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Study of the Convergence of the Laplace-Adomian Method Applied to the Telegraph Equation with Nonlinearity and Source Term

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Abstract. In this article, we study the convergence of the Laplace Adomian algorithm applied to the telegraph equation with a nonlinearity and a source term. To this end, we review some definitions and properties that enable us to establish the convergence proof, then we present a description of the method, followed by the convergence proof, and finally we propose an application of the algorithm.

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1. Introduction

In this article, we focus on the study of convergence and the application of the Laplace-Adomian algorithm to the telegraph equation model, incorporating non-linearity in order to take into account more complex effects. We recall that the study of the standard telegraph model equation for the linear case was the subject of a previous publication [1]. Vineet K Srivastava and his colleagues in [3] also proposed a description of the physical problem and suggested the solution of some applications of the model using the RDTM (Reduced Differential Transform Method). For the nonlinear telegraph equation with a source term, Abdul Rahmin Farhan Sabdin and his collaborators developed the MM-RDTM (Multistep Modified Reduced Differential Transform Method) in [5] and proposed an approximate solution for certain applications. In our study, we first review some definitions and properties, then present a description of the Laplace-Adomian algorithm, then study the convergence of the Laplace-Adomian algorithm applied to the nonlinear telegraph equation with a source term, taking into account certain conditions, and finally propose an application.

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2. Preliminary and definitions

In this section, we recall some definitions and theorems

Definition 1. *Let $u : \mathbb{R}^{+} \mapsto \mathbb{C}$ be a continuous function.*
The function $\mathcal{L}(u)$ defined by

$$\mathcal{L}(u) = \int_{0}^{\infty} u(t)e^{-st}dt.$$

This theorem is a result that proves the existence and uniqueness of the solution of the functional equation $Au = f$

Theorem 1. (*Minty-Browder*) [4]
Let E be a reflexive Banach space. Let $A : E \to E'$ an application (non-linear) continues such that :

$$\langle Au - Av; u - v\rangle > 0 \qquad \forall u, v \in E, \quad u \neq v$$

$$\lim_{\|v\|\to\infty} \frac{\langle Av; v\rangle}{\|v\|} = \infty$$

So for everything $f \in E'$, there is $u \in E$ a unique solution to the equation $Au = f$

Considering the functional equation in the form $u = Nu + f$, Cherruault and his colleagues establish the following result

Theorem 2. [2]
if

$$\forall n \geq 0, \left|\left|N^{(n)}(u_0)\right|\right| \leq M < \frac{1}{e}$$

then the decomposition series $\sum_{n\geq 0} u_n$ is absolutely convergent and we have the following relation :

$$||A_n|| \leq M^{n+1}e^{n+1}$$

3. The numerical Laplace-Adomian method

The standard telegraph equation is a partial differential equation given by :

$$\frac{\partial u^2}{\partial x^2} - \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u + \delta(u) = h(t, x)$$

where $u = u(t, x)$ is the resistance, and α,β and γ are constants related to the inductance, capacitance and conductance of the cable respectively.

Let us consider the following functional equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} & = & \alpha \dfrac{\partial^2 u}{\partial t^2} - \beta \dfrac{\partial u}{\partial t} - \gamma u - \delta\left(u\right) + h(t,x) \\ u\left(t,0\right) & = & f\left(t\right) \\ \dfrac{\partial u}{\partial x}\left(t,0\right) & = & g\left(t\right) \end{cases} \tag{1}$$

Taking $Lu = \dfrac{\partial^2 u}{\partial x^2}$, $Ru = \alpha \dfrac{\partial^2 u}{\partial t^2} - \beta \dfrac{\partial u}{\partial t} - \gamma u$ and $Nu = -\delta\left(u\right)$
We have :

$$Lu = Ru + Nu + h(t,x) \tag{2}$$

Where L is an invertible operator in the Adomian sense and R the linear remainder. Applying the laplace transform to the equation (2), we obtain :

$$\mathcal{L}_x\left(Lu\right) = \mathcal{L}_x\left(Ru\right) + \mathcal{L}_x\left(Nu\right) + \mathcal{L}_x\left[h\left(t,x\right)\right] \tag{3}$$

we deduce

$$s^2 \mathcal{L}_x\left(u\right) = sf\left(t\right) + g\left(t\right) + \mathcal{L}_x\left(Ru\right) + \mathcal{L}_x\left(Nu\right) + \mathcal{L}_x\left[h\left(t,x\right)\right] \tag{4}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$s^2 \sum_{n\geq 0} \mathcal{L}_x\left(u_n\right) = sf\left(t\right) + g\left(t\right) + \mathcal{L}_x\left[h\left(t,x\right)\right] + \sum_{n\geq 0} \mathcal{L}_x\left(Ru_n\right) + \mathcal{L}_x\left(N \sum_{n\geq 0} u_n\right) \tag{5}$$

