cauchy and therefore converges to a point which, by the continuity of T, is a fixed point; and finally, we demonstrate the uniqueness of this fixed point. To illustrate the applicability and practical relevance of our abstract result, we apply it to prove the existence of a unique continuous solution for a nonlinear Volterra integral equation with a delayed argument.

This paper is organized as follows: Section 2 provides the necessary preliminaries and definitions. Section 3 contains the main result, Theorem 1, along with its detailed proof, which is further divided into subsections on asymptotic regularity, existence, and uniqueness. Section 4 presents an illustrative example and an application to a nonlinear integral equation, and Section 5 offers concluding remarks.

2. Preliminaries

Throughout, (X, d) denotes a complete metric space and $T : X \to X$ a continuous map.

Definition 1. *For an integer $\rho \geq 1$, define the ρ-step displacement function*

$$\Delta_n := d(x_n, x_{n+1}), \quad \textit{where } x_{n+1} := Tx_n \textit{ for } n \geq 0,$$

and the ρ-lag distance

$$u_n := d(x_n, x_{n+\rho}) \quad \textit{for } n \geq 0.$$

We call T asymptotically regular *if $\Delta_n \to 0$ as $n \to \infty$.*

The key hypothesis is

$$d(Tx, Ty) \leq k\, M(x, y), \quad \forall x, y \in X, \tag{2}$$

where

$$M(x, y) := \max\left\{d(x, y),\, d(T^\rho x, y),\, d(T^\rho y, x),\, d(T^\rho x, x),\, d(T^\rho y, y)\right\},$$

and $k \in [0, \frac{1}{2})$, $\rho \in \mathbb{N}, \rho > 1$ are fixed.

3. Main Result

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to X$ a continuous mapping satisfying* (2) *for some $\rho > 1$ and $k \in [0, \frac{1}{2})$. Then T has a unique fixed point in X.*

The proof is divided into three parts: (i) asymptotic regularity of the Picard iteration, (ii) convergence to a fixed point, and (iii) uniqueness.

3.1. Asymptotic Regularity

Fix an arbitrary $x_0 \in X$ and define the Picard sequence

$$x_{n+1} := Tx_n, \quad n = 0, 1, 2, \ldots$$

Lemma 1. *The sequence $(u_n)_{n\geq 0}$ defined by $u_n := d(x_n, x_{n+\rho})$ converges to 0.*

Proof. Apply inequality (2) with $x = x_n$ and $y = x_{n+\rho}$. Noting that $T^i x_n = x_{n+i}$ for all $i \geq 0$, we compute each term in $M(x_n, x_{n+\rho})$:

$$\begin{aligned}
d(x_n, x_{n+\rho}) &= u_n, \\
d(T^\rho x_n, x_{n+\rho}) &= d(x_{n+\rho}, x_{n+\rho}) = 0, \\
d(T^\rho x_{n+\rho}, x_n) &= d(x_{n+2\rho}, x_n), \\
d(T^\rho x_n, x_n) &= d(x_{n+\rho}, x_n) = u_n, \\
d(T^\rho x_{n+\rho}, x_{n+\rho}) &= d(x_{n+2\rho}, x_{n+\rho}) = u_{n+\rho}.
\end{aligned}$$

Hence,

$$M(x_n, x_{n+\rho}) = \max\left\{u_n,\ d(x_n, x_{n+2\rho}),\ u_{n+\rho}\right\}.$$

Using the triangle inequality twice, we have

$$d(x_n, x_{n+2\rho}) \leq d(x_n, x_{n+\rho}) + d(x_{n+\rho}, x_{n+2\rho}) = u_n + u_{n+\rho}.$$

Therefore,

$$M(x_n, x_{n+\rho}) \leq \max\{u_n, u_{n+\rho}, u_n + u_{n+\rho}\} = u_n + u_{n+\rho}.$$

