EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 3, Article Number 7800
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Geometric Properties of Abel Differential Equations of the First and Second Kind

Paul Bracken

Department of Mathematics, University of Texas, Edinburg, TX 78540, USA

Abstract. A number of different procedures and techniques are used to study the Abel and some related equations of the first and second kind. The Abel equation of the first kind is related to a non-homogeneous second order Liénard equation. These equations are equivalent as far as algebraic and Liouvillian integrability go. Any Liouvillian solution to a Riccati equation induces a Liouvillian solution for the second order associated second order linear equation and conversely. Under suitable conditions on the coefficients which appear in the Abel equation, a Darboux first integral can be produced for the Abel equation of the first kind. Some results related to algebraic solutions to a form of the equation are developed at the end.

2020 Mathematics Subject Classifications: 35A30, 35A01, 34A05

Key Words and Phrases: Abel equation, first kind, projective, transformation, solution space, integrability

---

1. Introduction

There has over time been a great deal of interest for many reasons in equations that are called Abel differential equations [**1-4**]. They arise in many areas and have many applications. They provide a test for Hilbert's 16-th problem on the number and upper uniform limit for limit cycles of first order polynomial differential equations. as well as the center problem. Approaches to the study of Abel equations are necessary especially from the view-point of quasi-Lie schemes. These equations were first studied by Abel, and later by Appell and Liouville [$\mathbf{5-6}$]. At the present time, and here as well, Abel differential equations of the first and second kind are investigated systematically [**7-8**]. These equations [**9**] have the form

$$\begin{gathered} \frac{d\,y(x)}{dx} + a_1(x) y(x)^3 + 3a_2(x)\,y(x)^2 + 3a_3(x)\,y(x) + a_4(x) = 0, \\ (b_5(x) + b_6(x)\,z(x))\,\frac{dz(x)}{dx} + b_1(x) + 3b_2(x) z(x) + 3b_3(x)\,z(x)^2 + b_4(x)\,z(x)^3 = 0. \end{gathered} \tag{1.1}$$

DOI: https://doi.org/10.29020/nybg.ejpam.v19i3.7800

Email address: `paul.bracken@utrgv.edu` (P. Bracken)

Many ideas and procedures will be introduced and used to study equations of the type (1.1). The link between the Abel differential equation and some non autonomous Liénard equations is examined. The crux of the method of Darboux is the existence of finitely many Darboux invariant curves, or exponential factors, for a planar polynomial differential system, and forces it to possess a Liouvillian first integral. By obtaining some well-chosen transformations, it can be shown that if the Abel differential equation has a Liouvillian solution, then its associated Liénard equation also has a Liouvillian solution. Conversely, it holds that if the associated Liénard equation has a Liouvillian solution, then necessarily the Abel equation of the first kind has a Liouvillian solution. By exploiting a simple relation between the Abel differential equation of the first kind and the associated equation of the second order given by

$$\frac{d^2u}{dx^2}+a_1(x,u)(\frac{du}{dx})^3+3a_2(x,u)(\frac{du}{dx})^2+3a_3(x,u)\,\frac{du}{dx}+a_4(x,u)=0 \quad a_1(x,u)\neq 0, \tag{1.2}$$

under suitable conditions on the a_i, one finds a Darboux first integral of the Abel equation of the first kind. These equations have applications in describing geodesics of a projective connection in dimension two. Finally, some cases are investigated in which the Abel equation has algebraic solutions **[10-14]**.

Let us show the correspondence of these Abel equations under an invertible transformation given by

$$y(x)=\frac{1}{b_5(x)+b_6(x)\,z}, \tag{1.3}$$

and its inverse. These map the y equation in (1.1) to the z equation, and the inverse of (1.3) the z equation into the y equation. The correspondence of the coefficients under these transformations will be given. Putting (1.2) into the y equation in (1.1), the following result in z follows,

$$(b_5+b_6\,z)\,\frac{dz}{dx}-a_4b_6^2\,z^3-(3a_4b_5b_6+3a_3b_6-b_{6,x})\,z^2-(3a_1b_5^2+6a_3b_5-\frac{b_5}{b_6}\,b_{6,x}-b_{5,x}+3a_2)\,z$$
$$+\frac{1}{b_6}(a_1b_5^3-3a_3b_5^2+b_5\,b_{5,x}-3a_2b_5-a_1)=0. \tag{1.4}$$

When the transformation (1.3) is inverted to give $z(x)$, we have

$$z(x)=\big(\frac{1}{y(x)}-b_5(x)\big)\,\frac{1}{b_6(x)}. \tag{1.5}$$

Substitute (1.5) into the second equation in z in (1.1) to get the equation in y,

$$\frac{dy}{dx}+(-b_1b_6+3b_2b_5-3\frac{b_3b_5^2}{b_6}+\frac{b_4b_5^2}{b_6^2})y^3$$
$$+(b_{5,x}-3b_2-\frac{b_5}{b_6}b_{6,x}+6\frac{b_3b_5}{b_6}-3\frac{b_4b_5}{b_6})y^2+(\frac{b_{6,x}}{b_6}-3\,\frac{b_3}{b_6}+3\frac{b_4b_5}{b_6^2})\,y-\frac{b_4}{b_6^2}=0. \tag{1.6}$$

Results (1.5) and (1.6) relate the coefficents in one form of the equation to those in a related equation.

2. The Relationship of Abel Equation with Riccati and Liénard Equations.

If the Abel differential equation of the first kind has a Liouvillian solution, then necessarily its associated Liénard equation also has a Liouvillian solution. Conversely, if the associated Liénard equation has a Liouvillian solution, then necessarily the Abel differential equation of the first kind has a Liouvillian solution. The following definition of an *extension* iS used: Let $E \subset F$ of differential fields. An extension is called Liouvillian if F can be generated from E by adjoining a finite number of integrals, exponentials of integrals, and algebraic functions. An integral of an element κ is defined to be any solution of $y' = \zeta$. An exponential of an integral ζ is defined to be any solution of the equation $y' = \zeta\, y$. Darboux's method relies on the existence of finitely many Darboux invariant curves or exponential factors, for a planar polynomial differential system forces it to possess a Liouvillian first integral. The following two Lemmas will be subsequently be used to produce a Theorem.

Lemma 2.1. Define η as

$$\eta = -(a_1 y^2 + 2a_2 y), \tag{2.1}$$

and let L be given by

$$L = a_2 a_{1,x} - a_1 a_{2,x} + a_1\,(a_1 a_4 - a_2 a_3) + 2a_2(a_2^2 - a_1 a_3), \tag{2.2}$$

where y in (2.1) satisfies the equation

$$y_x + a_1(x) y^3 + 3a_2(x) y^2 + 3a_3(x) y + a_4(x) = 0. \tag{2.3}$$

Then provided y satisfies (2.3), η satisfies the following first order differential equation

$$\eta_x - 2\eta^2 - \mu(x)\,\eta - 2a_2\,a_4 - 2\,\frac{L}{a_1}\,y = 0, \tag{2.4}$$

where $\mu(x)$ is defined by

$$\mu(x) = \frac{a_{1,x} + 2a_2^2 - 6a_1 a_3}{a_1}, \tag{2.5}$$

If $L \neq 0$, the differential equation (2.3) is transformed into a Riccati system which admits the rational solution

$$\eta_0 = \frac{a_2^2}{a_1}.$$

This Riccati equation is integrable by means of quadratures, so the solutions reside in a Liouvillian extension of K.

Proof. Substitute η in (2.1) and L in (2.2) into (2.4) and simplify to find

$$(a_1\,y + a_2)(y_x + a_1 y^3 + 3a_2 y^2 + 3a_3 y + a_4) = 0. \tag{2.6}$$

Hence (2.4) holds whenever y satisfies (2.3), that is, when $y = -a_2/a_1$ which by (2.1), implies $\eta_0 = a_2^2/a_1$.

