EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 2, Article Number 7781
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Secure Distance-k Domination Number of Graphs

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Abstract. This paper introduces the concept of secure distance-k domination in graphs. Fundamental properties and bounds for the secure distance-k domination number are established, including its relationship with both the distance-k domination number and the classical domination number. We characterize graphs with secure distance-k domination number equal to 1 in terms of their diameter, and provide a complete characterization for when this parameter attains its maximum value. Additionally, the behavior of secure distance-k dominating sets on disconnected graphs is evaluated, establishing an additive property over graph components.
Furthermore, we derive a necessary and sufficient neighborhood containment condition for a distance-k dominating set to be secure, which yields a structural characterization of graphs where the distance-k domination number and its secure counterpart coincide. Finally, we determine exact values of the secure distance-k domination number for binary graph operations, specifically the join and corona products, and for the path and cycle graphs.
2020 Mathematics Subject Classifications: 05C12, 05C69, 05C76
Key Words and Phrases: Secure Domination, distance-k domination, secure distance-k domination, join, corona.

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1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, connected, and undirected graph. For the terminologies and notations, we refer to Haynes, Hedetniemi, and Henning [1]. The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the minimum length of a (u, v)-path of G if any, otherwise $d_G(u, v) = \infty$. The eccentricity $ecc_G(v)$ of a vertex v in G is the distance between v and a vertex farthest from v in G. That is,

$$ecc_G(v) = \max\{d_G(v, u) \mid u \in V(G)\}.$$

The minimum eccentricity among all vertices of G is the radius of G, denoted by $rad(G)$. The maximum eccentricity among all vertices of G is the diameter of G, denoted by $diam(G)$.

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i2.7781

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The *open neighborhood* of a vertex v in G, denoted by $N(v)$, is the set of all neighbors of v in G, while the *closed neighborhood* of v in G, denoted by $N[v]$, is the set $N[v] = N(v) \cup \{v\}$. For an integer $k \geq 1$, the *closed k-neighborhood* of v in G, denoted by $N_k[v]$, is the set of all vertices within distance k from v; that is, $N_k[v] = \{u \mid d_G(u,v) \leq k\}$. The *open k-neighborhood* of v, denoted by $N_k(v)$, is the set of all vertices different from v and at distance at most k from v in G; that is, $N_k(v) = N_k[v] \backslash \{v\}$. For a set $S \subseteq V(G)$, the *closed k-neighborhood* of S in G, denoted by $N_k[S]$, is the union of the closed k-neighborhoods of the vertices in S. Similarly, the *open k-neighborhood* of S in G, denoted by $N_k(S)$, is the union of the open k-neighborhoods of the vertices in S. A set $S \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N[S] = V(G)$.

The systematic integration of distance constraints into domination theory began in the mid-1970s. Moon and Meir (1975) introduced the notion of distance-k domination (referred to as a k-covering) in graphs [2], establishing fundamental bounds for tree structures. Independently and nearly concurrently, Slater (1976) evaluated r-domination in forest graphs [3]. A subset D of $V(G)$ is said to be a *distance-k dominating set* of G if for each $u \in V(G) \backslash D$, $d_G(u,v) \leq k$ for some v in D. The minimum cardinality of a distance-k dominating set is called the *distance-k domination number* of G, denoted by $\gamma_k(G)$. A distance-k dominating set D of G is said to be a γ_k-set if $|D| = \gamma_k(G)$. This core parameter has since been extensively investigated across various topologies, including recent work by Anand, Dayap, Casnillo, Pepper, and Nair (2025) [4] focusing on graph product operations.

Consequently, Cockayne, Favaron, and Mynhardt introduced the concept of secure domination [5] in 2003. A dominating set S is a *secure dominating set* if for every $u \in V(G) \backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S \backslash \{v\}) \cup \{u\}$ is a dominating set. The *secure domination number* of G, denoted by $\gamma_s(G)$, is the smallest cardinality of a secure dominating set. Castillano, Ugbinada, Canoy, and Go further studied this parameter in 2014 [6, 7] within the context of graph joins and compositions.

While secure domination and distance-k domination have been extensively studied independently, their isolation creates an operational gap in modeling systems that require both wide-area structural reach and local fault tolerance. Practically, classical secure domination ($\gamma_s(G)$) assumes that an active protecting node can only safeguard its immediate neighbors (distance 1) [8]. This fails to accurately model modern networks—such as wireless sensor grids, drone surveillance zones, or regional logistical hubs—where a single facility operates effectively up to an extended signal or physical travel radius of $k \geq 2$. Conversely, standard distance-k domination ($\gamma_k(G)$) assumes an idealized system with no disruptions; if a critical server or emergency facility at node w goes offline or is compromised, its entire k-neighborhood immediately loses coverage, precipitating systemic failure.

The secure distance-k domination parameter ($\gamma_k^s(G)$) explicitly bridges this gap. It models a cost-efficient, resilient network architecture where service nodes cover a wider neighborhood radius k, yet possess an innate "hot-swap" redundancy: if a facility at node w fails, a neighboring client node v can step in and assume the role of the provider, guaranteeing that the entire graph remains fully distance-k dominated without requiring expensive, global resource re-allocations. Theoretically, investigating $\gamma_k^s(G)$ allows us to quantify the exact structural premium of security over a distance radius, revealing how the network's underlying topology (such as its diameter, cycle lengths, or boundary features under join and corona operations) absorbs localized node disruptions.

In this paper, we introduce this new variant of domination that integrates secure domination and distance-k domination. We establish general bounds for the parameter and provide characterizations for its extremal values, particularly when it attains 1 and its maximum value. Furthermore, we derive necessary and sufficient conditions for secure distance-k dominating sets, determine when they coincide with classical domination numbers, and compute exact values for several standard graph classes and graph operations, including the join and corona operations, alongside structural bounds for path and cycle graphs.

2. Results

The terminologies, notations, and related results are included in this paper and individually cited.

Definition 1. A distance-k dominating set S is a *secure distance-k dominating set* if for each $v \in V(G)\backslash S$, there exist $w \in S \cap N_k(v)$ such that $(S\backslash\{w\}) \cup \{v\}$ is a distance-k dominating set in G. The minimum cardinality among all secure distance-k dominating sets in G, denoted by $\gamma_k^s(G)$, is called the *secure distance-k domination number* of G. Any secure distance-k dominating set with cardinality equal to $\gamma_k^s(G)$ is called γ_k^s-set.

Example 1. To illustrate the mechanics of a secure distance-k dominating set, we directly apply Definition 1 to the path P_5 and the cycle C_6 for the distance parameter $k = 2$.

1. The Path Graph $G = P_5$:
Let P_5 be the path graph with vertex set $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(P_5) = \{v_i v_{i+1} \mid 1 \leq i \leq 4\}$. Let $k = 2$.

First, we investigate whether a single-element set $S = \{x\}$ can be a secure distance-2 dominating set of P_5. If $S = \{v_1\}$, then $d(v_1, v_4) = 3 > 2$ and $d(v_1, v_5) = 4 > 2$. By symmetry, selecting $S = \{v_5\}$ leaves v_1 and v_2 undominated. If $S = \{v_2\}$, then $d(v_2, v_5) = 3 > 2$. By symmetry, selecting $S = \{v_4\}$ leaves v_1 undominated. Finally, if $S = \{v_3\}$, then $d(v_3, v_1) = 2 \leq 2$ and $d(v_3, v_5) = 2 \leq 2$, so $\{v_3\}$ is a distance-2 dominating set. However, for $\{v_3\}$ to be secure, it must satisfy Definition 1 for all vertices in $V(P_5) \setminus \{v_3\}$. For $v_1 \in V(P_5) \setminus \{v_3\}$, the only possible swap vertex in S is v_3. The swapped set is $S_{v_1} = (\{v_3\} \setminus \{v_3\}) \cup \{v_1\} = \{v_1\}$. As shown previously, $\{v_1\}$ is not a distance-2 dominating set since it fails to dominate v_4 and v_5. Thus, no single vertex forms a secure distance-2 dominating set, which means $\gamma_2^s(P_5) \geq 2$.