Using the decomposition relation

$$Nu = N\left(\sum_{n\geq 0} u_n\right) = \sum_{n\geq 0} A_n$$

This yields the following Adomian algorithm:

$$\begin{cases} s^2 \mathcal{L}_x\left(u_0\right) = sf\left(t\right) + g\left(t\right) + \mathcal{L}_x\left[h\left(t,x\right)\right] \\ s^2 \mathcal{L}_x\left(u_{n+1}\right) = \mathcal{L}_x\left(Ru_n\right) + \mathcal{L}_x\left(A_n\right); n \geq 0 \end{cases} \tag{6}$$

Applying the laplace transform to the equation (2), we obtain :

$$\begin{cases} u_0\left(t,x\right) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\left(sf\left(t\right) + g\left(t\right) + \mathcal{L}_x\left[h\left(t,x\right)\right]\right)\right] \\ u_{n+1}\left(t,x\right) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\left(Ru_n\right) + \dfrac{1}{s^2}\mathcal{L}_x\left(A_n\right)\right]; n \geq 0 \end{cases} \tag{7}$$

4. Algorithm of Laplace - ADM Convergence's

Considering the equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} & = \alpha \dfrac{\partial^2 u}{\partial t^2} - \beta \dfrac{\partial u}{\partial t} - \gamma u - \delta(u) + h(t,x) \\ u(t,0) & = f(t) \\ \dfrac{\partial u}{\partial x}(t,0) & = g(t) \end{cases}$$

With $(t,x) \in \Omega$ where $\Omega = [0; +\infty[\times [a,b]$

The application of the Laplace-ADM method gives

$$\begin{cases} u_0(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\left(sf(t) + g(t) + \mathcal{L}_x[h(t,x)]\right)\right] \\ u_{n+1}(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x(Ru_n) + \dfrac{1}{s^2}\mathcal{L}_x(A_n)\right]; n \geq 0 \end{cases} \tag{8}$$

Let us suppose :

· (H_1)
f is continuous then there is a real M_1 so that

$$|f(t)| \leq M_1 \text{ for all } t \in [0,T]$$

· (H_2)
g is continuous then there is a real M_2 so that

$$|g(t)| \leq M_2 \text{ for all } t \in [0,T]$$

· (H_3)
h is continuous then there is a real M_3 so that

$$|h(t,x)| \leq M_3 \text{ for all } (t,x) \in [0,T] \times [a,b] \text{ with } b \geq a \geq 0$$

However
R the linear remainder is continuous then there is a real $\theta > 0$ so that

$$||Ru|| \leq \theta \, ||u||$$

Indeed, we have :

$$\begin{cases} |u_0| & = \left|\mathcal{L}_x^{-1}\left[\dfrac{f(t)}{s}\right] + \mathcal{L}_x^{-1}\left[\dfrac{g(t)}{s^2}\right] + \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x(h(t,x))\right]\right| \\ |u_n| & = \left|\mathcal{L}_x^{-1}\left[\dfrac{\mathcal{L}_t(Ru_{n-1})}{s^2} + \dfrac{1}{s^2}\mathcal{L}_x(A_{n-1})\right]\right| ; n \geq 1 \end{cases}$$