Equation (2.4) can be produced in a constructive way starting with (2.1) to calculate $\eta_x - 2\eta^2$ as

$$\eta_x - 2\eta^2 = -a_{1,x}y^2 - 2a_{2,x}y + 6a_1a_3y^2 - 2a_1a_4y^2 + 6a_2a_3y + 2a_2a_4.$$

The derivative y_x has been eliminated by substituting (2.3) which appears when η_x is calculated. Adding $\mu\,\eta$ to this and collecting y terms, (2.4) follows

$$\eta_x - 2\eta^2 - \frac{a_{1,x} + 2(a_2^2 - 3a_1a_3)}{a_1}\,\eta = \frac{2}{a_1}(a_2\,a_{1,x} - a_1\,a_{2,x} + a_1^2a_4 + a_2^3 - 3a_1a_2a_3)\,y + 2a_2a_4$$
$$= 2\frac{L}{a_1}\,y + 2a_2\,a_4.$$

When $L=0$, this implies that η verifies a Riccati equation. Solving $L=0$ for $a_{2,x}$ and putting $\eta=\eta_0$ in (2.4) followed by $a_{2,x}$ the η equation simplifies to zero. So η_0 solves the η equation when $L=0$. Since the Riccati equation admits a rational solution, it is Liouville integrable. Set $\eta=\eta_0+1/\tau$, then τ satisfies a first order non-homogenous linear differential equation with coefficients in K. Thus τ lies in a Liouvillian extension of K as does η. □

Lemma 2.2. Suppose $L \neq 0$ and η defined by (2.1) satisfies the following equation

$$\eta_{xx} + 3\,(\delta_1\,\eta + \delta_2)\,\eta_x + \gamma_1\,\eta^3 + 3\gamma_2\,\eta^2 + 3\gamma_3\,\eta + \gamma_4 = 0, \tag{2.7}$$

where the quantities (δ_i, γ_i) must satisfy the system,

$$\delta_1 = -\frac{1}{6}\,\gamma_1 - \frac{4}{3}, \tag{2.8}$$

$$-2\delta_2 - \gamma_2 = \frac{1}{6}\,(4-\gamma_1)\,\mu(x), \tag{2.9}$$

$$\boldsymbol{\sigma}(x) = 3(\delta_2+\gamma_2) = \frac{L_x}{L} - (\frac{2}{a_1} - \frac{\gamma_1}{2a_1})\,a_{1,x} + 3a_3 - (3a_3 + \frac{a_2^2}{a_1})\,\gamma_1, \tag{2.10}$$

$$3\gamma_3 = 2(\gamma_1-3)\frac{a_2}{a_1^2}\,L - \mu'(x) + (1-\frac{\gamma_1}{2})\mu(x)^2 + \mu(x)\,\boldsymbol{\sigma} + \gamma_1a_2a_4, \tag{2.11}$$

$$\begin{aligned}\gamma_4 = 2a_2a_4\,\boldsymbol{\sigma} - \gamma_1\,a_1a_{2,x} - 2a_2a_{1,x} + 3\gamma_1\,a_2a_3a_4 - a_1a_4^2 - 6a_2a_3a_4 + (\gamma_1-1)\,a_1\,a_4^2 \\ +(4-\gamma_1)\frac{a_4}{a_1}\,L(x)\end{aligned} \tag{2.12}$$

where $\mu(x)$ is defined in (2.5).

Proof. Define η by (2.1) so it satisifies (2.4) with $\mu(x)$ defined by (2.5). Solve (2.1) for η', then differentiate it and eliminate y_x using (2.3) to give an expression for η_{xx}. To obtain this equation in the form (2.7), subtract this form for η_{xx} and (2.7) and then equate to zero. The coefficients of the powers of y produce a linear system for the set (δ_i, γ_i) which specifies (2.7). The coefficients of the two highest powers of y are proportional to the factor $\delta_1 + \gamma_1/6 + 4/3$. When this is set to zero we get (2.8), and the two highest powers of y are eliminated. Any ordered pair that satisfies (2.8) may be used. However we solve (2.8) for δ_1 in terms of γ_1 and replace δ_1 in the equations remaining in the set. There is a system of five equations but not all linearly independent. The first four are independent, and solving them yields δ_2 and the remaining γ_i in closed form. However, they are not yet in the form (2.9) or (2.10). Take δ_2 and γ_2, write the combination $-2\delta_1+\gamma_2$ and eliminate $a_{1,x}$ in terms of $\mu(x)$ using (2.5) to produce (2.9). Similarly, define $\boldsymbol{\sigma}$ to be $\boldsymbol{\sigma} = 3(\delta_2+\gamma_2)$ and simplify to get (2.10). Moreover, the derivative of $L(x)$ can be computed. It is found to be

$$L(x)_x = \frac{1}{2}\big[(4-\gamma_1)\frac{a_{1,x}}{a_1} + (6a_3+2\frac{a_2^2}{a_1})\,\gamma_1 - 6a_3 + 2\boldsymbol{\sigma} - 8\frac{a_2^2}{a_1}\big]\, L(x). \tag{2.13}$$

Eliminating $a_{1,x}$ in terms of $\mu(x)$ and substituting $L(x)_x$ from (2.13), the result (2.11) is found

$$3\gamma_3 = 2(\gamma_1-3)\frac{a_2}{a_1^2}\, L(x) - \mu(x)_x + (1-\frac{\gamma_1}{2})\,\mu(x)^2 + \mu(x)\,\boldsymbol{\sigma} + \gamma_1\, a_2 a_4$$

The function appears squared in γ_3 when $\gamma_1 \neq 2$. The last term to be calculated is γ_4. To finish this, obtain $a_{1,x}$ by solving (2.2) for it in terms of $L(x)$ and replace this form of $a_{1,x}$ as well as $L_x(x)$ from (2.13) into γ_4 to obtain the last of the γ_i as

$$\gamma_4 = 2a_2a_4\,\boldsymbol{\sigma} - \gamma_1\, a_1 a_2' - 2a_{1,x}a_2 + 3\gamma_1\, a_2a_3a_4 - a_1\, a_4^2 - 6a_2a_3a_4 + (\gamma-1)a_1a_4^2 + (4-\gamma_1)\,\frac{a_4}{a_1}\, L(x)$$

where $\boldsymbol{\sigma}$ is defined in (2.10). $\square$

The two lemmas which have been stated and proved above can be combined into a final Theorem.

Theorem 2.1. Suppose $\mathcal{F}$ is a differential field of characteristic zero and consider equation (2.3) over $\mathcal{F}$ with $(a_i)_{1\leq i\leq 4} \in \mathcal{F}$, $a_1 \neq 0$. Let η be given by (2.1) and L by (2.2) Then with $\mu(x)$ given by (2.5), η satisfies

$$\eta_x - 2\eta^2 - \mu(x)\,\eta - 2\,a_2a_4 - 2\frac{L}{a_1}\,a = 0,$$

where (δ_i, γ_i) are given in (2.8) to (2.12). $\square$

Proof. The first claim follows from (2.1) so $\eta = -(a_1\, y^2 + 2a_2\, y)$, and the second by invoking equation (2.4) as an identity. $\square$

3. Abel Equations and Projective Connections.

A holomorphic projective structure on an open set $U \subset \mathbb{C}^2$ is an equivalence class of torsion free affine connections $[\Gamma]$. Two connections Γ and $\hat{\Gamma}$, represented by their Christoffel symbols, are projectively equivalent if they share the same unparametrized geodesics. The analytic expression for this equivalence is

$$\hat{\Gamma}_{ab}^c = \Gamma_{ab}^c + \delta_a^c\,\omega_b + \delta_b^c\,\omega_a, \qquad a,b,c = 1,2.$$

for a one-form $\omega = \omega_a\, dx^a = \sum_{a=1}^2 \omega_a\, dx^a$.

In dimension 2, the projective structures or connections are equivalent to second order ordinary differential equations for $(x^1(t), x^2(t)) = (x(t), u(t))$ and the parameter t is eliminated between the two equations,

$$\ddot{x}^c(t) + \Gamma^c_{ab}\, \ddot{x}^a \dot{x}^b = \lambda\, \dot{x}^c,$$

where $\lambda \subset \mathbb{C}$. An ordinary differential equation similar to (1.2) is obtained,

$$\frac{d^2u}{dx^2} = \Gamma^1_{22}\, (\frac{du}{dx})^3 + (2\Gamma^1_{12} - \Gamma^2_{22})\, (\frac{du}{dx})^2 + (\Gamma^1_{11} - 2\Gamma^2_{12}) \frac{du}{dx} - \Gamma^2_{11}.$$

Conversely, any second order equation cubic in the first derivatives gives rise to some projective structure as the independent components of Γ^c_{ab} can be found from the coefficients of the powers of u_x.