Now, consider the set $S = \{v_2, v_4\}$. We verify that S is a secure distance-2 dominating set:

- **Distance-2 Domination Condition:** The vertices outside S are $V(P_5) \setminus S = \{v_1, v_3, v_5\}$. Since $d(v_1, v_2) = 1 \leq 2$, $d(v_3, v_2) = 1 \leq 2$, and $d(v_5, v_4) = 1 \leq 2$, every vertex in $V(P_5) \setminus S$ is within distance 2 of at least one vertex in S. Thus, S is a distance-2 dominating set.

- **Security Condition:** We evaluate the required swap for every vertex $v \in V(P_5) \setminus S$:
  - For v_1, we find $v_2 \in S \cap N_2(v_1)$ since $d(v_1, v_2) = 1 \leq 2$. The swapped set is $S_{v_1} = (S \setminus \{v_2\}) \cup \{v_1\} = \{v_1, v_4\}$. For the remaining vertices, $d(v_2, v_1) = 1$, $d(v_3, v_1) = 2$, and $d(v_5, v_4) = 1$. Since all vertices in P_5 are within distance 2 of S_{v_1}, the swapped set is a distance-2 dominating set.
  - For v_3, we can choose $v_2 \in S \cap N_2(v_3)$ since $d(v_3, v_2) = 1 \leq 2$. The swapped set is $S_{v_3} = (S \setminus \{v_2\}) \cup \{v_3\} = \{v_3, v_4\}$. The excluded vertices satisfy $d(v_1, v_3) = 2$, $d(v_2, v_3) = 1$, and $d(v_5, v_4) = 1$. Thus, S_{v_3} is a distance-2 dominating set.
  - For v_5, we find $v_4 \in S \cap N_2(v_5)$ since $d(v_5, v_4) = 1 \leq 2$. By symmetry with the first case, the swapped set is $S_{v_5} = (S \setminus \{v_4\}) \cup \{v_5\} = \{v_2, v_5\}$, which is a distance-2 dominating set.

Since all conditions of the definition are satisfied and $|S| = 2$, we have $\gamma_2^s(P_5) = 2$.

2. The Cycle Graph $G = C_6$:
Let C_6 be the cycle graph with vertex sequence $(v_1, v_2, v_3, v_4, v_5, v_6, v_1)$ and let $k = 2$.

First, we test if any single vertex set $S = \{x\}$ can be a secure distance-2 dominating set. For any chosen vertex, say v_1, its farthest vertex on the cycle is v_4 with $d(v_1, v_4) = 3 > 2$. Thus, no single vertex can even be a distance-2 dominating set of C_6. This means any valid secure distance-2 dominating set must contain at least two vertices, so $\gamma_2^s(C_6) \geq 2$.

Consider the set $S = \{v_1, v_4\}$. The remaining vertices are $V(C_6) \setminus S = \{v_2, v_3, v_5, v_6\}$.

- **Distance-2 Domination Condition:** The vertices v_2 and v_6 are at distance 1 from v_1, while v_3 and v_5 are at distance 1 from v_4. Since all distances to S are at most 2, S is a distance-2 dominating set.

- **Security Condition:** We check the swapping mechanics for each vertex outside S:
  - For v_2, we have $v_1 \in S \cap N_2(v_2)$ because $d(v_2, v_1) = 1 \leq 2$. The swapped set is $S_{v_2} = (S \setminus \{v_1\}) \cup \{v_2\} = \{v_2, v_4\}$. The vertices remaining outside S_{v_2} are $\{v_1, v_3, v_5, v_6\}$. Their distances to S_{v_2} are $d(v_1, v_2) = 1 \leq 2$, $d(v_3, v_4) = 1 \leq 2$, $d(v_5, v_4) = 1 \leq 2$, and $d(v_6, v_2) = 2 \leq 2$. Because every vertex is within distance 2 of S_{v_2}, the swapped set is a valid distance-2 dominating set.
  - By the structural symmetry of C_6, swapping any other non-dominating vertex with its closer dominating vertex results in an isomorphic configuration where every vertex remains within distance 2 of the swapped set.

Every vertex outside S successfully satisfies the security swap definition. Since $|S| = 2$, we conclude that $\gamma_2^s(C_6) = 2$.

Definition 2. [9] The join of two graph G_1 and G_2 , denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Definition 3. [4] Let G_1 and G_2 be two nontrivial connected graphs with orders m and n, respectively. The corona product $G_1 \circ G_2$ of G_1 and G_2 is obtained by taking one copy of G_1 and m copies of G_2 and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, where $1 \le i \le m$.

The following result establishes fundamental bounds relating the distance-k domination number, the secure distance-k domination number, and the classical domination number of a graph.

Theorem 1. *For any graph G of order n and integer $k \ge 2$,*

$$1 \le \gamma_k(G) \le \gamma_k^s(G) \le \gamma(G) \le n.$$

Proof. The inequality $1 \le \gamma_k(G)$ is immediate, since every distance-k dominating set is nonempty. Also, $\gamma_k(G) \le \gamma_k^s(G)$ follows directly from the fact that every secure distance-k dominating set is, by definition, a distance-k dominating set.

Next, we show that $\gamma_k^s(G) \le \gamma(G)$. Let $S \subseteq V(G)$ be a minimum dominating set of G. Since every vertex $v \in V(G) \setminus S$ is adjacent to some vertex in S, we have $d_G(v, u) = 1 \le k$ for some $u \in S$. Hence, S is a distance-k dominating set of G.

We now show that S is secure. Let $v \in V(G) \setminus S$. Since S is a dominating set, there exists a vertex $w \in S$ such that $d_G(v, w) = 1$. Clearly, $w \in S \cap N_k(v)$ because $1 \le k$. Define

$$S_v = (S \setminus \{w\}) \cup \{v\}.$$

We show that S_v is a distance-k dominating set of G.

Let $x \in V(G) \setminus S_v$. Since S is a dominating set of G, there exists a vertex $y \in S$ such that $d_G(x, y) = 1$. Consider the following cases:

Case 1: $y \ne w$.

In this case, $y \in S_v$, and thus $d_G(x, y) = 1 \le k$. Hence, x is distance-k dominated by a vertex in S_v.

Case 2: $y = w$.

Then $d_G(x, w) = 1$. Consider the subcases.

Subcase 2.1. If $v \in N(x)$. Then $d_G(x, v) = 1 \le k$.

Subcase 2.2. If $v \notin N(x)$. Then $d_G(x, v) \le d_G(x, w) + d_G(w, v) \le 1 + 1 = 2 \le k$.

In all subcases, this shows that $d_G(x, v) \le k$. Since $v \in S_v$, the vertex x is distance-k dominated by $v \in S_v$.

In either case, every vertex $x \in V(G) \setminus S_v$ is within distance at most k from some vertex in S_v. Therefore, S_v is a distance-k dominating set of G. Since $v \in V(G) \setminus S$ was arbitrary, S is a secure distance-k dominating set of G. Hence,

$$\gamma_k^s(G) \le \gamma(G).$$

Finally, since $V(G)$ is a dominating set of G, $\gamma(G) \leq n$. Therefore,

$$1 \leq \gamma_k(G) \leq \gamma_k^s(G) \leq \gamma(G) \leq n.$$

□

The following result shows that a single vertex is a secure distance-k dominating set if and only if the graph has diameter at most k.

Theorem 2. *Let G any connected graph and let $k \geq 2$ be an integer. Then*

$$\gamma_k^s(G) = 1 \quad \textit{if and only if} \quad diam(G) \leq k.$$

Proof. Let G be a connected graph and let $\gamma_k^s(G) = 1$. Then there exists a vertex $x \in V(G)$ such that $S = \{x\}$ is a secure distance-k dominating set. By the definition of security, for every $y \in V(G) \setminus \{x\}$, there must exist a vertex in S (which can only be x) within distance k of y such that $S_y = (S \setminus \{x\}) \cup \{y\} = \{y\}$ is a distance-k dominating set of G. Since $S_y = \{y\}$ is a distance-k dominating set for any chosen $y \in V(G) \setminus \{x\}$, it implies that every vertex $z \in V(G)$ satisfies $d_G(y, z) \leq k$. Thus, $ecc(y) \leq k$ for all $y \in V(G) \setminus \{x\}$. Additionally, since $S = \{x\}$ is a distance-k dominating set, we also have $ecc(x) \leq k$. Therefore, $ecc(v) \leq k$ for all $v \in V(G)$. Hence, $diam(G) = \max\{ecc(v) \mid v \in V(G)\} \leq k$.