There is a real $x_0 \in \mathbb{R}_*^+$ so that $\Re e\,(s) > x_0$, we deduce the following system :

$$\begin{cases} |u_0| & \leq & \left|\mathcal{L}_x^{-1}\left[\dfrac{f\,(t)}{s}\right]\right| + \left|\mathcal{L}_x^{-1}\left[\dfrac{g\,(t)}{s^2}\right]\right| + \left|\mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\,(h\,(t,x))\right]\right| \\ |u_n| & \leq & \mathcal{L}_x^{-1}\left[\dfrac{|Ru_{n-1}|}{|s^2|}\right] + \mathcal{L}_x^{-1}\left[\dfrac{1}{|s^2|}\mathcal{L}_x\,(|A_{n-1}|)\right]\ ;\ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M_1 + M_2 b + M_3 b^2, \\ |u_n| & \leq & \mathcal{L}_x^{-1}\left[\dfrac{\mathcal{L}_x\,(|Ru_{n-1}|)}{x_0^2}\right] + \mathcal{L}_x^{-1}\left[\dfrac{1}{x_0^2}\mathcal{L}_x\,(|A_{n-1}|)\right]\ ;\ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M_1 + M_2 b + M_3 b^2 \\ |u_n| & \leq & \dfrac{1}{x_0^2}\mathcal{L}_x^{-1}\left[\mathcal{L}_x\left(\left|Ru_{n-1}^1\right|\right) + \mathcal{L}_x\,(|A_{n-1}|)\right]\ ;\ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M_1 + M_2 b + M_3 b^2 \\ |u_n| & \leq & \dfrac{\theta}{x_0^2}\,||u_{n-1}|| + \dfrac{1}{x_0^2}\,||A_{n-1}||\ ;\ n \geq 1 \end{cases}$$

Step by step, we deduce :

$$\Rightarrow \begin{cases} |u_0| & \leq & M_1 + M_2 b + M_3 b^2 \\ |u_n| & \leq & \left(\dfrac{\theta}{x_0^2}\right)^n \left[M_1 + M_2 b + M_3 b^2\right] + \dfrac{1}{x_0^2} M^n e^n\ ;\ n \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} |u_0| & \leq & M_1 + M_2 b + M_3 b^2 \\ \sum \lim\limits_{n \geq 0} |u_n| & \leq & \dfrac{M_1 + M_2 b + M_3 b^2}{x_0^2 - \theta} + \dfrac{1}{x_0^2\,(1 - Me)} \end{cases}$$

Then the series $\sum\limits_{n \geq 0} u_n$ is convergent for $x \neq \sqrt{\theta}$, therefore this algorithm is convergent.

5. Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} + u^2 - xu\dfrac{\partial u}{\partial x} & = 0, x \geq 0 \\ u\,(t,0) = 0, \quad u_x\,(t,0) = e^t \end{cases} \tag{9}$$

Taking $Lu = \dfrac{\partial^2 u}{\partial x^2}$, $Ru = \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial u}{\partial t}$ and $Nu = u^2 - xu\dfrac{\partial u}{\partial x}$.

Where L is an invertible operator in the Adomian sense and R the linear remainder. Applying the laplace transform to the equation (9), we obtain :

$$\mathcal{L}_x\left(Lu\right) = \mathcal{L}_x\left(Ru + Nu\right) \tag{10}$$

$$s^2\mathcal{L}_x\left(u\right) - su\left(t,0\right) - \frac{\partial u}{\partial x}\left(0,x\right) = \mathcal{L}_x\left(Ru\right) + \mathcal{L}_x\left(Nu\right) \tag{11}$$

Using the decomposition series for the linear term $u(t,x)$ gives

$$s^2\sum_{n\geq 0}\mathcal{L}_x\left(u_n\right) = e^t + \sum_{n\geq 0}\mathcal{L}_x\left(Ru_n\right) + \sum_{n\geq 0}\mathcal{L}_x(A_n) \tag{12}$$

We deduce the following Laplace-Adomian algorithm

$$\begin{cases} s^2\mathcal{L}_x\left(u_0\right) = e^t \\ \\ s^2\mathcal{L}_x\left(u_{n+1}\right) = \mathcal{L}_x\left(Ru_n\right) + \mathcal{L}_x\left(A_n\right); n\geq 0 \end{cases} \tag{13}$$

We obtain

$$\begin{cases} u_0\left(t,x\right) = \mathcal{L}_x^{-1}\left[\dfrac{e^t}{s^2}\right] \\ \\ u_{n+1}\left(t,x\right) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\left(Ru_n\right)\right] + \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\left(A_n\right)\right]; n\geq 0 \end{cases}$$

Determinate $u_n(t,x)$, **for** $n \geq 0$

$u_0(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{e^t}{s^2}\right]$

$\Rightarrow u_0(t,x) = xe^t$

$u_1(t,x) = \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\left(Ru_0\right)\right] + \mathcal{L}_x^{-1}\left[\dfrac{1}{s^2}\mathcal{L}_x\left(A_0\right)\right]$