Abel differential equations of the first kind are of interest which are related to second order differential equations $u_{xx} = f(x, u, u_x)$, $f \in \mathbb{C}(x, u, u_x)$ in the ring of convergent power series in x, u, u_x which can be transformed under the pseudogroup of point tranformations $X = \phi(x, u)$, $U = \psi(x, u)$ whose functions $\phi(x, u)$, $\psi(x, u) \in \mathbb{C}(x, u)$ and the Jacobian $J \neq 0$ into the form $u_{xx} = 0$. Consider $\mathcal{P}$ the pseudogroup of transformations of the plane given by

$$X = \phi(x, u), \qquad U = \psi(x, u), \qquad J = \det \begin{pmatrix} \phi_x & \psi_x \\ \phi_u & \psi_u \end{pmatrix} \neq 0. \tag{3.1}$$

The action of $\mathcal{P}$ on (x, u) induces an action on (u_x, u_{xx}) given by the chain rule

$$U_X = \frac{dU}{dX} = \frac{U_x}{X_x} = \frac{\psi_x + u_x\psi_u}{\phi_x + u_x\phi_u}. \tag{3.2}$$

If this derivative is denoted $\tilde{U}_x$, then we can compute

$$\frac{d^2U}{dX^2} = \frac{\tilde{U}_x}{X_x}$$

$$= \frac{\psi_{xx} + u_x\psi_{xx} + u_{xx}\psi_x + u_x(\psi_{ux} + u_x\psi_{uu})}{\phi_x + u_x\phi_u} - \frac{(\psi_x + u_x\psi_u)}{(\psi_x + u_x\phi_u)^2} \cdot (\phi_{xx} + u_x\phi_{xu} + u_{xx}\phi_u + u_x(\phi_{ux} + u_x\, \phi_{uu})$$

$$= \frac{\lambda + \mu\, u_x + \nu u_x^2 + \xi\, u_x^3 + J\, u_{xx}}{(\phi_x + u_x\phi_u)^2}. \tag{3.3}$$

The coefficients in the numerator of (3.3) are given as follows

$$\begin{aligned} \lambda &= \phi_1\psi_{xx} - \psi_x\phi_{xx}, \\ \mu &= \phi_u\psi_{xx} + \phi_x\psi_{xu} + \phi_1\psi_{xx} - 2\phi_{xx}\psi_x - \psi_u\psi_{xx}, \\ \nu &= \phi_u\psi_{xu} + \phi_u\psi_{xx} - \phi_{uu}\psi_x + \psi_{uu}\phi_x - 2\psi_u\phi_{xu}, \\ \xi &= \phi_u\psi_{uu} - \psi_u\phi_{uu}, \end{aligned} \tag{3.4}$$

$$J = \phi_u \psi_x - \phi_x \psi_u.$$

Given two ordinary differential equations

$$u_{xx} = f(x, u, u_x), \qquad U_{xx} = F(X, U, U_x), \tag{3.5}$$

it is said they are point equivalent if there exists a transformation of the form (3.1) with induced action (3.2), which sends the source u_{xx} equation (3.5) into the target U_{xx} equation (3.5). In such case, using (3.2) and (3.3), there is the following identity between the functions $f(x, u, u_x)$ and $F(X, U, U_x)$

$$(\phi_x + u_x \phi_u)^3 F = J\, f + \lambda + \mu u_x + \nu u_x^2 + \xi u_x^3. \tag{3.6}$$

This class of projective connections is preserved under point transformations. Both Cartan and Liouville have shown that the necessary and sufficient condition for the existence of a point transformation which sends $u_{xx} = f(x, u, u_x)$ to $u_{xx} = 0$ is

$$f(x, u, u_x) = -(a_1(x, u)u_x^3 + 3a_2(x, u)u_x^2 + 3a_3(x, u)u_x + a_4(x, u)), \tag{3.7}$$

$(a_i)_{1 \le i \le 4} \in \mathbb{C}(x, u)$, along with two constraints denoted L_1 and L_2 given by

$$L_1 = \frac{\partial}{\partial x}\,(2\frac{\partial a_3}{\partial u} - \frac{\partial a_2}{\partial x} + a_1\, a_4) - \frac{\partial}{\partial u}(\frac{\partial a_4}{\partial u} + 3\, a_2 a_4) + a_4\,(\frac{\partial a_1}{\partial x} - 3a_1 a_3) + 3a_3(2\frac{\partial a_3}{\partial u} - \frac{\partial a_2}{\partial x} + a_1 a_4) = 0,$$

$$L_2 = \frac{\partial}{\partial u}\,(\frac{\partial a_3}{\partial u} - 2\frac{\partial a_2}{\partial x} + a_1 a_4) + \frac{\partial}{\partial x}(\frac{\partial a_1}{\partial x} - 3a_1 a_3) + a_1(\frac{\partial a_4}{\partial u} + 3a_2 a_4) - 3a_2(\frac{\partial a_3}{\partial u} - 2\frac{\partial a_2}{\partial x} + a_1 a_4) = 0. \tag{3.8}$$

Constraints (3.8) will play a large role here, such as when $\{a_k(x)\}_1^4$ just depend on x.

The Abel differential equation of the first kind can be related to second order equations by replacing $y = u_x$ in (2.3). for some function $u(x)$ which produces an equation which belongs to the family of projective connections,

$$u_{xx} + a_1(x)u_x^3 + 3a_2(x)\, u_x^2 + 3a_3(x)\, u_x + a_4(x) = 0. \tag{3.9}$$

Suppose $\{a_k(x, u)\} = \{a_k(x)\}_{k=1}^4 \in \mathbb{C}$, and $1 \le k \le 4$ are independent of u and $a_1(x) \ne 0$ in (2.3). Then conditions (3.8) simplify to the form

$$L_1 = (a_1 a_4 - \frac{da_2}{dx})_x - 3a_3\, \frac{da_2}{dx} + a_4\, \frac{da_1}{dx} = 0, \qquad L_2 = \frac{d}{dx}\big(\frac{da_1}{dx} + 3(a_2^2 - a_1 a_3)\big) = 0. \tag{3.10}$$

By starting with an equation for a projective connection (3.9) with $L_1 = L_2 = 0$, the coefficients $a_k(x)$ can be used to define a system of partial differential equations as follows,

$$\begin{aligned} &z_{uu} + a_1\, z_x - a_2\, z_u + (-a_{1,x} + 2(a_1 a_3 - a_2^2))\, z = 0, \\ &z_{xu} + a_2 z_x - a_3 z_u + (-a_{2,x} + a_1 a_4 - a_2 a_3)\, z = 0, \\ &z_{xx} + a_3\, z_x - a_4\, z_u + (-a_{3,x} + 2(a_2 a_4 - a_3^2))\, z = 0, \end{aligned} \tag{3.11}$$

$$L_1 = L_2 = 0.$$

It will be seen that system (3.11) leads directly to an equation for a projective connection of the form (3.9).

Lemma 3.1. The integrability conditions for system (3.11) given by

$$\frac{\partial}{\partial x}(z_{uu}) = \frac{\partial}{\partial u}(z_{xu}), \qquad \frac{\partial}{\partial u}(z_{xx}) = \frac{\partial}{\partial x}(z_{xu}), \tag{3.12}$$

are the constraints given in (3.10).

Proof. Isolate the second derivatives given in (3.11) and use them to compute first the compatibility $(z_{uu})_x = (z_{xu})_u$ to obtain

$$\frac{d^2 a_1}{dx^2} - 3\, a_1 a_{3,x} + 3(a_2^2)_x - 3a_{1,x} a_3 = 0,$$

which is $L_2 = 0$. Second compute the second relation in (3.12) which yields

$$\frac{d^2 a_1}{dx^2} - 2a_{1,x}\, a_4 + 3a_{2,x} a_3 - a_1 a_{4,x} = 0,$$

which is $L_1 = 0$ in (3.10). $\Box$

Let us return to the system (3.11) and consider the more general form in which the $\{a_k(x,u)\}_{k=1}^4$ depend on u. It given by

$$\begin{aligned} z_{uu} + a_1 z_x - a_2 z_u + \big(a_{2,u} - a_{1,x} + 2a_1 a_3 - 2a_2^2\big)\, z = 0, \\ z_{xu} + a_2 z_x - a_3 z_u + \big(a_{3,u} - a_{2,x} + a_1 a_4 - a_2 a_3\big)\, z = 0, \\ z_{xx} + a_3 z_x + a_4 z_u + \big(-a_{4,u} - a_{3,x} + 2a_2 a_4 - 2a_3^2\big)\, z = 0. \end{aligned} \tag{3.13}$$

Proposition 3.1. Let the functions $\{a_k(x,u)\}_{1 \le k \le 4}$ satisfy the integrability conditions (3.12), then the following statements hold:

(i) The vector space $\mathcal{S}$ of solutions for (3.13) is of dimension 3.

