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Closed-Form Solutions and Periodic Dynamics of a Four-Dimensional Eighth-Order System of Rational Difference Equations

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Abstract. This paper investigates the following eighth-order nonlinear system of rational difference equations for different cases:

$$\begin{cases} P_{n+1} = \frac{P_{n-7}}{\pm 1 \pm Q_{n-1}R_{n-3}S_{n-5}P_{n-7}}, \\ Q_{n+1} = \frac{Q_{n-7}}{\pm 1 \pm R_{n-1}S_{n-3}P_{n-5}Q_{n-7}}, \\ R_{n+1} = \frac{R_{n-7}}{\pm 1 \pm S_{n-1}P_{n-3}Q_{n-5}R_{n-7}}, \\ S_{n+1} = \frac{S_{n-7}}{\pm 1 \pm P_{n-1}Q_{n-3}R_{n-5}S_{n-7}}, \end{cases}$$

where n is a non-negative integer and the initial conditions $P_{-7}, ..., P_0, Q_{-7}, ..., Q_0, R_{-7}, ..., R_0$, $S_{-7}, ..., S_0$ are arbitrary real numbers. The study derives explicit closed-form solutions under specific conditions. These solutions facilitate rigorous analysis and provide an exact description of the long-term behavior of the system. Finally, numerical simulations are presented to illustrate key dynamical properties, including boundedness, periodicity, and sensitivity to initial conditions.

2020 Mathematics Subject Classifications: 39A10

Key Words and Phrases: Difference equations, systems of difference equations, recursive sequences, periodic solutions, numerical simulation

1. Introduction

Difference equations are an essential part of modern discrete mathematics due to their significance in modeling discrete-time evolving systems. They are naturally derived as discrete analogs of differential and delay differential equations and have been extensively

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used in a variety of fields such as population studies, epidemiology, ecology, economics, finance, genetics, neural networks, and engineering disciplines [1–3].

In recent decades, nonlinear rational difference equations and their systems have attracted considerable attention due to the richness and complexity of their dynamical behavior. Such systems may exhibit a wide range of qualitative properties, including boundedness, periodicity, stability, and chaos. Understanding these behaviors is essential for analyzing the long-term evolution of discrete dynamical systems.

A central topic in the study of difference equations is the determination of closed-form solutions, particularly for nonlinear systems. The derivation of explicit formulas provides significant insight into both the qualitative and quantitative behavior of the system. In particular, closed-form expressions facilitate the analysis of stability, the identification of periodic patterns, and a deeper understanding of the underlying dynamical structure. Consequently, the search for analytical solutions of nonlinear difference equations remains an active area of research.

Several researchers have investigated the analytical solutions and qualitative behavior of nonlinear rational systems of difference equations. For instance, the authors in [4] studied periodic solutions of a nonlinear system and derived explicit expressions describing the evolution of the system under arbitrary real initial conditions. The author in [5] analyzed a class of fourth-order rational systems in a four-dimensional setting, examining the existence of solutions and qualitative properties such as periodicity, and supporting their results with numerical simulations. In [6], explicit solution formulas were obtained for nonlinear systems, and periodic behavior with different periods was established and verified numerically. Furthermore, the authors in [7] investigated systems of order $6k + 3$, proposing a systematic approach for deriving closed-form solutions and analyzing several distinct cases. The study in [8] focused on the solvability in closed form of three-dimensional nonlinear systems, where explicit solutions were obtained using analytical transformations. More recently, the authors in [9] examined a three-dimensional second-order rational system, deriving explicit solutions and illustrating various dynamical behaviors through numerical simulations. These works contribute to a deeper understanding of the analytical structure and dynamics of nonlinear difference systems. For further related studies, see [10–17].

The remainder of this paper is organized as follows. Section 2 introduces the first system, derives its closed-form solutions, and studies their basic properties, including boundedness and non-increasing behavior, as well as the case where one of the initial values is zero. Section 3 presents the second system, obtains its closed-form solutions, and examines the same case. Section 4 analyzes the third system, derives its closed-form solutions, determines the conditions for periodic behavior, and examines the case where one of the initial values is zero. Section 5 studies the fourth system in a similar manner, including its solutions and periodicity conditions, and also considers the case where one of the initial values is zero. Section 6 presents numerical simulations to illustrate and verify the theoretical results. Finally, Section 7 summarizes the main conclusions of the paper.