We have

$Ru_0 = \dfrac{\partial^2 u_0}{\partial t^2} - \dfrac{\partial u_0}{\partial t}$

$A_0 = u_0^2 - xu_0\dfrac{\partial u_0}{\partial x} = 0$

$\Rightarrow u_1(t,x) = 0$

$$u_2(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{s^2}\mathcal{L}_x(Ru_1)\right] + \mathcal{L}_x^{-1}\left[\frac{1}{s^2}\mathcal{L}_x(A_1)\right]$$

We have

$$Ru_1 = \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial u_1}{\partial t} = 0$$

$$A_1 = xu_0\frac{\partial u_1(t,x)}{\partial x} + xu_1\frac{\partial u_0(t,x)}{\partial x} - 2u_0u_1 = 0$$

We deduce : $u_2(t,x) = 0$

$$u_3(t,x) = \mathcal{L}_x^{-1}\left[\frac{1}{s^2}\mathcal{L}_x(Ru_2)\right] + \mathcal{L}_x^{-1}\left[\frac{1}{s^2}\mathcal{L}_x(A_2)\right]$$

$$Ru_2 = \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial u_2}{\partial t} = 0$$

$$\Rightarrow A_2 = xu_0\frac{\partial u_2(t,x)}{\partial x} + xu_1\frac{\partial u_1(t,x)}{\partial x} + xu_2\frac{\partial u_0(t,x)}{\partial x} - u_1^2 - 2u_0u_2 = 0$$

We deduce : $u_3(t,x) = 0$

In recursive way, we deduce

$$u_n(t,x) = 0, \text{ for } n \geq 1$$

Then

$$u(x,t) = \sum_{n\geq 0} u_n(x,t) = xe^t + \sum_{n\geq 1} u_n(t,x) = xe^t$$

Conclusion : The exact solution to the problem is

$$u(x,t) = xe^t$$

6. Conclusion

This document studies the convergence of the Laplace-Adomian method applied to the telegraph equation with non-linearity and source term, providing rigorous proof and practical application. This study establishes the effectiveness of the Laplace-ADM method for numerically solving nonlinear partial differential equations, particularly those of the telegraph, by accelerating convergence towards the solution.

References

[1] M Youssouf, B Moussa and P Youssouf. Resolution of the standard telegraph equation by the Laplace-Adomian method. *Journal of Advances in Mathematics and Computer Science (JAMCS)*, vol 39, issue 3 : 63-71, 2024.

[2] K Abbaoui and Y Cherruault, Convergence of the Adomian method applied to the nonlinear equations. *Math. Comput. Modelling*, (20,9),pp60-73,1994.

[3] K Vineet, Srivastava, M K Awasthi, R K Chaurasia, and M Tamsir. The Telegraph Equation and Its Solution by Reduced Differential Transform Method. *Hindawi Publishing Corporation*, vol 2013, pp 1-6, 2013. http://dx.doi.org/10.1155/2013/746351.

[4] H Brezis. *Analyse fonctionnelle: Théorie et Applications.* Masson, Paris, 1983.

[5] A Rahman, F Sabdin, C H C Hussi, A Mandangan, G B Ekal, J Sulaiman. *Journal of Advanced Research in Applied Sciences and Engineering Technology* Vol 31, Issue 3 : 238-248, 2023.

[6] R K Chaurasia, R Mishra, R G Mishra, D Kumar,A K Shrivastava. Numerical Modelling And Solution Of Microstrip Antenna By Using Caupto Time Fraction RDT Method. *IJSTR*, Vol 8(08) : pp 135 - 138, 2019.

[7] Y Paré, , Y Minoungou, A W Nébié. Resolution of nonlinear convection - diffusion - reaction equations of Cauchy's kind by the Laplace SBA method. *EJPAM*, Vol. 12, No. 3 : 742-760, 2019.

[8] Y. Pare, B. Saley, B. Some, S Ousseini and A Bakari. A numerical method for solving Cauchy evolution problem of partial differential equations (PDEs) for several variables, *Int. J. Numer. Methods Appl*, Vol 1(1) : 87-100, 2009.

[9] A M Wazwaz,*Partial differential equations and applications*, Netherland Balkema Publisher, 2002.