(ii) Every basis $\{\omega_1, \omega_2, \omega_3\}$ of $\mathcal{S}$ determines a local isomorphism $\sigma(C^2, 0) \to (P^2, \sigma(0))$ where $\sigma = \{\omega_1, \omega_2, \omega_3\}$. Moreover, if $\tilde{\sigma} = \{\tilde{\omega_1}, \tilde{\omega}_2, \tilde{\omega}_3\}$ is associated to another basis of $\mathcal{S}$, then σ and $\tilde{\sigma}$ are projectively equivalent, that is, $\tilde{\sigma} = M\,\sigma$ where M is a matrix which is an element of $GL(3, \mathbb{C})$.

Proof. (i) Let B be the ring of differential operators which are linear and have coefficients in $\mathbb{C}(x,u)$ and $B/\mathcal{S}$ denotes the left B module associated to (3.13). Since the a_k satisfy the compatibility or integrability conditions (3.12), the ideal $\mathcal{O}[\zeta, \eta]$ generated by the principal symbol of elements of $\mathcal{S}$ is $(\zeta^2, \zeta\eta, \eta^2) \circ (\zeta, \eta)$, which shows that $B/\mathcal{S}$ is B-isomorphic to $\mathcal{O}^3$ and one has in particular that,

$$\dim_{\mathbb{C}} \mathcal{S} = \dim_{\mathbb{C}}\, Hom_B(B/\mathcal{S}, \mathcal{O}) = 3. \tag{3.14}$$

(ii) Let $\{\omega_1, \omega_2, \omega_3\}$ be a basis of $\mathcal{S}$. It may be verified that the determinant

$$W(x,u) = \det \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_{1,x} & \omega_{2,x} & \omega_{3,x} \\ \omega_{1,u} & \omega_{2,u} & \omega_{3,x} \end{pmatrix}$$

is a non-null element of $\mathbb{C}(x,u)$, unless the ω_j are linearly independent over $\mathbb{C}$. In addition, system (3.13) demonstrates that $\partial_x W(x,u) = \partial_u W(x,u) = 0$. As one of the $\omega_j(0)$ is non-null, this implies the existence of a local isomorphism $\phi = (\omega_1, \omega_2, \omega_3)$ of the preceding, and projective equivalence is consequently a consequence of (i). □

Corollary 3.1. The system of partial differential equations (3.13) has three linearly independent solutions in $\mathbb{C}(x,u)$ over $\mathbb{C}$, that is, the dimension of the solution space $\mathcal{S} \subset \mathbb{C}(x,u)$ over $\mathbb{C}$ is three.

Proof. This is a consequence of Proposition 3.1 where all functions $(a_i)_{1 \le i \le 4}$ are independent of the u variable. In that case, (3.13) reduces to the integrability conditions (3.10) $L_1 = L_2 = 0$. □

Theorem 3.1. Let $z_1, z_2, z_3 \in C(x,u)$ be three linearly independent solutions to system (3.11) and $z = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ the general solution. Putting $z = 0$ allows us to define u implicitly in terms of x. Using the system (3.11) with $L_1 = L_2 = 0$, an integral curve to differential equation (3.9) is obtained. Conversely, given (3.9), its associated system (3.13) can be constructed and based on that obtain solutions to (3.9).

Proof. The plane with a fixed origin is called $(\mathbb{C}^2, 0)$, $\mathbb{C}P_1$ the complex projective line and $\mathbb{C}P_2$ the complex projective plane. When the set of functions z_1, z_2, z_3 is linearly independent, the Wronskian determinant is non-vanishing for all $(x,u) \in (\mathbb{C}, 0)$,

$$W(x,u) = \det \begin{pmatrix} z_1 & z_2 & z_3 \\ z_{1,x} & z_{2,x} & z_{3,x} \\ z_{1,u} & z_{2,u} & z_{3,u} \end{pmatrix} (x,u) \neq 0.$$

Let $(x_0, u_0) \in (\mathbb{C}, 0)$, then outside the projective line $\mathbb{C}P_1 = \{[\alpha_1 \, \alpha_2 \, \alpha_3] \in \mathbb{C}P_2 : z_{1,u}(x_0,u_0)\,\alpha_1 + z_{2,u}(x_0,u_0)\,\alpha_2 + z_{3,x}(x_0,u_0)\,\alpha_3 = 0\}$, it must be that $z_u(x_0, u(x)) \neq 0$. This means the implicit function theorem can be used to define a holomorphic function $u(x)$ where $z(x,u(x)) = 0$. Under implicit differentiation, the relation $z(x,u(x)) = 0$ implies that

$$z_u \, u_x + z_x = 0, \qquad z_{xx} + 2 z_{xu} \, u_x + z_{uu} u_x^2 + z_u \, u_{xx} = 0. \tag{3.15}$$

System (3.13) and the first equation of (3.15) to replace $z_x, z_{xu}, z_{xx}, z_{uu}$ in the second equation of (3.15) by their values in terms of z, z_u, u_x and the set of coefficient functions $(a_i)_{1 \le i \le 4}$ and their derivatives. The result of this is

$$(u_{xx} + a_1 \, u_x^3 + 3 a_2 \, u_x^2 + 3 a_3 \, u_x + a_4) \, z_u$$

$$+ [(a_{3,x} - 2 a_2 \, a_4 + 2 a_3^2) + 2\,(a_{2,x} + a_2 a_3 - a_1 a_4 \, u_x + (a_{1,x} + 2 a_2^2 - 2 a_1 a_3) \, u_x^2] \, z(x,u(x)) = 0.$$

Along the trajectory, $z(x,u(x)) = 0$, this is equivalent to $(u_{xx} + a_1(x)\, u_x^3 + 3\, a_2(x)\, u_x^2 + 3 a_3(x)\, u_x + a_4(x))\, z_u(x,u(x)) = 0$. However, since $z_u(x,u(x)) \neq 0$, by the implicit function theorem, we conclude this equation is equivalent to (3.9),

$$u_{xx} + a_1(x)\, u_x^3 + 3 a_2(x)\, u_x^2 + 3 a_3(x)\, u_x + a_4(x) = 0.$$

The set of integral curves of (3.9) and the set of functions $u(x)$ defined implicitly through $z(x,u) = 0$ for z the general solution of system (3.11) coincide as there is a neighborhood of a point where the elements of the equivalence class coincide. This follows from the generalized Cauchy-Lipschitz theorem. Thus given a point (x_0, u_0) and a direction denoted u_0', there is a unique integral curve $u(x)$ of (3.9) which satisfies $u(x_0) = u_0$ and $u_x(x_0) = u_0'$. Given such a point (x_0, u_0), apply the implicit function theorem at (x_0, u_0), $v(x_0) = u_0$, $z(x, v(x)) = 0$ and the two further conditions

$$v_x\, z_u(x, v(x)) + z_x(x, v(x)) = 0, \qquad z_{xx}(x, v(x)) + 2z_{xu}v_x + z_{uu}\, v_x^2 + z_u v_{xx} = 0. \quad (3.16)$$

Set $V = \mathbb{C}P_2 \backslash \{\mathbb{C}P_1\} = \mathbb{C}P_2\ \{[\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{C}P_2 : z_{1,u}(x_0, u_0)\, \alpha_1 + z_{2,u}\,(x_0, u_0)\, \alpha_2 + z_{3,u}(x_0, u_0)\, \alpha_3 = 0\}$.