Conversely, suppose that $diam(G) \leq k$. Let $x \in V(G)$ and set $S = \{x\}$. Since $ecc(x) \leq diam(G) \leq k$, we have $d_G(x, y) \leq k$ for every $y \in V(G) \setminus \{x\}$. Thus, S is a distance-k dominating set of G. We now show that S is secure. Let $y \in V(G) \setminus S$. Since x is the only vertex of S, the only possible defending move is to replace x by y. Define

$$S_y = (S \setminus \{x\}) \cup \{y\} = \{y\}.$$

We must show that S_y is a distance-k dominating set of G. Suppose $z \in V(G) \setminus S_y$. Since $diam(G)$ is the maximum distance between any two distinct vertices of G,

$$d_G(y, z) \leq diam(G) \leq k.$$

Therefore, for every $z \in V(G) \setminus S_y$ belongs to the closed k-neighborhood of y, i.e.,

$$V(G) \setminus S_y \subseteq N_k[y].$$

Since $y \in S_y$, it follows that

$$V(G) = N_k[S_y].$$

Hence, S_y is a distance-k dominating set of G and so S is secure distance-k dominating set of G. Hence, $\gamma_k^s(G) = 1$. □

The next result are the bounds number for path and cycle graph.

Theorem 3. *For $k \geq 2$, the secure distance-k dominating set of Path Graph P_n is*

$$\gamma_k^s(P_n) = \begin{cases} 1, & \textit{if} \quad 1 \leq n \leq k+1 \\ 2, & \textit{if} \quad k+2 \leq n \leq 2k+2 \\ 3, & \textit{if} \quad 2k+3 \leq n \leq 4k+3 \end{cases}$$

and

$$\gamma_k^s(P_n) \leq \left\lfloor \frac{n}{k+1} \right\rfloor, \qquad \textit{if} \quad n \geq 4k+4.$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ where $v_i v_{i+1} \in E(P_n)$ for all $1 \leq i \leq n-1$. The distance between v_i and v_j is given by $d(v_i, v_j) = |i-j|$. We analyze the problem by partitioning n into distinct cases based on the path length.

Case 1: $1 \leq n \leq k+1$

Note that the diameter of the path graph P_n is $diam(P_n) = n-1$. Since $n \leq k+1$, it follows that $diam(P_n) \leq k$. By Theorem 2, $\gamma_k^s(P_n) = 1$ if and only if $diam(P_n) \leq k$. Therefore, we immediately obtain $\gamma_k^s(P_n) = 1$.

Case 2: $k+2 \leq n \leq 2k+2$

Consider the set $S = \{v_1, v_n\}$. Note that $N_k[v_1] = \{v_i \mid 1 \leq i \leq k+1\}$ and $N_k[v_n] = \{v_i \mid n-k \leq i \leq n\}$. Since $n \leq 2k+2$, we have $(k+1) \geq n-(k+1)+1 = n-k$, which implies $N_k[v_1] \cup N_k[v_n] = V(P_n)$. Thus, S is a distance-k dominating set.

To prove S is secure, let $v_i \in V(P_n) \setminus S$:

- **Subcase 2.1:** If $1 < i \leq k+1$, then $d(v_i, v_1) = i-1 \leq k$, so $v_1 \in S \cap N_k[v_i]$. Let $S' = (S \setminus \{v_1\}) \cup \{v_i\} = \{v_i, v_n\}$. Since v_i dominates $\{v_1, \ldots, v_{i+k}\} \cap V(P_n)$ and v_n dominates $\{v_{n-k}, \ldots, v_n\}$, and because $n-k \leq k+2 \leq i+k$ holds for all $i > 1$, we have $N_k[S'] = V(P_n)$.

- **Subcase 2.2:** If $k+2 \leq i < n$, then $d(v_i, v_n) = n-i \leq k$, so $v_n \in S \cap N_k[v_i]$. Let $S' = (S \setminus \{v_n\}) \cup \{v_i\} = \{v_1, v_i\}$. By symmetric reasoning, v_1 dominates up to v_{k+1} and v_i dominates down to v_{i-k}. Since $i-k \leq (n-1)-k \leq k+1$, their neighborhoods overlap, ensuring $N_k[S'] = V(P_n)$.

Thus, S is a secure distance-k dominating set, showing $\gamma_k^s(P_n) \leq 2$.

To show minimality, note that for $n \geq k+2$, the diameter of the path graph is $diam(P_n) = n-1 \geq k+1 > k$. By Theorem 2, $\gamma_k^s(P_n) = 1$ if and only if $diam(P_n) \leq k$. Since $diam(P_n) > k$, it must hold that $\gamma_k^s(P_n) \neq 1$, which implies $\gamma_k^s(P_n) \geq 2$. Combining this with our upper bound, we conclude that $\gamma_k^s(P_n) = 2$.

Case 3: $2k+3 \leq n \leq 4k+3$

Consider the set $S = \{v_{k+1}, v_{\lceil \frac{n}{2} \rceil}, v_{n-k}\}$. By checking the intervals, $N_k[v_{k+1}] = \{v_1, \ldots, v_{2k+1}\}$, $N_k[v_{n-k}] = \{v_{n-2k}, \ldots, v_n\}$, and $N_k[v_{\lceil \frac{n}{2} \rceil}]$ covers the central region. Because $n \leq 4k+3$, these three neighborhoods successfully cover all vertices, so S is a distance-k dominating set.

To show S is secure, let $v_i \in V(P_n) \setminus S$:

- **Subcase 3.1:** If $1 \leq i \leq 2k+1$ and $i \neq k+1$, v_i is within distance k of v_{k+1}. We swap v_{k+1} for v_i. The new set $S' = \{v_i, v_{\lceil \frac{n}{2} \rceil}, v_{n-k}\}$ remains dominating because v_i still covers the left edge of the path up to at least v_{k+1}, and any coverage lost on the right side of v_{k+1}'s neighborhood is perfectly subsumed by $v_{\lceil \frac{n}{2} \rceil}$.

- **Subcase 3.2:** If v_i is closer to the center, it lies within $N_k[v_{\lceil \frac{n}{2} \rceil}]$. Swapping $v_{\lceil \frac{n}{2} \rceil}$ for v_i yields $S' = \{v_{k+1}, v_i, v_{n-k}\}$. Since v_{k+1} and v_{n-k} fixedly dominate the outer flanks $\{v_1, \ldots, v_{2k+1}\}$ and $\{v_{n-2k}, \ldots, v_n\}$, the shift of the center vertex to v_i maintains coverage over the middle gap because $d(v_{k+1}, v_{n-k}) = n - 2k - 1 \leq 2k + 2$, which means the gap between the fixed outer sets is at most $2k$, completely coverable by any single vertex placed between them.

- **Subcase 3.3:** If $n - 2k \leq i \leq n$ and $i \neq n - k$, v_i swaps with v_{n-k} symmetrically to Subcase 3.1.