2. Formulation of the First System

In this section, we examine the behavior and solution of the following four-dimensional nonlinear system:

$$\begin{cases} P_{n+1} = \frac{P_{n-7}}{1+Q_{n-1}R_{n-3}S_{n-5}P_{n-7}}, \\ Q_{n+1} = \frac{Q_{n-7}}{1+R_{n-1}S_{n-3}P_{n-5}Q_{n-7}}, \\ R_{n+1} = \frac{R_{n-7}}{1+S_{n-1}P_{n-3}Q_{n-5}R_{n-7}}, \\ S_{n+1} = \frac{S_{n-7}}{1+P_{n-1}Q_{n-3}R_{n-5}S_{n-7}}, \end{cases} \tag{1}$$

where $n \in \mathbb{N}_0$ and the initial conditions $P_{-7}, ..., P_0, Q_{-7}, ..., Q_0, R_{-7}, ..., R_0, S_{-7}, ..., S_0$ are arbitrary non-negative real numbers.

Theorem 1. *Let $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ be a solution of system (1). Then, for every non-negative integer n, the solution is given by the following closed-form expressions:*

$$P_{8n-7} = P_{-7} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)\, P_{-7}Q_{-1}R_{-3}S_{-5}+1)}, \ P_{8n-6} = P_{-6} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+1)\, P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$P_{8n-5} = P_{-5} \prod_{i=0}^{n-1} \frac{((4i+1)\, P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+2)\, P_{-5}Q_{-7}R_{-1}S_{-3}+1)}, \ P_{8n-4} = P_{-4} \prod_{i=0}^{n-1} \frac{((4i+1)\, P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+2)\, P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$P_{8n-3} = P_{-3} \prod_{i=0}^{n-1} \frac{((4i+2)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+3)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}, \ P_{8n-2} = P_{-2} \prod_{i=0}^{n-1} \frac{((4i+2)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+3)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$P_{8n-1} = P_{-1} \prod_{i=0}^{n-1} \frac{((4i+3)\, P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+4)\, P_{-1}Q_{-3}R_{-5}S_{-7}+1)}, \ P_{8n} = P_{0} \prod_{i=0}^{n-1} \frac{((4i+3)\, P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+4)\, P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$Q_{8n-7} = Q_{-7} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)\, P_{-5}Q_{-7}R_{-1}S_{-3}+1)}, \ Q_{8n-6} = Q_{-6} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+1)\, P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$Q_{8n-5} = Q_{-5} \prod_{i=0}^{n-1} \frac{((4i+1)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+2)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}, \ Q_{8n-4} = Q_{-4} \prod_{i=0}^{n-1} \frac{((4i+1)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+2)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$Q_{8n-3} = Q_{-3} \prod_{i=0}^{n-1} \frac{((4i+2)\, P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+3)\, P_{-1}Q_{-3}R_{-5}S_{-7}+1)}, \ Q_{8n-2} = Q_{-2} \prod_{i=0}^{n-1} \frac{((4i+2)\, P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+3)\, P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$Q_{8n-1} = Q_{-1} \prod_{i=0}^{n-1} \frac{((4i+3)\, P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+4)\, P_{-7}Q_{-1}R_{-3}S_{-5}+1)}, \ Q_{8n} = Q_{0} \prod_{i=0}^{n-1} \frac{((4i+3)\, P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+4)\, P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$R_{8n-7} = R_{-7} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)\, P_{-3}Q_{-5}R_{-7}S_{-1}+1)}, \ R_{8n-6} = R_{-6} \prod_{i=0}^{n-1} \frac{((4i)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+1)\, P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$R_{8n-5}=R_{-5}\prod_{i=0}^{n-1}\frac{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+2)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ R_{8n-4}=R_{-4}\prod_{i=0}^{n-1}\frac{((4i+1)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+2)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$R_{8n-3}=R_{-3}\prod_{i=0}^{n-1}\frac{((4i+2)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+3)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ R_{8n-2}=R_{-2}\prod_{i=0}^{n-1}\frac{((4i+2)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+3)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$R_{8n-1}=R_{-1}\prod_{i=0}^{n-1}\frac{((4i+3)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+4)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ R_{8n}=R_{0}\prod_{i=0}^{n-1}\frac{((4i+3)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+4)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$S_{8n-7}=S_{-7}\prod_{i=0}^{n-1}\frac{((4i)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ S_{8n-6}=S_{-6}\prod_{i=0}^{n-1}\frac{((4i)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+1)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$S_{8n-5}=S_{-5}\prod_{i=0}^{n-1}\frac{((4i+1)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+2)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ S_{8n-4}=S_{-4}\prod_{i=0}^{n-1}\frac{((4i+1)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+2)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$S_{8n-3}=S_{-3}\prod_{i=0}^{n-1}\frac{((4i+2)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+3)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ S_{8n-2}=S_{-2}\prod_{i=0}^{n-1}\frac{((4i+2)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+3)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$S_{8n-1}=S_{-1}\prod_{i=0}^{n-1}\frac{((4i+3)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+4)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},\ S_{8n}=S_{0}\prod_{i=0}^{n-1}\frac{((4i+3)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+4)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}.$$