Since the Wronskian is non-zero, there exists the following isomorphism $V \to \mathbb{C}^2$ which takes $[\alpha_1 : \alpha_2 : \alpha_3] \to (\Psi_1([\alpha_1 : \alpha_2 : \alpha_3]), \Psi_2([\alpha_1 : \alpha_2 : \alpha_3])$ such that

$$\Psi_1([\alpha_1 : \alpha_2 : \alpha_3]) = (\frac{z_1(x_0, u_0)\, \alpha_1 + z_2(x_0, u_0)\, \alpha_2 + z_3(x_0, u_0)\, \alpha_3}{z_{1,u}(x_0, u_0)\, \alpha_1 + z_{2,u}(x_0, u_0)\, \alpha_2 + z_{3,u}(x_0, u_0)\, \alpha_3}),$$

$$\Psi_2([\alpha_1 : \alpha_2 : \alpha_3]) = (\frac{z_{1,x}(x_0, u_0)\, \alpha_1 + z_{2,x}(x_0, u_0)\alpha_2 + z_{3,x}(x_0, u_0)\alpha_3}{z_{1,u}(x_0, u_0)\alpha_1 + z_{2,u}(x_0, u_0)\, \alpha_2 + z_{3,u}x_0, u_0)\, \alpha_3}). \quad (3.17)$$

For the same reason that $[\alpha_1 : \alpha_2 : \alpha_3] \to (\alpha_1/\alpha_3, \alpha_2/\alpha_3)$ is an isomorphism on $U_3 = \{[\alpha_1, \alpha_2, \alpha_3] \in \mathbb{C}P_2, \alpha_3 \neq 0\}$ onto $\mathbb{C}^2$, such an isomorphism exists.

This says for example that by using the first equation in (3.16), the value of $v' : v'(x_0) = -\Psi_2([\alpha_1 : \alpha_2 : \alpha_3])$ at x_0 can take any given value $\delta \in \mathbb{C}$. Taking $v'(x_0) = u_0'$, there is equality of the elements in equivalence classes of integral curves of (3.9) in a neighborhood of the point and the set of functions $u(x)$ defined implicitly by $z(x, u) = 0$ for z the general solution of the system of equations in (3.11). □

4. First Integral for the Abel Equation.

A first integral of the first Abel equation we started with in (1.1) is now formulated such that its coefficients satisfy (3.10). Once we have arrived at (3.10), some of the quantities introduced already turn out to be conserved or constant with respect to x,

Lemma 4.1. The two quatities $3\,\tau_1$ and τ_2 defined as

$$3\tau_1 = a_{1,x} + 3(a_1^2 - a_1 a_3), \quad (4.1)$$

and

$$\tau_2 = L = a_2 a_{1,x} - a_1 a_{2,x} + a_1^2 a_4 + 2a_2^3 - 3a_1 a_2 a_3, \quad (4.2)$$

are conserved, that is constant, with respect to differentiation by x. Writing the last three terms in (4.2) as $a_1^2 a_4 - a_1 a_2 a_3 + 2a_2^3 - 2a_1 a_2 a_3$ and then factoring, it is clear that L in (4.2) is the same as (2.2). Also $\mu(x)$ in (2.5) is constant provided the a_k satisfy the constraint.

$$3\tau_1\,(\frac{1}{a_1})_x = 3a_{1,x} + 3a_3 - 2\frac{a_2^2}{a_1}.$$

Proof. The derivative $(3\tau_1)_x$ is proportional to $L_2 = 0$ in (3.10). To see that τ_1 is constant, write the derivative of L as

$$L_x = a_1(2a_4 a_{1,x} - 3a_3 a_{2,x} + a_1 a_{4,x} - a_{2,xx} + a_2(a_{1,x} + 3(a_2^2 - a_1\, a_3)),$$

which is $L_1 = 0$ in (3.10). The last result is shown by putting $\mu(x)$ in the form,

$$\mu(x) = \frac{3\tau_1}{a_1} - 3a_1 - 3a_3 + 2\frac{a_2^2}{a_1}$$

differentiating both sides with respect to x and using (4.1) with $\mu(x)_x = 0$. □

Theorem 4.1. Consider Abel equation (2.3) where coefficients a_i satisfy constraints (3.11), and (3.9) is the associated second order differential equation, the corresponding differential system in terms of z is (3.11). Then the general solution of (3.11) for z is given by

$$z = \alpha_1\, \varphi_1\, e^{m_1 u} + \alpha_2\, \varphi_2\, e^{m_2 u} + \alpha_3\, \varphi_3\, e^{m_3 u}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \tag{4.3}$$

where φ_i for $1 \leq i \leq 3$ satisfy the following equations with $3\tau_1$ given in (4.2)

$$\varphi_i' = \frac{3\tau_1 + a_1 a_3 - a_2^2 - a_2\, m_i - m_i^2}{a_1}\, \varphi_i \tag{4.4}$$

The m_i with $1 \leq i \leq 3$ are the roots of the cubic polynomial

$$m_i^3 - 3\tau_1\, m_i - \tau_2 = 0, \tag{4.5}$$

The $\tau_1, \tau_2 \in \mathbb{C}$ given in (4.2) are chosen so the roots m_i are distinct, non-zero and τ_2 is just L in (2.2). Define the set of $\{\omega_i\}_{i=1}^3$ as

$$\omega_1 = (m_3 - m_2)\, \varphi_2 \varphi_3\, y + \varphi_2 \varphi_3' - \varphi_2' \varphi_3, \quad \omega_2 = (m_1 - m_3)\, \varphi_1 \varphi_3\, y + \varphi_1' \varphi_3 - \varphi_3' \varphi_1, \tag{4.6}$$

$$\omega_3 = (m_2 - m_1)\, \varphi_1 \varphi_2\, y + \varphi_1' \varphi_2 - \varphi_2' \varphi_1.$$

Then it follows the product

$$\omega_1^{m_3 - m_2}\, \omega_2^{m_1 - m_3}\, \omega_3^{m_2 - m_1} \tag{4.7}$$

is constant along all integral curves of y equation (2.3) which is developed from solutions $u(x)$ of the implicit equation with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$,

$$z = \alpha_1\, \varphi_1\, e^{m_1 u} + \alpha_2\, \varphi_2 e^{m_2} + \alpha_3\, \varphi_3\, e^{m_3 u}, \tag{4.8}$$

by setting $y = u_x$.

Proof. The idea is to start with the system for z in (3.11) and obtain a solution of the form (4.3) with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and $(m_i)_{1 \leq i \leq 3}$ constants which are nonzero, distinct and satisfy (4.4). To get (4.3), it suffices to show that for $1 \leq i \leq 3$,

$$z_i = \varphi_i(x)\, e^{m_i u}, \tag{4.9}$$

verifies the system. The first step is to substitute (4.9) into the first equation of (3.11) to get

$$m_i^2\,\varphi_i + a_1\,\varphi_{i,x} - a_2\,m_i\,\varphi_i + (-a_{1,x} + 2(a_1a_3 - a_2^2))\varphi_i = 0.$$

From this, isolate the derivative $\varphi_{i,x}$ to obtain (4.4). Put (4.4), in the second equation of z system (3.11) and (4.4) appears

$$m_i^3 - (a_{1,x} + 3(a_2^2 + 3(a_2^2 - a_1a_3))\,m_i - (a_{1,x}a_2 - a_1a_{2,x} + a_1^2a_4 + 2a_2^2 - 3a_1a_2a_3) = 0.$$

In fact, the second equation of (3.11) is

$$(m_i + a_2)\,\varphi_{i,x} + (a_{2,x} + a_2a_3 + m_ia_3 - a_1a_4)\varphi_i(x) = 0.$$

Replacing $\varphi_{i,x}$ in this, it takes the form

$$a_1^{-1}\,[-m_i^3 + 3\tau_1\,m_i + 3a_2\tau_1 + a_4a_1^2 - a_2^3 - a_{2,x}] = 0.$$

Differentiate $\varphi_{i,x}$ with respect to x and substitute $\varphi_{i,x}$ using (4.4), an expression for φ_{xx} results

$$\varphi_{i,xx} = \Big(\frac{3\tau_1 + a_1a_3 - a_2^2 + a_2m_i - m_i^2}{a_1}\Big)_x\,\varphi_i + \Big(\frac{3\tau_1 + a_1a_3 - a_2^2 + a_2m_i - m_i^2}{a_1}\Big)^2\,\varphi_i. \quad (4.10)$$

The Wronskian determinant has to be nonzero to proceed. Thus at this point it is calculated. Computing $z_{i,x}$ and $z_{i,u}$ using (4.9), W is given by

$$W = \det\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_{1,x} & \varphi_{2,x} & \varphi_{3,x} \\ m_1\varphi_1 & m_2\varphi_2 & m_3\varphi_3 \end{pmatrix}$$

Using (4.4), this determinant is found to be

$$W = \frac{1}{a_1}\,(m_2 - m_1)(m_3 - m_1)(m_3 - m_2)\,\varphi_1\,\varphi_2\,\varphi_3.$$

This is nonzero when the m_i are distinct.