Hence, S is secure, so $\gamma_k^s(P_n) \leq 3$. To prove minimality, suppose by contradiction that there exists a secure distance-k dominating set $S^* = \{v_a, v_b\}$ of size 2, where $1 \leq a < b \leq n$. To ensure v_1 is dominated by S^*, we must have $d(v_1, v_a) \leq k$, which implies $a \leq k + 1$. Similarly, to dominate v_n, we must have $d(v_b, v_n) \leq k$, which implies $b \geq n - k$. Now, consider the vertex v_1. If $v_1 \notin S^*$, then by the security of S^*, there must exist a vertex in $S^* \cap N_k[v_1]$ that can be replaced by v_1 such that the resulting set remains a distance-k dominating set. Since $b \geq n-k \geq (2k+3)-k = k+3$, we have $d(v_1, v_b) = b-1 \geq k+2 > k$. Thus, $v_b \notin N_k[v_1]$, meaning v_a is the unique vertex in S^* capable of swapping with v_1. Swapping v_a for v_1 yields the shifted dominating set $S_1 = \{v_1, v_b\}$. For S_1 to successfully dominate the entire path P_n, the neighborhoods $N_k[v_1]$ and $N_k[v_b]$ must overlap or be perfectly adjacent. The maximum index dominated by v_1 is $1 + k$, and the minimum index dominated by v_b is $b - k$. Thus, to avoid leaving an undominated gap between them, we must satisfy:

$$(1 + k) \geq (b - k) - 1 \implies b \leq 2k + 2.$$

By symmetric reasoning, considering the vertex $v_n \notin S^*$ and its mandatory swap with v_b to form the dominating set $S_n = \{v_a, v_n\}$, the condition that no undominated gap forms between v_a and v_n forces:

$$(a + k) \geq (n - k) - 1 \implies a \geq n - 2k - 1.$$

Combining our upper bound $a \leq k + 1$ with this new lower bound yields:

$$n - 2k - 1 \leq k + 1 \implies n \leq 3k + 2.$$

While this establishes a contradiction for $n \geq 3k + 3$, we must also ensure S^* is secure against internal shifts when $2k+3 \leq n \leq 3k+2$. Consider the vertex v_{k+2}. Since $a \leq k+1$, we have $d(v_a, v_{k+2}) \geq (k + 2) - (k + 1) = 1$. If $v_{k+2} \notin S^*$, any valid secure swap with v_{k+2} would shift a vertex away from defending an outer boundary. Specifically, if v_a swaps with v_{k+2} to form $S' = \{v_{k+2}, v_b\}$, the vertex v_1 is left completely undominated since

$d(v_1, v_{k+2}) = k+1 > k$ and $d(v_1, v_b) > k$. If v_b swaps with v_{k+2} to form $S'' = \{v_a, v_{k+2}\}$, the vertex v_n is left undominated because $b \geq n-k \geq k+3$, meaning

$$d(v_{k+2}, v_n) = n-k-2 \geq (2k+3)-k-2 = k+1 > k.$$

Thus, no secure distance-k dominating set of cardinality 2 can exist when $n \geq 2k+3$. We conclude that $\gamma_k^s(P_n) \geq 3$, which completes the proof that $\gamma_k^s(P_n) = 3$.

Case 4: $n \geq 4k+4$

Let $M = \left\lfloor \frac{n}{k+1} \right\rfloor$ and define $S = \{v_{m(k+1)} \mid 1 \leq m \leq M\}$. For each m, the neighborhood is $N_k[v_{m(k+1)}] = \{v_j \mid m(k+1)-k \leq j \leq m(k+1)+k\} \cap V(P_n)$.

Consecutive elements in S are separated by a distance of exactly $k+1$. Because their right and left coverage boundaries intersect via $m(k+1)+k \geq (m+1)(k+1)-k$, adjacent neighborhoods overlap continuously. Furthermore, the largest index in S is $M(k+1)$, which dominates up to $M(k+1)+k$. Since $n - M(k+1) \leq k$, the final block dominates up to v_n. Thus, $\bigcup_{v \in S} N_k[v] = V(P_n)$, confirming S is a distance-k dominating set.

To show security, let $v_i \in V(P_n) \setminus S$. We consider two possibilities for its position:

- **Internal interval:** If $m(k+1) < i < (m+1)(k+1)$ for some $1 \leq m < M$, then v_i is within distance k of both bounding vertices. Swap $v_{m(k+1)}$ for v_i to form $S' = (S \setminus \{v_{m(k+1)}\}) \cup \{v_i\}$. The shifted vertex v_i dominates down to $i-k$. Since $i \leq (m+1)(k+1)-1$, we have $i-k \leq m(k+1)+1$, ensuring no gap opens with the left neighbor $v_{(m-1)(k+1)}$ (which dominates up to $m(k+1)-1$). Symmetrically, v_i dominates up to $i+k \geq m(k+1)+1+k = (m+1)(k+1)$, maintaining continuous overlap with the right neighbor $v_{(m+1)(k+1)}$.
- **End interval:** If $M(k+1) < i \leq n$, then $d(v_i, v_{M(k+1)}) = i - M(k+1) \leq k$. Swapping $v_{M(k+1)}$ for v_i yields $S' = (S \setminus \{v_{M(k+1)}\}) \cup \{v_i\}$. The new vertex v_i dominates down to $i-k \leq n-k \leq M(k+1)$, preserving the boundary overlap with $v_{(M-1)(k+1)}$ which covers up to $M(k+1)-1$.

In all scenarios, S' remains a distance-k dominating set. Thus, S is secure, yielding the upper bound:

$$\gamma_k^s(P_n) \leq \left\lfloor \frac{n}{k+1} \right\rfloor.$$

□

Theorem 4. *For $n \geq 3$, the secure distance-k domination number of Cycle Graph C_n is*

$$\gamma_k^s(C_n) = \begin{cases} 1, & \text{if} \quad 3 \leq n \leq 2k+1 \\ 2, & \text{if} \quad 2k+2 \leq n \leq \lfloor \frac{5k}{2} \rfloor + 2 \\ 3, & \text{if} \quad \lfloor \frac{5k}{2} \rfloor + 3 \leq n \leq 4k+3 \end{cases}$$

and

$$\gamma_k^s(C_n) \leq \left\lfloor \frac{n}{k+1} \right\rfloor, \qquad \text{if} \quad n \geq 4k+4.$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ with edges v_iv_{i+1} for $1 \le i \le n-1$ and v_nv_1. The distance between two vertices is defined cyclically as $d(v_i, v_j) = \min(|i-j|, n-|i-j|)$.

Case 1: $3 \le n \le 2k+1$

The diameter of a cycle graph is $diam(C_n) = \lfloor \frac{n}{2} \rfloor$. For $n \le 2k+1$, we have $diam(C_n) \le \lfloor \frac{2k+1}{2} \rfloor = k$. By Theorem 2, $\gamma_k^s(G) = 1$ if and only if $diam(G) \le k$. Therefore, $\gamma_k^s(C_n) = 1$.

Case 2: $2k+2 \le n \le \lfloor \frac{5k}{2} \rfloor + 2$

Consider the set $S = \{v_1, v_{k+2}\}$. The distance between these two vertices along the shorter path is $k+1$. The closed neighborhood $N_k[v_1]$ covers $2k+1$ vertices. The remaining gap on the cycle consists of at most $n-(2k+1)$ vertices. Since $n \le \lfloor \frac{5k}{2} \rfloor + 2 \le 2k+1+k$, the remaining gap is easily covered by $N_k[v_{k+2}]$. Thus, S is a distance-k dominating set.

To show S is secure, let $v_i \in V(C_n) \setminus S$.

- If $2 \le i \le k+1$, then $d(v_i, v_1) \le k$, meaning $v_1 \in S \cap N_k[v_i]$. Swapping v_1 for v_i yields $S' = \{v_i, v_{k+2}\}$. The distance between these two elements satisfies $d(v_i, v_{k+2}) = (k+2) - i \le k$. On a cycle, two vertices whose distance is at most k can completely dominate up to $2k+2+k = 3k+2$ vertices, which safely exceeds n.

- If $k+3 \le i \le n$, v_i swaps with v_{k+2} or v_1 depending on proximity. By cyclic symmetry, the distance between the shifted element and the remaining element never exceeds $k+1$, keeping the maximum distance between adjacent code-defenders within the required threshold to maintain global coverage.

Thus, $\gamma_k^s(C_n) \le 2$. Since $n \ge 2k+2$, $diam(C_n) \ge k+1 > k$. Applying Theorem 2, $\gamma_k^s(C_n) \ne 1$, forcing $\gamma_k^s(C_n) = 2$.

Case 3: $\lfloor \frac{5k}{2} \rfloor + 3 \le n \le 4k+3$

Consider the configuration $S = \{v_1, v_{k+2}, v_{2k+3}\}$. Each consecutive pair of vertices in S is separated by a distance of exactly $k+1$. * $N_k[v_1]$ covers indices from $n-k+1$ up to $k+1$. * $N_k[v_{k+2}]$ covers indices from 2 up to $2k+2$. * $N_k[v_{2k+3}]$ covers indices from $k+3$ up to $3k+3$.