Proof. The result is immediate for $n=0$. For $n>0$, assume by induction that it holds for $n-1$ and that it can be written in the following form:

$$P_{8n-15}=P_{-7}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ P_{8n-14}=P_{-6}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+1)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$P_{8n-13}=P_{-5}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+2)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ P_{8n-12}=P_{-4}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+2)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$P_{8n-11}=P_{-3}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+3)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},\ P_{8n-10}=P_{-2}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+3)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$P_{8n-9}=P_{-1}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+4)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ P_{8n-8}=P_{0}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+4)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$Q_{8n-15}=Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ Q_{8n-14}=Q_{-6}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+1)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$Q_{8n-13}=Q_{-5}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+2)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},\ Q_{8n-12}=Q_{-4}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+2)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$Q_{8n-11}=Q_{-3}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+3)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ Q_{8n-10}=Q_{-2}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+3)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$Q_{8n-9}=Q_{-1}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+4)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ Q_{8n-8}=Q_{0}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+4)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$R_{8n-15}=R_{-7}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},\ R_{8n-14}=R_{-6}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+1)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)},$$

$$R_{8n-13}=R_{-5}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+2)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ R_{8n-12}=R_{-4}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+2)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$R_{8n-11}=R_{-3}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+3)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ R_{8n-10}=R_{-2}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+3)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$R_{8n-9}=R_{-1}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+4)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ R_{8n-8}=R_{0}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+4)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$S_{8n-15}=S_{-7}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)},\ S_{8n-14}=S_{-6}\prod_{i=0}^{n-2}\frac{((4i)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)}{((4i+1)\,P_{0}Q_{-2}R_{-4}S_{-6}+1)},$$

$$S_{8n-13}=S_{-5}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+2)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)},\ S_{8n-12}=S_{-4}\prod_{i=0}^{n-2}\frac{((4i+1)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)}{((4i+2)\,P_{-6}Q_{0}R_{-2}S_{-4}+1)},$$

$$S_{8n-11}=S_{-3}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+3)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},\ S_{8n-10}=S_{-2}\prod_{i=0}^{n-2}\frac{((4i+2)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)}{((4i+3)\,P_{-4}Q_{-6}R_{0}S_{-2}+1)},$$

$$S_{8n-9}=S_{-1}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+4)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},\ S_{8n-8}=S_{0}\prod_{i=0}^{n-2}\frac{((4i+3)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}{((4i+4)\,P_{-2}Q_{-4}R_{-6}S_{0}+1)}.$$

Now, it follows from system (1) that

$$\begin{aligned}
P_{8n-7} &= \frac{P_{8n-15}}{1+Q_{8n-9}R_{8n-11}S_{8n-13}P_{8n-15}}\\
&= \frac{\left(P_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)}{1+\left(\begin{array}{c}\left(Q_{-1}\prod\limits_{i=0}^{n-2}\frac{((4i+3)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+4)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)\cdot\left(R_{-3}\prod\limits_{i=0}^{n-2}\frac{((4i+2)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+3)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)\cdot\\ \left(S_{-5}\prod\limits_{i=0}^{n-2}\frac{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+2)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)\cdot\left(P_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)\end{array}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(P_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)}{1+\left(P_{-7}Q_{-1}R_{-3}S_{-5}\prod_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+4)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)} \\
&= \frac{\left(P_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)}{1+\left(\frac{P_{-7}Q_{-1}R_{-3}S_{-5}}{((4n-4)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)} \\
&= \frac{((4n-4)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)\left(P_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\right)}{(4n-4)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1+P_{-7}Q_{-1}R_{-3}S_{-5}} \\
&= \frac{P_{-7}\left((4n-4)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1\right)}{((4n-3)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)} \\
&= P_{-7}\prod_{i=0}^{n-1}\frac{((4i)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}{((4i+1)\,P_{-7}Q_{-1}R_{-3}S_{-5}+1)}
\end{aligned}
$$