Substitute z_i into the third equation of the system (3.11) and simplify it. The result is found to be equivalent to the following second order equation for φ_i

$$\varphi_{i,xx} + a_3\,\varphi_{i,x} + (-a_4m_i - a_{3,x} + 2(a_2a_4 - a_3^2))\,\varphi_i = 0. \quad (4.11)$$

The first term on the right of (4.9) can simplified by computing the derivative before φ_i. The derivative of $a_{1,xx}(x)$ appears. It can be obtained by solving the constraint $L_2 = 0$ for this derivative and substituted to find

$$\Big(\frac{3\tau_1 + a_1a_3 - a_2^2 + a_2m_i - m_i^2}{a_1}\Big)_x\varphi_i$$

$$= [(a_1(a_{1,x}a_3 + a_1a_{3,x} - (a_2^2)_x + m_ia_{2,x}) - a_{1,x}(a_{1,x} + 2(a_2^2 - a_1a_3) + a_2\,m_i - m_i^2)]\,\frac{\varphi_i}{a_1^2}. \quad (4.12)$$

Expanding the second term in (4.10), and replacing the powers of m_i by $m_i^3 = 3\tau_1\, m_i + \tau_2$, $m_i^4 = 3\tau_2\, m_i^2 + \tau_2 m_i$ as well as the expressions for $3\tau_1$ and τ_2 from (4.2) in terms of the a_i and their derivatives, this second term is

$$
\begin{aligned}
&(\frac{3\tau_1 + a_1 a_3 - a_2^2 + a_2\, m_i - m_i^3}{a_1})^2\, \varphi_i \\
&= [a_{1,x}^2 + 4a_{1,x}(a_2^2 - a_1 a_3) + 4(a_2^2 - a_1 a_3)^2 - (a_{1,x} - a_1 a_3)\, m_i^2 \qquad (4.13)\\
&+ a_2(a_2 a_{1,x} - a_1 a_{2,x} + a_1^2 a_4 - a_1 a_2 a_3) m_i - 2a_2(a_2 a_{1,x} - a_1 a_{2,x} + a_1^2 a_4 + 2a_2^3 - a_1 a_2 a_3)]\, \frac{\varphi_i}{a_1^2}.
\end{aligned}
$$

Combining (4.12) and (4.13), the following expression for the second derivative $\varphi_{i,xx}$ results from (4.8)

$$
\varphi_{i,xx} = \frac{1}{a_3}\big(- a_{1,x} a_3 + a_1 a_{2,x} + 4a_1\, a_3^2 + a_1^2(m_i - 2a_2) + a_3(m_i + a_2)(m_i - a_2)\big)\, \varphi_i. \quad (4.14)
$$

The second term in (4.11) is

$$
a_3\, \varphi_{i,x} = a_3(\frac{3\tau_1 + a_1 a_3 - a_2^2 + a_2 m_i - m_i^2}{a_1})\, \varphi_i = (a_{1,x} + 2(m_i - 2a_2 a_3) + a_2 m_i - m_i^2)\, \frac{a_1 a_3}{a_4}\, \varphi_i, \quad (4.15)
$$

and the last term in (4.11) can be used unmodified,

$$
(-a_4 m_i - a_{3,x} + 2(a_2 a_4 - a_3^2))\, \varphi_i \quad (4.16)
$$

Substituting (4.14), (4.15) and (4.16) into differential equation (4.11), all terms on the left collapse to zero, so the equation is satisfied identically.

This means that when φ_i satisfies (4.4) and the m_i satisfiy (4.5), functions $z_i = \varphi_i\, e^{m_i u}$ satisfies system (3.11). The constant τ_i are chosen so that cubic (4.5) has three distinct roots.

To get the first integral, the general solution u of (3.9) is obtained by setting $z = 0$ in (4.3). Thus $u(x)$ is obtained from the following expression where all $\alpha_i \neq 0$,

$$
\alpha_1\, \varphi_1(x)\, e^{m_1 u} + \alpha_2\, \varphi_2(x)\, e^{m_2 u} + \alpha_3\, \varphi_3(x)\, e^{m_3 u} = 0. \quad (4.17)
$$

Upon solving (4.17) $u(x)$ is now a function of x, so it may be differentiated with respect to x to produce

$$
\alpha_1\, (m_1 u_x\, \varphi_1 + \varphi_{1,x})\, e^{m_1 u} + \alpha_2\, (m_2 u_x \varphi_2 + \varphi_{2,x})\, e^{m_2 u} + \alpha_3\, (m_3 u_x \varphi_3 + \varphi_{3,x})\, e^{m_3 u} = 0. \quad (4.18)
$$

Multiply (4.17) by $(m_3 u_x\, \varphi_3 + \varphi_{3,x})$ and (4.18) by φ_3, calculate their difference, then the following is the result

$$
\alpha_1\, ((m_3 - m_1) u_x \varphi_1 \varphi_3 + \varphi_1 \varphi_{3,x} - \varphi_{1,x} \varphi_3)\, e^{m_1 u} + \alpha_2\, ((m_3 - m_2)\, u_x \varphi_2 \varphi_3 + \varphi_2 \varphi_{3,x} - \varphi_{2,x} \varphi_3)\, e^{m_2 u} = 0. \quad (4.19)
$$

With the $\{\omega_i\}_{i=1}^3$ defined in (4.6), (4.19) can be presented as

$$\frac{\alpha_1}{\omega_1}\, e^{m_1 u} = \frac{\alpha_2}{\omega_2}\, e^{m_2 u}. \tag{4.20}$$

In a similar way α_1 can be eliminated from the pair (4.17) and (4.18), and combined with (4.20). These results are summarized now as

$$\frac{\alpha_1}{\omega_1}\, e^{m_1 u} = \frac{\alpha_2}{\omega_2}\, e^{m_2 u} = \frac{\alpha_3}{\omega_3}\, e^{m_3 u}. \tag{4.21}$$

The three roots of cubic (4.5) must add to zero, so multiplying the three terms of the system in (4.21) and using $m_1 + m_2 + m_3 = 0$ gives

$$\frac{\alpha_1}{\omega_1}\frac{\alpha_2}{\omega_2}\frac{\alpha_3}{\omega_3}\, e^{(m_1+m_2+m_3)u} = \frac{\alpha_1\,\alpha_2\,\alpha_3}{\omega_1\omega_2\omega_3}. \tag{4.22}$$

Substituting (4.21) repeatedly, $e^{3m_i u}$ can be calculated for $i = 1, 2, 3$,

$$e^{3m_1 u} = \kappa\, \frac{\omega_1^3}{\alpha_1^3}, \qquad e^{3m_2 u} = \kappa\, \frac{\omega_2^3}{\alpha_2^3}, \qquad e^{3m_3 u} = \kappa\, \frac{\omega_3^3}{\alpha_3^3}, \tag{4.23}$$

where κ is defined to be

$$\kappa = \frac{\alpha_1\alpha_2\alpha_3}{\omega_1\omega_2\omega_3}. \tag{4.24}$$

The equations in (4.24) can be expressed equivalently by solving for the ω_i^3,

$$\omega_1^3 = \frac{\alpha_1^3}{\kappa}\, e^{3m_1 u}, \qquad \omega_2^3 = \frac{\alpha_2^3}{\kappa}\, e^{3m_2 u}, \qquad \omega_3^3 = \frac{\alpha_3^3}{\kappa}\, e^{3m_3 u}. \tag{4.25}$$

Raise the first expression in (4.25) to the power $m_3 - m_2$, the second to the power $m_1 - m_3$ and the last to $m_2 - m_1$ and forming the product, it is found to be a product of the α_i raised to a power

$$\omega_1^{3(m_3-m_2)} \cdot \omega_2^{3(m_1-m_3)} \cdot \omega_3^{3(m_2-m_1)} = \alpha_1^{3(m_3-m_2)} \cdot \alpha_2^{3(m_1-m_2)} \cdot \alpha_3^{3(m_2-m_1)}. \tag{4.26}$$