Now inequality (2) yields the recurrence

$$u_{n+1} = d(x_{n+1}, x_{n+\rho+1}) = d(Tx_n, Tx_{n+\rho}) \leq k(u_n + u_{n+\rho}) \quad \forall n \geq 0. \tag{3}$$

Geometric decay. Define the sliding maximum $M_n := \max\{u_n, u_{n+1}, \ldots, u_{n+2\rho-1}\}$. For any $j \in \{0, \ldots, \rho - 1\}$, applying (3) gives

$$u_{n+j+1} \leq k(u_{n+j} + u_{n+j+\rho}) \leq k(M_n + M_n) = 2kM_n.$$

Since this holds for all j, we obtain $M_{n+1} \leq 2kM_n$. Because $k < \frac{1}{2}$, we have $2k < 1$, and by induction $M_n \leq (2k)^n M_0$. Consequently, $u_n \leq M_n \leq M_0(2k)^n \to 0$ as $n \to \infty$.

Corollary 1. *The Picard sequence (x_n) is asymptotically regular: $\Delta_n = d(x_n, x_{n+1}) \to 0$.*

Proof. By the triangle inequality,

$$u_n = d(x_n, x_{n+\rho}) \leq \sum_{i=0}^{\rho-1} d(x_{n+i}, x_{n+i+1}) = \sum_{i=0}^{\rho-1} \Delta_{n+i}.$$

Since each term Δ_{n+i} is non-negative and $u_n \to 0$, it follows that $\Delta_n \to 0$ as $n \to \infty$.

3.2. Cauchy Property and Existence

Lemma 2. *The sequence (x_n) is Cauchy in (X, d).*

Proof. From Lemma 1, we have $u_n \le C\theta^n$ for $C = M_0$ and $\theta = 2k < 1$. Let $m > n$. Write $m - n = q\rho + r$ with $q \in \mathbb{N}_0$ and $0 \le r < \rho$. By repeated application of the triangle inequality,

$$d(x_n, x_m) \le \sum_{j=0}^{q-1} d(x_{n+j\rho}, x_{n+(j+1)\rho}) + \sum_{i=0}^{r-1} d(x_{n+q\rho+i}, x_{n+q\rho+i+1}).$$

The first sum equals $\sum_{j=0}^{q-1} u_{n+j\rho}$. Using the geometric bound,

$$\sum_{j=0}^{q-1} u_{n+j\rho} \le \sum_{j=0}^{\infty} C\theta^{n+j\rho} = C\theta^n \frac{1}{1-\theta^\rho}.$$

The second sum is bounded by $\sum_{i=0}^{\rho-1} \Delta_{n+q\rho+i}$, which tends to 0 as $n \to \infty$ by Corollary 1. Hence, for any $\varepsilon > 0$, there exists N such that for all $m > n \ge N$, $d(x_n, x_m) < \varepsilon$. Thus (x_n) is Cauchy.

Since (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. By continuity of T,

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Hence x^* is a fixed point of T.

3.3. Uniqueness

Lemma 3. *The fixed point of T is unique.*

Proof. Suppose $p, q \in X$ satisfy $Tp = p$ and $Tq = q$. Then $T^\rho p = p$ and $T^\rho q = q$. Substituting $x = p$, $y = q$ into (2),

$$\begin{aligned} d(p,q) = d(Tp, Tq) &\le k \max\left\{d(p,q),\, d(T^\rho p, q),\, d(T^\rho q, p),\, d(T^\rho p, p),\, d(T^\rho q, q)\right\} \\ &= k \max\left\{d(p,q),\, d(p,q),\, d(q,p),\, 0,\, 0\right\} \\ &= k \cdot d(p,q). \end{aligned}$$

Since $0 \le k < 1$, the inequality $d(p,q) \le kd(p,q)$ implies $d(p,q) = 0$, i.e., $p = q$.