Together, their neighborhoods cover up to index $3k+3$ from the left side, and wrap around from the right edge down to $n-k+1$. The remaining unaddressed gap has a length of $n-(3k+3)$. Since $n \le 4k+3$, the gap size is at most $(4k+3)-(3k+3) = k$. This remaining gap falls fully within the wrap-around reach of $N_k[v_1]$, so S is a distance-k dominating set.

To establish security, notice that any vertex $v_i \notin S$ is at most a distance k away from its nearest element in S. Swapping $v_j \in S$ for v_i shifts one boundary wall by at most k units. Because the maximum span between any two elements in S is bounded by $k+1$, shifting one element closer to another increases the opposite gap to at most $2k+1$. Any gap of size $2k+1$ or less on a cycle remains completely dominated by the outer flanks of the remaining unchanged elements. Thus, S is secure, yielding $\gamma_k^s(C_n) \le 3$.

To prove minimality, suppose there exists a secure distance-k dominating set $S^* = \{v_a, v_b\}$ of size 2. To avoid leaving any element completely undominated, the two vertices must split the cycle such that neither of the two clear paths between them exceeds a distance of $2k+1$. However, if an internal vertex changes places with one of these components during a security shift, the remaining element is left to defend a path segment larger than $2k+1$ alone. This creates an immediate coverage collapse because a single vertex can only dominate a span of $2k+1$ vertices. Thus, no secure set of size 2 can exist for $n \geq \lfloor \frac{5k}{2} \rfloor + 3$, confirming $\gamma_k^s(C_n) = 3$.

Case 4: $n \geq 4k+4$

Let $M = \left\lfloor \frac{n}{k+1} \right\rfloor$ and define $S = \{v_{m(k+1)} \mid 1 \leq m \leq M\}$. For any vertex $v_j \in V(C_n)$, indices are evaluated modulo n. The neighborhood of each dominating element is $N_k[v_{m(k+1)}] = \{v_j \mid |j - m(k+1)| \pmod{n} \leq k\}$.

For any consecutive pair $v_{m(k+1)}, v_{(m+1)(k+1)} \in S$ (where $1 \leq m < M$), their distance along the cycle is exactly $k+1$. Their respective coverage blocks overlap continuously because $m(k+1) + k \geq (m+1)(k+1) - k$. For the wrap-around interval between the final element $v_{M(k+1)}$ and the first element v_{k+1}, the cyclic distance between them is given by $d(v_{M(k+1)}, v_{k+1}) = n - M(k+1) + (k+1)$. Since $n - M(k+1) \leq k$, we have $d(v_{M(k+1)}, v_{k+1}) \leq 2k+1$. A cyclic gap of length at most $2k+1$ between two dominating elements is completely covered by their combined neighborhoods, ensuring $\bigcup_{v \in S} N_k[v] = V(C_n)$. Hence, S is a distance-k dominating set.

To establish security, let $v_i \in V(C_n) \setminus S$. Due to the cyclic partition, v_i must lie within some interval $m(k+1) < i < (m+1)(k+1)$ (interpreting $M+1 \equiv 1$). Since the distance between these bounding vertices is at most $2k+1$, v_i is at a distance of at most k from at least one of them. We swap that closer vertex, say $v_{m(k+1)}$, for v_i to form $S' = (S \setminus \{v_{m(k+1)}\}) \cup \{v_i\}$.

Because $d(v_{m(k+1)}, v_i) \leq k$, replacing $v_{m(k+1)}$ with v_i shifts the coverage boundary by at most k units. In a cycle where the distance between adjacent dominating vertices is at most $k+1$, their respective closed neighborhoods share an overlap of at least k vertices. The substitution of $v_{m(k+1)}$ with v_i preserves domination over $V(C_n)$ if the adjacent neighborhood intersections remain nonempty; that is, we must verify that $N_k[v_i] \cap N_k[v_{(m-1)(k+1)}] \neq \emptyset$ and $N_k[v_i] \cap N_k[v_{(m+1)(k+1)}] \neq \emptyset$.

Since $m(k+1) < i < (m+1)(k+1)$, it follows that $m(k+1)+1 \leq i \leq (m+1)(k+1)-1$. To verify the clockwise intersection, notice that the maximum index dominated by v_i is $i+k$. Substituting the lower bound for i yields:

$$i + k \geq m(k+1) + 1 + k = (m+1)(k+1)$$

Since $v_{(m+1)(k+1)}$ dominates down to $(m+1)(k+1) - k$, their neighborhoods overlap, ensuring continuous coverage along the clockwise path segment.

Symmetrically, for the counterclockwise intersection, the minimum index dominated by v_i is $i - k$. Substituting the upper bound for i yields:

$$i - k \leq (m+1)(k+1) - 1 - k = m(k+1) + k$$

Since $v_{(m-1)(k+1)}$ dominates up to $(m-1)(k+1) + k = m(k+1) - 1$, and the vertex $v_{m(k+1)}$ (which is the upper bound of the gap) is within distance k of v_i, no vertex between $v_{(m-1)(k+1)}$ and v_i is left undominated.

Thus, $N_k[S'] = V(C_n)$, proving that S' is a distance-k dominating set, confirming that S is secure and establishing the upper bound:

$$\gamma_k^s(C_n) \leq \left\lfloor \frac{n}{k+1} \right\rfloor.$$

□

The following result establishes that the parameter $\gamma_k^s(G)$ is additive, being equal to the sum of the secure distance-k domination numbers of its components.

Theorem 5. *Let $G_1, G_2, \ldots, G_m$ be the components of the graph G and let $k \geq 2$ be an integer. Then S is a secure distance-k dominating set in G if and only if $S_i = S \cap V(G_i)$ is a secure distance-k dominating set in G_i for every $i = 1, 2 \ldots, m$. Moreover,*

$$\gamma_k^s(G) = \sum_{i=1}^{m} \gamma_k^s(G_i).$$

Proof. Let S be a secure distance-k dominating set in G. Let $S_i = S \cap V(G_i)$ for every $i = 1, 2 \ldots, m$. Then $S = \bigcup_{i=1}^m S_i$.

Choose i arbitrarily. Let $x \in V(G_i) \setminus S_i \subseteq V(G)$. Then $x \in V(G) \setminus S$. This implies that there exists $y \in S$ such that $d_G(x, y) \leq k$. So y must be a vertex in G_i, otherwise, $d_G(x, y) = \infty$. Hence, $y \in S_i$ and $d_{G_i}(x, y) \leq k$. That is, $y \in N_k(x) \cap S$. This shows that S_i is distance-k dominating set in G_i. Since S is a secure distance-k dominating set in G, $T = (S \setminus \{y\}) \cup \{x\}$ is a distance-k dominating set of G. Let $z \in V(G_i) \setminus T_i$, where $T_i = (S_i \setminus \{y\}) \cup \{x\}$. Then, $z \in V(G) \setminus T$. This implies that there exists $r \in N_k(z) \cap T$ such that $d_G(r, z) \leq k$. Note that $r \in V(G_i)$, otherwise, $d_G(r, z) = \infty$. Now,

$$T = (S \setminus \{y\}) \cup \{x\} = \left(\bigcup_{\substack{1 \leq j \leq m \\ j \neq i}} S_j \right) \cup T_i.$$

Since $r \in T \cap V(G_i)$, $r \in T_i$. This shows that there exists $r \in T_i$ such that $d_{G_i}(r, z) \leq k$. This implies that S_i is a secure distance-k dominating set in G_i. Since i is arbitrary, S_i is a secure distance-k dominating set in G_i for every $i = 1, 2, \ldots, m$.

Conversely, suppose that $S_i = S \cap V(G_i)$ is a secure distance-k dominating set in G_i for every $i \in \{1, 2, \ldots, m\}$. Then $S = \bigcup_{i=1}^m S_i$.