The last thing to do is to take implicit solution $u(x)$ from (4.17) and require $y = u_x$. $\square$

5. Constructing Algebraic Solutions for an Abel Equation.

The following Abel differential equation composed of linear powers of v has some interesting properties and is investigated here:

$$v\,\frac{dv}{dx} = b_1 + b_2\, v, \quad b_2 \neq 0, \quad b_1, b_2 \in \mathbb{C}(x). \tag{5.1}$$

Theorem 5.1. Let $n > 1$ and $\{f_i\}_{1\le i\le n} \in \mathbb{C}(x)$, and let F be a nontrivial polynomial of degree $n > 1$ of the form

$$F(x, v) = v^n + f_1\, v^{n-1} + f_2\, v^{n-2} + \cdots + f_{n-1}\, v + f_n. \tag{5.2}$$

For all the roots of (5.2) to satisfy (5.1), it is necessary that the following differential equations be satisfied:

$$\begin{aligned} f_{2,x}(x) - (b_2 + f_{1,x})\, f_1 + nb_1 &= 0, \\ f_{3,x}(x) - (2b_2 + f_{1,x})\, f_2 + (n-1)\, b_1\, f_1 &= 0, \\ f_{4,x}(x) - (3b_2 + f_{1,x})\, f_3 + (n-2) b_1\, f_2 &= 0, \\ &\vdots \\ f_{n-1,x}(x) - ((n-2)b_2 + f_{1,x})\, f_{n-2} + 3b_1 f_{n-3} &= 0, \\ f_{n,x}(x) - ((n-1)b_2 + f_{1,x})\, f_{n-1} + 2b_1 f_{n-2} &= 0, \\ f_{n-1}(x)\, b_1 - (nb_2 + f_{1,x})\, f_n &= 0. \end{aligned} \tag{5.3}$$

Proof. Let $(v_i(x))_{1 \le i \le n}$ be the roots of the $F(x, v) = 0$ where F is defined by (5.2). A polynomial vanishes only at its roots and there are exactly n roots, where n is the degree of the polynomial. Setting $v = v_i(x)$ in (5.2) yields

$$F(x, v_i(x)) = v_i^n + f_1\, v_i^{n-1} + f_2\, v_2^{n-2} + \cdots + f_{n-1}\, v_i + f_n = 0 \tag{5.4}$$

Each $v_i(x)$ is constant with respect to differentiation with respect to v at fixed x. Differentiating (5.4) with respect to x and multiplying both sides by v_i, we get

$$\begin{aligned} & v_i \frac{\partial F}{\partial x}(x, v_i) + \frac{\partial v}{\partial x} \frac{\partial F}{\partial v}(x, v_i) \\ &= (nb_2 + f_{1,x})\, v^n + (nb_1 + (n-1)b_2 f_1 + f_{2,x})\, v_i^{n-1} + ((n-1)\, b_1 f_1 + (n-2) b_2 f_2 + f_{3,x})\, v_i^{n-2} \\ &\quad + ((n-2) b_1 f_2 + (n-3) b_2 f_2 + f_{4,x})\, v_i^{n-3} + \cdots + (3b_1 f_{n-3} + 2b_2 f_{n-2} + f_{n-1,x})\, v_i^2 \\ &\quad + (2b_1 f_{n-2} + b_2 f_{n-1} + f_{n,x})\, v_i + b_1\, f_{n-1}. \end{aligned} \tag{5.5}$$

If $P(x, v) = 0$ then all its coefficients vanish. This gives in addition to the constraint $f_{n-1} = 0$, n differential equations for the f_k. If the opposite holds $P(x, v) \neq 0$, then the functions $v_1, \ldots, v_n$ which are also roots of $F(x, v)$ are roots of $P(x, v)$ as well. When $F(x, v)$ is non-trivial, so that $f_{1,x} + nb_2 \neq 0$, then $P(x, v_i)$ must be proportional to $F(x, v_i)$ and the proportionality factor is $nb_2 + f_{1,x}$. Collecting like powers of v_i in this, their coefficients produce system (5.3). □

It is worth developing results for the cases in which F is linear or quadratic. The results for the quadratic case are the same as those obtained from (5.3).

Theorem 5.2. (a) Supppose (5.4) is a linear polynomial $F(x, v) = v - f(x)$. Then one of the b_i in (5.1) can be expressed in terms of the other and (5.1) such as

$$v \frac{dv}{dx} = b_1 + (f_x - \frac{b_1}{f})\, v. \tag{5.6}$$

(b) Suppose $F(x,v) = v^2 + 2f\,v + g$ be quadratic with $f^2 \neq g$. Then the two roots of $F(x,v) = 0$ satisfy (5.1) where b_1, b_2 are given by

$$b_1 = -\frac{1}{2}\,g\,\frac{2f\,f_x - g_x}{f^2 - g}, \qquad b_2 = -\frac{1}{2}\,\frac{4f^2 f_x - f\,g_x - 2f_x g}{f^2 - g}. \tag{5.7}$$

Proof. (a) Substitute the root $v_1 = f(x)$ into (5.1) and solve for the term b_2, for example. Then in terms of b_1, we have $b_2 = f_x - b_1/f$. Putting b_2 into (5.1) we obtain (5.6).

(b) The roots of the quadratic $F(x,v)$ can be computed explicitly, $v_{1,2} = -f(x) \pm \sqrt{f(x)^2 - g(x)}$ and provided $f^2 \neq g$, the roots are distinct. Substitute the roots into (5.1) to produce a system of two equations for b_1, b_2 in (5.1). Solving this system produces results (5.7).

□

Two examples are presented now to show how to apply the results and are not hard to check. Let $f(x) = -2(x+1)(2x-1)$ and $g(x) = (x+1)^2(x-1)(8x-1)$ in $F(x,v)$ where the roots are $v_{1,2} = (x+1)(4x - 2 \pm \sqrt{3-6x})$. Substituting f and g into (5.7) gives b_1 and b_2, so (5.1) takes the form

$$v\,\frac{dv}{dx} = -3x(x+1)(2x-1) + (14x-2)\,v.$$

A quadratic F is defined by taking $f = -(x^2+2x+1)$ and $g = x^3 + 3x + 1$. The roots of the polynomial are $v_{1,2} = -x^2 - 2x - 1 \pm \sqrt{x^4 + 3x^3 + 6x^2 + x}$ and putting f and g into (5.7) we get

$$v\,\frac{dv}{dx} = -\frac{(x^3+3x+1)(4x^3+9x^2+12x+1)}{2x(x^3+3x^2+6x+1)} - \frac{(x+1)(8x^4+25x^3+45x^2+17x+1)}{2x(x^3+3x^2+6x+1)}\,v.$$

We present system (5.3) for the case $n = 3$ where $F(x,v) = v^3 + f_1 v^2 + f_2 v + f_3$. The system is

$$f_{2,x} - (b_2 + f_{1,x})\,f_1 + 3b_2 = 0, \quad f_{3,x} - (2b_2 + f_{1,x})\,f_2 + 2b_1 = 0, \quad f_2 b_1 - (3b_2 + f_{1,x})\,f_3 = 0.$$

An example of a cubic $F(x,v) = v^3 + f_1(x)v^2 - (4/27)\,(f_1(x))^3$ has roots which satisfy (5.1) given by,

$$v\,\frac{dv}{dx} = \frac{1}{9}(f_1(x))_x^2 - \frac{1}{2}\,f_{1,x}(x)\,v.$$

Theorem 5.3. Consider Abel equation (5.1) where b_1, b_2 are determined by (5.7) and the roots of $v^2 + 2fv + g$ are distinct $f^2 \neq g$. If $k > 1$ and b_2 can be factored with respect to $(\beta_i)_{1 \leq j \leq k-1}$ distinct complex numbers

$$b_2 = c\,(x - \beta_1)\dots(x - \beta_{k-1}) \tag{5.8}$$

then the remaining functions b_1, f and g can be found explicitly.