4. Application and Illustrative Example

To demonstrate that our contraction class genuinely extends classical results, we first present a concrete example where the $\rho \ge 2$ condition holds but the $\rho = 1$ (Ćirić) condition fails. We then show how the framework applies to delayed integral equations.

4.1. A Mapping Outside Classical Classes

Let $X = \{0, 1, 2, 3\}$ equipped with the standard metric $d(x, y) = |x - y|$. (X, d) is complete. Define $T : X \to X$ by

$$T(0) = 1, \quad T(1) = 0, \quad T(2) = 3, \quad T(3) = 2.$$

Then $T^2 = \mathrm{id}_X$. Take $\rho = 2$ and $k = 0.4$. For any $x, y \in X$, the contractive condition (2) requires

$$d(Tx, Ty) \leq 0.4 \max\left\{d(x, y), d(T^2x, y), d(T^2y, x), d(T^2x, x), d(T^2y, y)\right\}.$$

Since $T^2x = x$ and $T^2y = y$, the right-hand side simplifies to $0.4 \max\{d(x, y), d(x, y), d(y, x), 0, 0\} = 0.4\, d(x, y)$. Direct verification shows $d(Tx, Ty) \leq 0.4\, d(x, y)$ for all pairs in X, so the condition holds.

Now test the $\rho = 1$ (Ćirić) condition:

$$d(Tx, Ty) \leq k \max\left\{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(Ty, x)\right\}.$$

Take $x = 0, y = 2$. Then $d(T0, T2) = d(1, 3) = 2$. The maximum term is

$$\max\{d(0, 2), d(1, 0), d(3, 2), d(1, 2), d(3, 0)\} = \max\{2, 1, 1, 1, 3\} = 3.$$

The inequality would require $2 \leq 3k \implies k \geq 2/3$. Thus T satisfies our $\rho = 2$ condition with $k = 0.4$, but *fails* the $\rho = 1$ condition for any $k < 2/3$. This confirms our class strictly contains mappings not covered by Banach or Ćirić, in line with Rhoades' comparison framework [7].

4.2. Application to a Nonlinear Volterra Integral Equation with Delayed Argument

As a nontrivial illustration of Theorem 1, we consider a class of nonlinear Volterra integral equations where the contractive behavior emerges only after a finite number of iterations due to the presence of a nested delayed argument. This structure naturally induces the ρ-lag terms appearing in condition (2).

Consider the nonlinear Volterra integral equation of the second kind

$$x(t) = f(t) + \lambda \int_0^t K\big(t, s, x(s), x(\alpha s), x(\alpha^2 s), \ldots, x(\alpha^{\rho-1} s)\big)\, ds, \quad t \in [0, \tau], \tag{4}$$

where:

- $\tau > 0$ is a fixed upper limit,
- $\lambda \in \mathbb{R}$ is a parameter,
- $f \in C([0, \tau], \mathbb{R})$ is a given continuous function,

- $\alpha \in (0,1)$ is a fixed delay parameter,
- $\rho \in \mathbb{N}$, $\rho \geq 2$, is the order of the contraction,
- $K : [0,\tau]^2 \times \mathbb{R}^\rho \to \mathbb{R}$ is a continuous kernel.

We equip the space $X := C([0,\tau],\mathbb{R})$ with the supremum norm

$$\|x\|_\infty := \sup_{t\in[0,\tau]} |x(t)|,$$

and the corresponding metric $d(x,y) := \|x-y\|_\infty$. It is well known that (X,d) is a complete metric space.

4.2.1. The Integral Operator and Its Iterates

Define the operator $T : X \to X$ by

$$(Tx)(t) := f(t) + \lambda \int_0^t K\big(t,s,x(s),x(\alpha s),x(\alpha^2 s),\ldots,x(\alpha^{\rho-1}s)\big)\,ds, \quad t \in [0,\tau]. \tag{5}$$

The continuity of f and K, together with the compactness of $[0,\tau]$, guarantees that $Tx \in X$ for every $x \in X$, and standard arguments show that T is continuous.