Let $w \in V(G) \setminus S$. Then $w \in V(G_i) \setminus S_i$ for some $i \in \{1, 2, \ldots, m\}$. Since S_i is a distance-k dominating set in G_i, there exists $r \in S_i$ such that $d_{G_i}(w, r) \leq k$. Because G_i is a subgraph of G, $d_G(w, r) \leq d_{G_i}(w, r) \leq k$ where $r \in S$. Hence, S is a distance-k dominating set in G.

We now show that S is secure. Since S_i is a secure distance-k dominating set in G_i, there exists $z \in S_i \cap N_k(w)$ such that $T_i = (S_i \setminus \{z\}) \cup \{w\}$ is a distance-k dominating set in G_i. Define $T = (S \setminus \{z\}) \cup \{w\}$. To prove that T is a distance-k dominating set in G, let $r \in V(G) \setminus T$. We consider two cases based on the location of r:

Case 1: $r \in V(G_i) \setminus T_i$.

Since T_i is a distance-k dominating set in G_i, there exists $s \in T_i \subseteq T$ such that $d_G(r, s) \leq d_{G_i}(r, s) \leq k$.

Case 2: $r \in V(G_j)$ for some $j \neq i$.

In this case, $r \notin T_i$. Since $r \in V(G_j) \setminus S_j$ and S_j is a distance-k dominating set in G_j, there exists $s \in S_j$ such that $d_{G_j}(r, s) \leq k$. Since $j \neq i$, $S_j \subseteq T$. Thus, $s \in T$ and $d_G(r, s) \leq d_{G_j}(r, s) \leq k$.

In either case, r is distance-k dominated by a vertex in T. Thus, T is a distance-k dominating set in G, which implies S is a secure distance-k dominating set in G.

Finally, we establish the equality of the domination numbers. Let S be a $\gamma_k^s(G)$-set. Then $S_i = S \cap V(G_i)$ is a secure distance-k dominating set in G_i for all $i = 1, 2, \ldots, m$. This implies that

$$\gamma_k^s(G) = |S| = \sum_{i=1}^{m} |S_i| \geq \sum_{i=1}^{m} \gamma_k^s(G_i). \tag{1}$$

Let Q_i be a $\gamma_k^s(G_i)$-set for every $i = 1, 2, \ldots, m$. Then $R = \bigcup_{i=1}^{m} Q_i$ is a secure distance-k dominating set in G. Thus,

$$\gamma_k^s(G) \leq |R| = \sum_{i=1}^{m} |Q_i| = \sum_{i=1}^{m} \gamma_k^s(G_i). \tag{2}$$

Combining equations (1) and (2), we obtain

$$\gamma_k^s(G) = \sum_{i=1}^{m} \gamma_k^s(G_i).$$

□

The next result characterizes graphs whose secure distance-k domination number attains its maximum value. It shows that this occurs precisely when the graph is edgeless, where every vertex must be included in any secure distance-k dominating set.

Theorem 6. *Let G be any graph of order n and let $k \geq 2$ be an integer. Then $\gamma_k^s(G) = n$ if and only if $G \cong \overline{K}_n$*

Proof. Let $\gamma_k^s(G) = n$ and suppose, on the contrary, that $G \ncong \overline{K}_n$. Then G has at least one edge, meaning there exist two distinct vertices $x, y \in V(G)$ such that $xy \in E(G)$, which implies $d_G(x, y) = 1 \leq k$.

Consider the set $S = V(G) \setminus \{y\}$. We will show that S is a secure distance-k dominating set of G. First, let $z \in V(G) \setminus S$, which means $z = y$. Since $x \in S$ and $d_G(y, x) = 1 \leq k$, the vertex y is distance-k dominated by S. Since all other vertices of G are already contained within S, they trivially dominate themselves. Thus, S is a distance-k dominating set of G.

Next, we verify that S is secure. The only vertex in $V(G) \setminus S$ is y. Since $d_G(y,x) = 1 \leq k$, we have $x \in S \cap N_k(y)$. Define the swapped set by replacing x with y:

$$S_y = (S \setminus \{x\}) \cup \{y\} = (V(G) \setminus \{x,y\}) \cup \{y\} = V(G) \setminus \{x\}.$$

By a symmetric argument to the one used for S, since $y \in S_y$ and $d_G(x,y) = 1 \leq k$, the vertex x is distance-k dominated by S_y, and all other vertices dominate themselves. Hence, $S_y = V(G) \setminus \{x\}$ is a valid distance-k dominating set of G.

This confirms that S is a secure distance-k dominating set of G. Since $|S| = n-1$, it follows that $\gamma_k^s(G) \leq n-1$, which directly contradicts the assumption that $\gamma_k^s(G) = n$.

Conversely, suppose $G \cong \overline{K}_n$. Then G can be expressed as the disjoint union of isolated vertices, $G = \bigcup_{i=1}^{n} K_1$. Applying Theorem 5, we obtain:

$$\gamma_k^s(G) = \sum_{i=1}^{n} \gamma_k^s(K_1) = \sum_{i=1}^{n} 1 = n.$$

□

The following result provides a necessary and sufficient condition for a distance-k dominating set to be secure.

Theorem 7. *Let G be a connected graph and let $k \geq 2$ be an integer. Let $S \subseteq V(G)$ be a distance-k dominating set. Then S is a secure distance-k dominating set if and only if for every $v \in V(G) \setminus S$, there exists a vertex $w \in S \cap N_k(v)$ such that*

$$N_k[w] \subseteq N_k[v] \cup N_k[S \setminus \{w\}].$$

Proof. Assume that S is a secure distance-k dominating set of G. Let $v \in V(G) \setminus S$. By definition, there exists a vertex $w \in S \cap N_k(v)$ such that the set $S' = (S \setminus \{w\}) \cup \{v\}$ is a distance-k dominating set of G. Let $x \in N_k[w]$, which implies $d_G(x,w) \leq k$. Since S' is a distance-k dominating set of G, there must exist some vertex $y \in S'$ such that $d_G(x,y) \leq k$. We analyze the position of y within S':

- If $y = v$, then $d_G(x,v) \leq k$, meaning $x \in N_k[v]$.
- If $y \in S \setminus \{w\}$, then x is within distance k of a vertex in $S \setminus \{w\}$, meaning $x \in N_k[S \setminus \{w\}]$.

In either case, $x \in N_k[v] \cup N_k[S \setminus \{w\}]$. Since x was chosen arbitrarily from $N_k[w]$, we conclude that

$$N_k[w] \subseteq N_k[v] \cup N_k[S \setminus \{w\}].$$

Conversely, suppose that for every $v \in V(G) \setminus S$, there exists a vertex $w \in S \cap N_k(v)$ such that $N_k[w] \subseteq N_k[v] \cup N_k[S \setminus \{w\}]$.

Fix an arbitrary vertex $v \in V(G) \setminus S$, choose its corresponding vertex $w \in S \cap N_k(v)$, and define the swapped set $S' = (S \setminus \{w\}) \cup \{v\}$. To show that S is secure, we must verify that S' is a distance-k dominating set of G. Let $x \in V(G) \setminus S'$. Since the original set S is a distance-k dominating set of G, there exists some vertex $z \in S$ such that $d_G(x,z) \leq k$. We consider two cases based on z:

Case 1: $z \neq w$.

In this case, $z \in S \setminus \{w\} \subseteq S'$. Since $d_G(x,z) \leq k$, the vertex x is distance-k dominated by $z \in S'$.

Case 2: $z = w$.

In this case, $d_G(x,w) \leq k$, which means $x \in N_k[w]$. Since $N_k[w] \subseteq N_k[v] \cup N_k[S \setminus \{w\}]$, it follows that $x \in N_k[v] \cup N_k[S \setminus \{w\}]$. This yields two possibilities: either $d_G(x,v) \leq k$ (where $v \in S'$), or there exists some vertex $u \in S \setminus \{w\} \subseteq S'$ such that $d_G(x,u) \leq k$. In both situations, x is safely distance-k dominated by a vertex belonging to S'.