$$
\begin{aligned}
Q_{8n-7} &= \frac{Q_{8n-15}}{1+R_{8n-9}S_{8n-11}P_{8n-13}Q_{8n-15}} \\
&= \frac{\left(Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)}{1+\left(\begin{array}{c}\left(R_{-1}\prod_{i=0}^{n-2}\frac{((4i+3)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+4)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)\cdot\left(S_{-3}\prod_{i=0}^{n-2}\frac{((4i+2)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+3)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)\cdot \\ \left(P_{-5}\prod_{i=0}^{n-2}\frac{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+2)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)\cdot\left(Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)\end{array}\right)} \\
&= \frac{\left(Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)}{1+\left(P_{-5}Q_{-7}R_{-1}S_{-3}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+4)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)} \\
&= \frac{\left(Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)}{1+\left(\frac{P_{-5}Q_{-7}R_{-1}S_{-3}}{((4n-4)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)} \\
&= \frac{((4n-4)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)\left(Q_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\right)}{(4n-4)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1+P_{-5}Q_{-7}R_{-1}S_{-3}} \\
&= \frac{Q_{-7}\left((4n-4)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1\right)}{((4n-3)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}
\end{aligned}
$$

$$= Q_{-7}\prod_{i=0}^{n-1}\frac{((4i)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)}{((4i+1)\,P_{-5}Q_{-7}R_{-1}S_{-3}+1)},$$

$$\begin{aligned}
R_{8n-7} &= \frac{R_{8n-15}}{1+S_{8n-9}P_{8n-11}Q_{8n-13}R_{8n-15}}\\
&= \frac{\left(R_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}{1+\left(\begin{array}{c}\left(S_{-1}\prod\limits_{i=0}^{n-2}\frac{((4i+3)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+4)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)\cdot\left(P_{-3}\prod\limits_{i=0}^{n-2}\frac{((4i+2)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+3)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)\cdot\\ \left(Q_{-5}\prod\limits_{i=0}^{n-2}\frac{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+2)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)\cdot\left(R_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)\end{array}\right)}\\
&= \frac{\left(R_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}{1+\left(P_{-3}Q_{-5}R_{-7}S_{-1}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+4)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}\\
&= \frac{\left(R_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}{1+\left(\frac{P_{-3}Q_{-5}R_{-7}S_{-1}}{((4n-4)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}\\
&= \frac{((4n-4)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)\left(R_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\right)}{(4n-4)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1+P_{-3}Q_{-5}R_{-7}S_{-1}}\\
&= \frac{R_{-7}\,((4n-4)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4n-3)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}\\
&= R_{-7}\prod_{i=0}^{n-1}\frac{((4i)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)}{((4i+1)\,P_{-3}Q_{-5}R_{-7}S_{-1}+1)},
\end{aligned}$$

$$\begin{aligned}
S_{8n-7} &= \frac{S_{8n-15}}{1+P_{8n-9}Q_{8n-11}R_{8n-13}S_{8n-15}}\\
&= \frac{\left(S_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)}{1+\left(\begin{array}{c}\left(P_{-1}\prod\limits_{i=0}^{n-2}\frac{((4i+3)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+4)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)\cdot\left(Q_{-3}\prod\limits_{i=0}^{n-2}\frac{((4i+2)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+3)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)\cdot\\ \left(R_{-5}\prod\limits_{i=0}^{n-2}\frac{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+2)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)\cdot\left(S_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)\end{array}\right)}\\
&= \frac{\left(S_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)}{1+\left(P_{-1}Q_{-3}R_{-5}S_{-7}\prod\limits_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+4)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(S_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)}{1+\left(\frac{P_{-1}Q_{-3}R_{-5}S_{-7}}{((4n-4)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)} \\
&= \frac{((4n-4)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)\left(S_{-7}\prod_{i=0}^{n-2}\frac{((4i)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\right)}{(4n-4)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1+P_{-1}Q_{-3}R_{-5}S_{-7}} \\
&= \frac{S_{-7}\,((4n-4)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4n-3)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}\prod_{i=0}^{n-2}\frac{((4i)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)} \\
&= S_{-7}\prod_{i=0}^{n-1}\frac{((4i)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}{((4i+1)\,P_{-1}Q_{-3}R_{-5}S_{-7}+1)}.
\end{aligned}
$$

Similarly, the other relations can also be established. Therefore, by induction, the stated formulas hold for all $n \geq 0$.