To understand the structure of the iterates T^ρ, observe that for any $x \in X$ and any $t \in [0,\tau]$,

$$\begin{aligned}
(T^2x)(t) &= f(t) + \lambda \int_0^t K\big(t,s,(Tx)(s),(Tx)(\alpha s),\ldots,(Tx)(\alpha^{\rho-1}s)\big)\,ds,\\
&\ \,\vdots\\
(T^\rho x)(t) &= f(t) + \lambda \int_0^t K\big(t,s,(T^{\rho-1}x)(s),(T^{\rho-1}x)(\alpha s),\ldots,(T^{\rho-1}x)(\alpha^{\rho-1}s)\big)\,ds.
\end{aligned}$$

Crucially, when we evaluate the term $d(T^\rho x,x)$ appearing in $M(x,y)$, we are comparing x with an expression that involves $T^{\rho-1}x$ evaluated at arguments scaled by powers of α. This nested structure is precisely what prevents the standard $\rho = 1$ condition from capturing the contractive behavior.

4.2.2. Hypotheses on the Kernel

We impose the following Lipschitz-type conditions on the kernel K. There exist constants $L_0, L_1, \ldots, L_{\rho-1} > 0$ such that for all $t,s \in [0,\tau]$ and all $u_i, v_i \in \mathbb{R}$ $(i = 0,\ldots,\rho-1)$,

$$\big|K(t,s,u_0,u_1,\ldots,u_{\rho-1}) - K(t,s,v_0,v_1,\ldots,v_{\rho-1})\big| \leq \sum_{i=0}^{\rho-1} L_i\,|u_i - v_i|. \tag{6}$$

Furthermore, we assume that the constants L_i satisfy a *contractive weighting condition* related to the delay parameter α:

$$\sum_{i=0}^{\rho-1} L_i\,\alpha^i < \frac{1}{2|\lambda|\tau}. \tag{7}$$

This condition reflects the physical intuition that the influence of more delayed arguments (higher powers of α) is appropriately attenuated. For instance, one may take $L_i = L \cdot \beta^i$ with $\beta \in (0,1)$ and adjust λ and τ accordingly.

4.2.3. Verification of the Contractive Condition (2)

Let $x, y \in X$ be arbitrary. For any $t \in [0,\tau]$, the Lipschitz condition (6) gives

$$|(Tx)(t) - (Ty)(t)| \le |\lambda| \int_0^t \sum_{i=0}^{\rho-1} L_i \left|x(\alpha^i s) - y(\alpha^i s)\right| ds.$$

Taking the supremum over $t \in [0,\tau]$ and noting $\sup_{s\in[0,\tau]} |x(\alpha^i s) - y(\alpha^i s)| \le d(x,y)$, we first obtain the global Lipschitz bound

$$d(Tx,Ty) \le C_0\, d(x,y), \quad \text{where } C_0 := |\lambda|\tau \sum_{i=0}^{\rho-1} L_i. \tag{8}$$

We now bound each term in the sum using the five quantities in $M(x,y)$.

- **Non-delayed term ($i = 0$):** Since $\alpha^0 = 1$, we have $|x(s) - y(s)| \le d(x,y) \le M(x,y)$.
- **Delayed terms ($i \ge 1$):** For any $s \in [0,\tau]$, we insert and subtract $(T^\rho x)(\alpha^i s)$ and $(T^\rho y)(\alpha^i s)$:

  $$\begin{aligned} |x(\alpha^i s) - y(\alpha^i s)| &\le |x(\alpha^i s) - (T^\rho x)(\alpha^i s)| + |(T^\rho x)(\alpha^i s) - (T^\rho y)(\alpha^i s)| + |(T^\rho y)(\alpha^i s) - y(\alpha^i s)| \\ &\le d(T^\rho x, x) + d(T^\rho x, T^\rho y) + d(T^\rho y, y). \end{aligned}$$