Thus, every vertex $x \in V(G) \setminus S'$ is distance-k dominated by S', making S' a valid distance-k dominating set. Consequently, S is a secure distance-k dominating set of G. □

Corollary 1. *Let G be a connected graph and let $k \geq 2$ be an integer. Then $\gamma_k(G) = \gamma_k^s(G)$ if and only if there exists a γ_k-set S such that for every $v \in V(G) \setminus S$, there exists $w \in S \cap N_k(v)$ satisfying*

$$N_k[w] \subseteq N_k[v] \cup N_k[S \setminus \{w\}].$$

Proof. Suppose $\gamma_k(G) = \gamma_k^s(G)$. Then there exists a minimum secure distance-k dominating set S with $|S| = \gamma_k(G)$. Since S is secure distance-k dominating set, it follows that S is a distance-k dominating set of G. Note that $|S| = \gamma_k(G)$. Then S is a $\gamma_k(G)$-set of G. Thus, S is both a γ_k-set and a secure distance-k dominating set. The result follows from Theorem 7.

Conversely, suppose there exists a γ_k-set S satisfying the stated condition. Then by Theorem 7, S is secure. Hence

$$\gamma_k^s(G) \leq |S| = \gamma_k(G).$$

Since always $\gamma_k(G) \leq \gamma_k^s(G)$, equality follows. □

The following result provides a sufficient condition for a secure distance-k dominating set to be a dominating set.

Theorem 8. *Let G be a connected graph of order n, and let $k \geq 1$ be an integer. Let $S \subseteq V(G)$ be a secure distance-k dominating set of G. Then S is a dominating set of G if and only if for every vertex $u \in V(G) \setminus S$,*

$$N_k[w] \subseteq N_1[v] \cup N_k[S \setminus \{w\}]$$

for some vertex $w \in S \cap N_1(u)$ and some vertex $v \in V(G) \setminus S$.

Proof. Assume that S is a dominating set of G. By definition, every vertex $u \in V(G) \setminus S$ is adjacent to at least one vertex in S. Let $w \in S$ be a neighbor of u, which implies $d_G(u,w) = 1$. Since $1 \leq k$, we trivially have $w \in S \cap N_1(u) \subseteq S \cap N_k(u)$.

Because S is also given as a secure distance-k dominating set, there exists a vertex $v \in V(G) \setminus S$ such that when w and v switch places, the new set $S' = (S \setminus \{w\}) \cup \{v\}$ remains a valid distance-k dominating set of G.

Now, let $x \in N_k[w]$, meaning $d_G(x, w) \leq k$. Since S' is a distance-k dominating set, x must be distance-k dominated by some element $y \in S'$. We look at the two possibilities for y:

(i) If $y = v$, then $d_G(x, v) \leq k$. Since S is a dominating set, v is structurally forced to dominate its immediate neighborhood at distance 1, implying that $x \in N_1[v]$.

(ii) If $y \in S \setminus \{w\}$, then $x \in N_k[S \setminus \{w\}]$.

Combining these cases, we get $x \in N_1[v] \cup N_k[S \setminus \{w\}]$. Since x was chosen arbitrarily from $N_k[w]$, we have

$$N_k[w] \subseteq N_1[v] \cup N_k[S \setminus \{w\}].$$

Conversely, suppose that for every $u \in V(G) \setminus S$, there exists a vertex $w \in S \cap N_1(u)$ satisfying the neighborhood containment condition. By definition, if $w \in S \cap N_1(u)$, then w is a vertex in S whose distance to u is exactly 1 ($d_G(u, w) = 1$). This means that u is directly adjacent to $w \in S$. Since this condition holds for every choice of $u \in V(G) \setminus S$, it implies that every single vertex outside of S has at least one neighbor inside S. Therefore, S is a dominating set of G. □

Corollary 2. *Let G be a connected graph of order n, and let $k \geq 1$ be an integer. If for every minimum secure distance-k dominating set (γ_k^s-set) S of G and every vertex $u \in V(G) \setminus S$, there exists a vertex $w \in S \cap N_1(u)$ such that*

$$N_k[w] \subseteq N_1[v] \cup N_k[S \setminus \{w\}]$$

for some $v \in V(G) \setminus S$, then

$$\gamma_k^s(G) = \gamma(G).$$

Proof. Let S be a γ_k^s-set of G, which implies that S is a secure distance-k dominating set and $|S| = \gamma_k^s(G)$. By Theorem 8, the given neighborhood condition implies that S is structurally forced to be a dominating set of G. Thus,

$$\gamma(G) \leq |S| = \gamma_k^s(G).$$

To establish equality, we must show the reverse inequality, $\gamma_k^s(G) \leq \gamma(G)$, holds for all $k \geq 1$. We consider two cases based on the value of k:

Case 1: $k \geq 2$.
By Theorem 1, the inequality $\gamma_k^s(G) \leq \gamma(G)$ holds for any graph when $k \geq 2$. Combining this with our previous inequality yields $\gamma_k^s(G) = \gamma(G)$.

Case 2: $k = 1$.
We want to show that $\gamma_1^s(G) \leq \gamma(G)$. Let D be a minimum dominating set of G, meaning

$|D| = \gamma(G)$. By definition, every vertex $u \in V(G) \setminus D$ satisfies $d_G(u, D) = 1$, making D a distance-1 dominating set.

Now, we show that D is secure under $k = 1$. Let $u \in V(G) \setminus D$. By the given hypothesis for $k = 1$, there exists a vertex $w \in D \cap N_1(u)$ such that:

$$N_1[w] \subseteq N_1[v] \cup N_1[D \setminus \{w\}]$$

for some $v \in V(G) \setminus D$. Let $D' = (D \setminus \{w\}) \cup \{u\}$. To prove D is secure, we must verify that D' is a distance-1 dominating set. Let $x \in V(G) \setminus D'$. Since D is a dominating set, there exists $z \in D$ such that $d_G(x, z) \leq 1$, meaning $x \in N_1[z]$.

- If $z \neq w$, then $z \in D \setminus \{w\} \subseteq D'$. Thus, x is distance-1 dominated by D'.
- If $z = w$, then $x \in N_1[w]$. By setting the vertex $v = u$ in our hypothesis containment, we obtain $x \in N_1[u] \cup N_1[D \setminus \{w\}]$. If $x \in N_1[D \setminus \{w\}]$, it is dominated by D'. If $x \in N_1[u]$, it is directly distance-1 dominated by $u \in D'$.

Thus, D' is a distance-1 dominating set, which proves D is a secure distance-1 dominating set. Since $\gamma_1^s(G)$ is the minimum size of all such sets, we have:

$$\gamma_1^s(G) \leq |D| = \gamma(G).$$

In both cases, we have established that $\gamma_k^s(G) \leq \gamma(G)$. Combining this with the lower bound $\gamma(G) \leq \gamma_k^s(G)$, we conclude by double inequality that $\gamma_k^s(G) = \gamma(G)$ for all $k \geq 1$. □

Example 2 below illustrates the results above.

Example 2. Consider the graph G shown in Figure 1. We will show that the secure distance-2 domination number is equal to the classical domination number, specifically $\gamma_2^s(G) = \gamma(G) = 3$. First, we determine the classical domination number $\gamma(G)$. Let $S = \{v_2, v_6, v_8\}$. We observe the following standard open neighborhoods for these vertices:

- $N[v_2] = \{v_1, v_2, v_3, v_4, v_5\}$
- $N[v_6] = \{v_5, v_6, v_7\}$
- $N[v_8] = \{v_1, v_8, v_9, v_{10}\}$

Taking the union, we see that $N[v_2] \cup N[v_6] \cup N[v_8] = V(G)$. Thus, S is a standard distance-1 dominating set of G, which implies that $\gamma(G) \leq |S| = 3$.

To see that $\gamma(G)$ cannot be smaller than 3, suppose there exists a dominating set S' with $|S'| \leq 2$.

- To dominate the vertices v_9 and v_{10}, S' must contain at least one vertex from $\{v_8, v_9, v_{10}\}$.
- To dominate the vertex v_7, S' must contain at least one vertex from $\{v_5, v_6, v_7\}$.