Proof. By differentiation it may be verified that b_1 and b_2 may be expressed using $\Delta = 4(f^2 - g)$ as

$$b_1 = -\frac{g}{2}\,\frac{d\log(\Delta)}{dx}, \qquad b_2 = \frac{1}{2}f\,\frac{d\log(f^2\Delta)}{dx}. \tag{5.9}$$

By assumption b_2 in (5.8) admits a pole at ∞ and when f is polynomial it can be factored as well

$$f = -\zeta(x - r_1)(x - r_2)\cdots(x - r_k) \tag{5.10}$$

where $r_k \in \mathbb{C}$ and ζ non-zero. Using (5.8) in (5.9) and integrating we solve the second equation in (5.9) for $\log(f^2\Delta)$

$$\log(f^2\Delta) = \frac{2c}{\zeta}\int \frac{(x-\beta_1)(x-\beta_2)\cdots(x-\beta_{k-1})}{(x-r_1)(x-r_2)\cdots(x-r_k)}\,dx.$$

The roots r_i can be chosen to be distinct from the set $(\beta_i)_{i=1}^{k-1}$, so a partial fraction decomposition can be produced with $(s_i)_{1\leq i\leq 4} \in \mathbb{C}$

$$\frac{2c}{\zeta}\,\frac{(x-\beta_1)(x-\beta_2)\cdots(x-\beta_{k-1})}{(x-r_1)(x-r_2)\cdots(x-r_k)} = \frac{s_1}{x-r_1} + \frac{s_2}{x-r_2}\cdots + \frac{s_k}{x-r_k}, \tag{5.11}$$

This means Δ admits the factorization

$$\Delta = 4C\,(x-r_1)^{s_1-2}\cdots(x-r_k)^{s_k-2}.$$

By hypothesis $s_i \in \mathbb{N}_{\geq 2}$ since $\Delta \in \mathbb{C}[x]$. Solving for g in terms of Δ yields,

$$g = f^2 - \frac{\Delta}{4} = a^2\,(x-r_1)^2\cdots(x-r_k)^2 - C\,(x-r_1)^{s_1-2}\cdots(x-r_k)^{s_k-2}.$$

Introduce $\Lambda = (f^2/2)\,(\log(\Delta))_x$ where

$$(\log(\Delta))_x = \sum_{j=1}^{k}\frac{s_j - 2}{x - r_j}.$$

then Λ can be computed as

$$\Lambda = \frac{a^2}{2}(x-r_1)^2(x-r_2)^2\cdots(x-r_k)^2\cdot\Big(\sum_{j=1}^{k}\frac{s_j-2}{x-r_j}\Big).$$

This permits us to calculate b_1 as

$$b_1 = -\Lambda\,\frac{g}{f^2} = -\Lambda\,\Big(1 - \frac{C}{a^2}(x-r_1)^{s_1-4}\cdots(x-r_k)^{s_k-4}\Big).$$

This equation requires that $s_i \in \mathbb{N}_{\geq 3}$ for $1 \leq i \leq k$. Substitute $x = \beta_i$ into partial fraction decomposition (5.11) to produce the following system of equations $1 \leq i \leq k$,

$$\frac{s_1}{\beta_i - r_1} + \frac{s_2}{\beta_i - r_2} + \cdots + \frac{s_k}{\beta_i - r_k} = 0,$$

Multiply the partial fraction decomposition (5.11) by $(x-r_1)\cdots(x-r_k)$ and collect the coefficient of x^{k-1} on both sides of the equality. This produces a equation that relates c and ζ

$$c = \frac{\zeta}{2}\,(s_1 + \cdots + s_k). \tag{5.12}$$

□

It is worth remarking that the Abel equation of the first kind of the form $y_x = -b_1 y^3 + b_2 y^2$ with $b_1, b_2 \in \mathbb{C}[x]$. can be studied an upper bound to its rational solutions can be located according to $\deg(b_1)$. Also the substitution $y = v^{-1}$ maps this to an equation of the form (5.1).

6. Conclusions.

To summarize the main results, algebraic integrability of some Abel differential equations of the first and second kind has been investigated here. An Abel equation can be associated to a Riccati equation when its Liouville invariant given by (2.2) vanishes. When L is not zero a non-autonomous Liénard equation can be associated to the Abel equation of the first kind. Louvillian integrability of of the Abel equation is equivalent to Liouvillian integrability of the corresponding Liénard equation. Abel equations will continue to have many applications and be worthy of study. For example, such an equation can result geometrically. To see this, let $P(x) = (G(x), H(x))$ and $Q(x) = (I(x), J(x))$ be two possible rational curves. For given $a(x), b(x), c(x), p(x) \in \mathbb{C}(x)$ find a function $\xi(x)$ such that if

$$U(x) = (G(x) + a(x)\,\xi(x), H(x) + b(x)\,\xi(x))$$

then $U(x)N(x)$ is parallel to $(c(x)(G(x)+a(x)\xi(x))_x, p(x)(H(x)+b(x)\xi(x)_x)$ for any x. The condition that ray $\overrightarrow{U(x)N(x)} = (I(x) - G(x) - a(x)\xi(x), J(x) - H(x)\xi(x))$ be parallel to $(c(x)(G(x)+a(x)\xi(x))_x, p(x)(H(x)+b(x)\xi(x))$ is exactly that

$$\frac{J(x) - H(x) - b(x)\xi(x)}{I(x) - G(x) - a(x)\xi(x)} = \frac{p(x)(H(x)+b(x)\xi(x))_x}{c(x)(G(x)+a(x)\xi(x))_x}.$$

This is equivalent to an Abel equation of the second kind

$$(b_5(x) + b_6(x)\xi)\xi_x + b_1(x) + 3b_2(x)\xi + 3b_3(x)\,\xi^2 + b_4(x)\,\xi^3 = 0$$

provided the $(b_k)_{k=1}^6$ are given as $b_1 = (I-G)\cdot p\cdot H_x - (J-H)\,cG_x$, $3b_2 = (I-G)\,p\,b_x - (J-H)\,ca_x + b\,c\,G_x - ap\,H_x$, $3b_3 = bca_x - ap\,b_x$, $b_4 = 0$, $b_5 = (I-G)\,b\,p - (J-H)\,a\,c$ and $b_6 = a\,b(c-p)$, where $a, b, p \neq 0$. It is very likely more applications of these equations will emerge in the future.

□

Conflict of Interest Statement: I declare all work invested in this paper to be mine and there are no conflict of interests.

References

[1] N. Abel. *Sur l'équation* $(y + s)\,dy + (p + qy + xy^2)\,dx = 0$*, Oeuvres Complète de N. H. Abel.* Cambridge University Press, Cambridge, 2012.

[2] P. Appell. Sur les invariants de quelques équations differentials. *J. Math. Pure Appl. (4)*, V:361–423, 1889.

[3] P. Appell. Sur les fonctions hypergéometriques de deux variables. *J. Math. Pure Appl.*, 3 (8):173–216, 1882.

[4] E. Cartan. Sur les variés à connexion projective. *Bull. Soc. Math Fr.*, 52:205–241, 1924.

[5] R. Liouville. Sur une classe d'équations différentielles du premier ordre et les formations invariants qui s'y rapportent. *C. R. Acad. Sci.*, 105:460–463, 1887.

[6] R. Liouville. Sur une classe d'équations différentialles non-linéaires. *C. R. Acad. Sci.*, 103:520–528, 1887.

[7] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, vol. I-II.* Interscience, New York, 1969.

[8] E. L. Ince. *Ordinary Differential Equations.* Dover Publications, New York, 1956.

[9] O. Wone. Abel ordinary differential equations differential algebra and projective connections. *Bull. des Sci. Math.*, 199:103567, 2025.

[10] N. H. Ibragimov and F. Magri. Geometric proof of lie's linearization theorem. *Nonlinear Dynamics*, 36(1):41–46, 2004.

[11] S. S. Chern, W. H. Chen, and K. S. Lam. *Lectures on Differential Geometry.* World Scientific, Singapore, 1999.

[12] N. Hintchin. Complex manifolds and einstein equations. *Lecture Notes in Mathematics*, 970:73–88, 1982.

[13] A. Hénault. Sur la linéarization des tissus de ^{2}. *Topology*, 32(3):531–542, 1993.

[14] B. Chow and Y. Chow. *Lectures on Differential Geometry.* Graduate Studies in Mathematics, vol. 245. AMS, Providence, RI, 2024.