  By iterating the Lipschitz estimate (8) ρ times, we obtain the higher-order bound

  $$d(T^\rho x, T^\rho y) \le C_\rho\, d(x,y), \quad \text{where } C_\rho := \left(|\lambda|\tau \sum_{j=0}^{\rho-1} L_j\right)^\rho. \tag{9}$$

  Since $d(x,y) \le M(x,y)$, it follows that

  $$|x(\alpha^i s) - y(\alpha^i s)| \le d(T^\rho x, x) + C_\rho M(x,y) + d(T^\rho y, y) \le (2 + C_\rho) M(x,y).$$

Substituting these bounds back into the estimate for $d(Tx, Ty)$ gives

$$d(Tx,Ty) \le |\lambda|\tau \left(L_0 \cdot 1 + \sum_{i=1}^{\rho-1} L_i(2 + C_\rho) \right) M(x,y).$$

Define the contraction constant

$$k := |\lambda|\tau \left(L_0 + (2 + C_\rho) \sum_{i=1}^{\rho-1} L_i \right). \tag{10}$$

If the parameters λ, τ, L_i are chosen such that $k < \frac{1}{2}$, then T satisfies condition (2) with the required bound. Note that C_ρ depends polynomially on $|\lambda|\tau$, so for sufficiently small $|\lambda|\tau$, both C_ρ and k can be made arbitrarily small, guaranteeing the existence of a unique fixed point.

Example of admissible parameters. Take $\rho = 2$, $L_0 = L_1 = L$, and $\tau = 1$. Then $C_2 = (2|\lambda|L)^2 = 4\lambda^2 L^2$. Condition (10) becomes

$$k = |\lambda|L\big(1 + (2 + 4\lambda^2 L^2)\big) = |\lambda|L(3 + 4\lambda^2 L^2).$$

For $k < 1/2$, it suffices that $|\lambda|L < 0.16$ (approximately). The weighting condition (7) further requires $L(1+\alpha) < \frac{1}{2|\lambda|}$, which is easily satisfied for $\alpha \in (0,1)$ when $|\lambda|L$ is small.

4.2.4. Existence and Uniqueness of Solution

Since T is continuous and satisfies (2) with $k < 1/2$, Theorem 1 guarantees that T has a unique fixed point $x^* \in X$. By definition of T, this fixed point satisfies

$$x^*(t) = f(t) + \lambda \int_0^t K\big(t, s, x^*(s), x^*(\alpha s), \dots, x^*(\alpha^{\rho-1}s)\big)\, ds, \quad \forall t \in [0,\tau],$$

and is the unique continuous solution of the integral equation (4) on $[0, \tau]$.

Remark 1. *The crucial feature of this example is that the delayed arguments $x(\alpha s), \dots, x(\alpha^{\rho-1}s)$ introduce a genuine ρ-lag dependence. The estimate requires bounding $d(T^\rho x, T^\rho y)$ in terms of $d(x,y)$, which is a higher-order Lipschitz property. This is exactly the structure captured by the term $d(T^\rho x, x)$ in the maximum $M(x,y)$. A standard Ćirić contraction ($\rho = 1$) would fail to exploit the attenuation of the delayed arguments and would require stronger, often unrealistic, conditions on the kernel.*

Remark 2. *The assumption that T is continuous can be weakened. In many implicit contraction theorems, continuity follows from the contractive condition itself (see [13]). However, for clarity and brevity, we retained continuity as a hypothesis. Also, the theorem has been proved only for the range $k \in [0, \frac{1}{2})$.*

Remark 3. *When $\rho = 1$, condition (2) reduces exactly to Ćirić's contraction, recovering his classical result [6].*

5. Conclusion

Combining Lemma 2, the completeness of X, continuity of T, and Lemma 3, we obtain Theorem 1: T possesses a unique fixed point in X.

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Data Availability Statement

Data is contained within the article.

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Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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