Lemma 1. *Let $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ be a solution of system (1). Then every solution of system (1) is bounded and consists of non-increasing subsequences.*

Proof. It follows from system (1) that

$$0 \leq P_{n+1} = \frac{P_{n-7}}{1+Q_{n-1}R_{n-3}S_{n-5}P_{n-7}} \leq P_{n-7}$$

$$0 \leq Q_{n+1} = \frac{Q_{n-7}}{1+R_{n-1}S_{n-3}P_{n-5}Q_{n-7}} \leq Q_{n-7}$$

$$0 \leq R_{n+1} = \frac{R_{n-7}}{1+S_{n-1}P_{n-3}Q_{n-5}R_{n-7}} \leq R_{n-7}$$

$$0 \leq S_{n+1} = \frac{S_{n-7}}{1+P_{n-1}Q_{n-3}R_{n-5}S_{n-7}} \leq S_{n-7}$$

Then, the subsequences $\{P_{8n-7}\}_{n=0}^{\infty}$, $\{P_{8n-6}\}_{n=0}^{\infty}$,$\{P_{8n-5}\}_{n=0}^{\infty}$,$\{P_{8n-4}\}_{n=0}^{\infty}$,$\{P_{8n-3}\}_{n=0}^{\infty}$,$\{P_{8n-2}\}_{n=0}^{\infty}$, $\{P_{8n-1}\}_{n=0}^{\infty}$, and $\{P_{8n}\}_{n=0}^{\infty}$ are non-increasing and bounded above by

$$L = \max\{P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0\}.$$

Similarly, the subsequences $\{Q_{8n-7}\}_{n=0}^{\infty}$, $\{Q_{8n-6}\}_{n=0}^{\infty}$,$\{Q_{8n-5}\}_{n=0}^{\infty}$,$\{Q_{8n-4}\}_{n=0}^{\infty}$,$\{Q_{8n-3}\}_{n=0}^{\infty}$,$\{Q_{8n-2}\}_{n=0}^{\infty}$, $\{Q_{8n-1}\}_{n=0}^{\infty}$, and $\{Q_{8n}\}_{n=0}^{\infty}$ are non-increasing and bounded above by

$$M = \max\{Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0\}.$$

Moreover, $\{R_{8n-7}\}_{n=0}^{\infty}$, $\{R_{8n-6}\}_{n=0}^{\infty}$,$\{R_{8n-5}\}_{n=0}^{\infty}$,$\{R_{8n-4}\}_{n=0}^{\infty}$,$\{R_{8n-3}\}_{n=0}^{\infty}$,$\{R_{8n-2}\}_{n=0}^{\infty}$, $\{R_{8n-1}\}_{n=0}^{\infty}$, and $\{R_{8n}\}_{n=0}^{\infty}$ are non-increasing and bounded above by

$$N = \max\{R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0\}.$$

Furthermore, $\{S_{8n-7}\}_{n=0}^{\infty}$, $\{S_{8n-6}\}_{n=0}^{\infty}$,$\{S_{8n-5}\}_{n=0}^{\infty}$,$\{S_{8n-4}\}_{n=0}^{\infty}$,$\{S_{8n-3}\}_{n=0}^{\infty}$,$\{S_{8n-2}\}_{n=0}^{\infty}$, $\{S_{8n-1}\}_{n=0}^{\infty}$, and $\{S_{8n}\}_{n=0}^{\infty}$ are non-increasing and bounded above by

$$K = \max\{S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0\}.$$

Corollary 1. *Let* $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ *be a solution of system* (1)*. Then the following statements hold:*

(i) If $P_{-7} = 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $P_{8n-7} = 0$, $Q_{8n-1} = Q_{-1}$, $R_{8n-3} = R_{-3}$, $S_{8n-5} = S_{-5}$.

(ii) If $P_{-6} = 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $P_{8n-6} = 0$, $Q_{8n} = Q_0$, $R_{8n-2} = R_{-2}$, $S_{8n-4} = S_{-4}$.

(iii) If $P_{-5} = 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $P_{8n-5} = 0$, $Q_{8n-7} = Q_{-7}$, $R_{8n-1} = R_{-1}$, $S_{8n-3} = S_{-3}$.