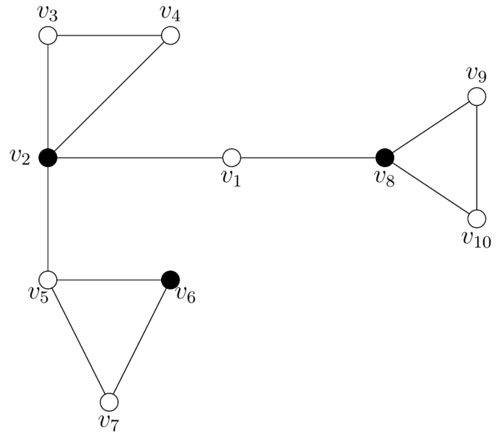

Figure 1: Graph G with $\gamma_2^s(G) = 3$.

Since these two sets of vertices are completely disjoint and share no common neighbors, any dominating set must pick at least one vertex from each group, exhausting the 2 vertices. However, any such pair (for example, $S' = \{v_6, v_8\}$) completely fails to dominate the upper vertices like v_3 and v_4, because $d(v_3, \{v_6, v_8\}) = 3 > 1$. Thus, no dominating set of size 2 exists, establishing that $\gamma(G) = 3$.

Next, we verify that $S = \{v_2, v_6, v_8\}$ is a secure distance-2 dominating set of G. S is a distance-2 dominating set since it is already a distance-1 dominating set. To show that S is *secure*, let $u \in V(G) \setminus S$. We must find a vertex $w \in S \cap N_2(u)$ such that the set $S_u = (S \setminus \{w\}) \cup \{u\}$ remains a distance-2 dominating set.

- If $u = v_3$, we choose $w = v_2 \in S \cap N_2(v_3)$. The set is $S_{v_3} = \{v_3, v_6, v_8\}$. For every vertex $u' \in V(G) \setminus S_{v_3}$, $d(u', S_{v_3}) \leq 2$. Thus, S_{v_3} is a distance-2 dominating set.
- If $u = v_1$, we can choose $w = v_8 \in S \cap N_2(v_1)$. The set is $S_{v_1} = \{v_2, v_6, v_1\}$. It is easy to verify that $d(v_9, S_{v_1}) = 2$ and $d(v_{10}, S_{v_1}) = 2$, keeping all vertices within distance 2 of S_{v_1}.

A similar valid swap can be established for all other vertices in $V(G) \setminus S$. Hence, S is a secure distance-2 dominating set, which implies $\gamma_2^s(G) \leq 3$.

Finally, because any secure distance-2 dominating set must fundamentally be a distance-2 dominating set, and since $\mathrm{diam}(G) = 5 > 4$, a single vertex can cover a maximum distance path of 4. Thus, a single vertex cannot dominate the graph at distance 2, meaning $\gamma_2^s(G) \geq 2$.

Furthermore, if we test any potential secure distance-2 dominating set of size 2, such as $S' = \{v_2, v_8\}$, it fails security. For instance, if $u = v_4 \in V(G) \setminus S'$, the only available vertex is v_2. Swapping them gives $S'_{v_4} = \{v_4, v_8\}$, which leaves v_7 stranded since $d(v_7, S'_{v_4}) = 3 > 2$. Symmetrical checks rule out all other pairs. Thus, $\gamma_2^s(G) = 3$. Consequently, we have shown that $\gamma_2^s(G) = \gamma(G) = 3$.

The succeeding results are the secure distance-k domination number of the join and corona of graphs and special graphs.

Theorem 9. *Let G and H be any two connected graphs and let $k \geq 2$ be an integer.. Then*

$$\gamma_k^s(G + H) = 1.$$

Proof. Since $diam(G + H) \leq 2$, by Theorem 2, $\gamma_k^s(G + H) = 1$. □

Corollary 3. *Let $k \geq 2$ be an integer. Then,*

(i) for the Complete Graph K_n, $\gamma_k^s(K_n) = 1$,

(ii) for the Fan Graph F_n, $\gamma_k^s(F_n) = 1$,

(iii) for the Wheel Graph W_n, $\gamma_k^s(W_n) = 1$,

(iv) for the Star Graph S_n, $\gamma_k^s(S_n) = 1$, and

(v) for the Complete Bipartite Graph $K_{m,n}$, $\gamma_k^s(K_{m,n}) = 1$.

Theorem 10. *Let G be a nontrivial complete graph of order n and H be any graph of order m, and let $k \geq 3$. Then $\gamma_k^s(G \circ H) = 1$.*

Proof. We first determine the diameter of the corona product $G \circ H$. By definition, $G \circ H$ is formed by taking one copy of G and n copies of H (denoted $H_1, H_2, \ldots, H_n$), then joining each vertex $v_i \in V(G)$ to every vertex in the corresponding copy H_i.

Let $x, y \in V(G \circ H)$ be any two distinct vertices. We analyze the distance $d_{G \circ H}(x, y)$ by considering all possible configurations of these vertices:

Case 1: $x, y \in V(G)$.
Since G is a complete graph ($G \cong K_n$) and nontrivial ($n \geq 2$), any two distinct vertices in G are directly adjacent. Thus, $d_{G \circ H}(x, y) = 1$.

Case 2: $x \in V(G)$ and $y \in V(H_i)$ for some $i \in \{1, 2, \ldots, n\}$.
Let $x = v_j \in V(G)$.

- If $j = i$, then v_i is adjacent to every vertex in H_i by construction, so $d_{G \circ H}(v_i, y) = 1$.
- If $j \neq i$, then v_j is adjacent to v_i in G (since G is complete), and v_i is adjacent to y in H_i. This forms a path $v_j \to v_i \to y$ of length 2. Because v_j and y share no direct edge, $d_{G \circ H}(v_j, y) = 2$.

Case 3: $x, y \in V(H_i)$ for the same copy H_i.
Both x and y are adjacent to the same base vertex $v_i \in V(G)$. This creates a path $x \to v_i \to y$ of length 2. Thus, $d_{G \circ H}(x, y) \leq 2$.

Case 4: $x \in V(H_i)$ and $y \in V(H_j)$ where $i \neq j$.
The vertex x is adjacent to $v_i \in V(G)$, and y is adjacent to $v_j \in V(G)$. Since G is a complete graph, v_i and v_j are directly adjacent. This forms a shortest path $x \to v_i \to v_j \to y$ of length 3. Because there are no edges connecting different copies of H, and no

edges connecting a vertex in H_i to a vertex in G other than v_i, no shorter path can exist. Thus, $d_{G \circ H}(x, y) = 3$. Taking the maximum distance over all possible pairs of vertices, we obtain:

$$\mathrm{diam}(G \circ H) = \max_{x,y \in V(G \circ H)} d_{G \circ H}(x, y) = 3.$$

Since we are given that $k \geq 3$, it follows that $\mathrm{diam}(G \circ H) \leq k$. By Theorem 2, a connected graph has a secure distance-k domination number of 1 if and only if its diameter is at most k. Therefore, $\gamma_k^s(G \circ H) = 1$. □

3. Conclusion

This study introduces a systematic framework for secure distance-k domination in graphs and establishes its fundamental structural properties. Sharp bounds relating the distance-k domination number, the secure distance-k domination number, and the classical domination number are established, together with structural characterizations for their extremal values. In particular, graphs with secure distance-k domination number equal to 1 are completely characterized by their diameter, while those attaining the maximum value are characterized as edgeless graphs.

An additive property over graph components is established, allowing the parameter to be computed explicitly on disconnected topologies. Furthermore, a necessary and sufficient neighborhood containment condition for a distance-k dominating set to be secure is derived, yielding a clear structural criterion for when the distance-k domination number and its secure counterpart coincide ($\gamma_k(G) = \gamma_k^s(G)$). We also establish necessary and sufficient conditions for a secure distance-k dominating set to be a classical dominating set, which underscores the mathematical transition from local to distance-based domination.

Finally, exact values of the secure distance-k domination number are determined for standard graph families, and binary graph operations, specifically the join and corona products. These results extend existing domination theory and provide a solid foundation for further investigations into secure domination under distance constraints, particularly regarding more complex graph products, graph layers, and specialized structural families.

Acknowledgements

The authors expressed their sincere gratitude to Department of Science and Technology-Science Education Institute (DOST-SEI) for funding this study.

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