(iv) If $P_{-4} = 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $P_{8n-4} = 0$, $Q_{8n-6} = Q_{-6}$, $R_{8n} = R_0$, $S_{8n-2} = S_{-2}$.

(v) If $P_{-3} = 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $P_{8n-3} = 0$, $Q_{8n-5} = Q_{-5}$, $R_{8n-7} = R_{-7}$, $S_{8n-1} = S_{-1}$.

(vi) If $P_{-2} = 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $P_{8n-2} = 0$, $Q_{8n-4} = Q_{-4}$, $R_{8n-6} = R_{-6}$, $S_{8n} = S_0$.

(vii) If $P_{-1} = 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $P_{8n-1} = 0$, $Q_{8n-3} = Q_{-3}$, $R_{8n-5} = R_{-5}$, $S_{8n-7} = S_{-7}$.

(viii) If $P_0 = 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $P_{8n} = 0$, $Q_{8n-2} = Q_{-2}$, $R_{8n-4} = R_{-4}$, $S_{8n-6} = S_{-6}$.

(ix) If $Q_{-7} = 0$, $P_{-5} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $Q_{8n-7} = 0$, $P_{8n-5} = P_{-5}$, $R_{8n-1} = R_{-1}$, $S_{8n-3} = S_{-3}$.

(x) If $Q_{-6} = 0$, $P_{-4} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $Q_{8n-6} = 0$, $P_{8n-4} = P_{-4}$, $R_{8n} = R_0$, $S_{8n-2} = S_{-2}$.

(xi) If $Q_{-5} = 0$, $P_{-3} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $Q_{8n-5} = 0$, $P_{8n-3} = P_{-3}$, $R_{8n-7} = R_{-7}$, $S_{8n-1} = S_{-1}$.

(xii) If $Q_{-4} = 0$, $P_{-2} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $Q_{8n-4} = 0$, $P_{8n-2} = P_{-2}$, $R_{8n-6} = R_{-6}$, $S_{8n} = S_0$.

(xiii) If $Q_{-3} = 0$, $P_{-1} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $Q_{8n-3} = 0$, $P_{8n-1} = P_{-1}$, $R_{8n-5} = R_{-5}$, $S_{8n-7} = S_{-7}$.

(xiv) If $Q_{-2} = 0$, $P_0 \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $Q_{8n-2} = 0$, $P_{8n} = P_0$, $R_{8n-4} = R_{-4}$, $S_{8n-6} = S_{-6}$.

(xv) If $Q_{-1} = 0$, $P_{-7} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $Q_{8n-1} = 0$, $P_{8n-7} = P_{-7}$, $R_{8n-3} = R_{-3}$, $S_{8n-5} = S_{-5}$.

(xvi) If $Q_0 = 0$, $P_{-6} \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $Q_{8n} = 0$, $P_{8n-6} = P_{-6}$, $R_{8n-2} = R_{-2}$, $S_{8n-4} = S_{-4}$.

(xvii) If $R_{-7} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $S_{-1} \neq 0$ then we have $R_{8n-7} = 0$, $P_{8n-3} = P_{-3}$, $Q_{8n-5} = Q_{-5}$, $S_{8n-1} = S_{-1}$.

(xviii) If $R_{-6} = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $S_0 \neq 0$ then we have $R_{8n-6} = 0$, $P_{8n-2} = P_{-2}$, $Q_{8n-4} = Q_{-4}$, $S_{8n} = S_0$.

(xix) If $R_{-5} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $S_{-7} \neq 0$ then we have $R_{8n-5} = 0$, $P_{8n-1} = P_{-1}$, $Q_{8n-3} = Q_{-3}$, $S_{8n-7} = S_{-7}$.

(xx) If $R_{-4} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $S_{-6} \neq 0$ then we have $R_{8n-4} = 0$, $P_{8n} = P_0$, $Q_{8n-2} = Q_{-2}$, $S_{8n-6} = S_{-6}$.

(xxi) If $R_{-3} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $S_{-5} \neq 0$ then we have $R_{8n-3} = 0$, $P_{8n-7} = P_{-7}$, $Q_{8n-1} = Q_{-1}$, $S_{8n-5} = S_{-5}$.

(xxii) If $R_{-2} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $S_{-4} \neq 0$ then we have $R_{8n-2} = 0$, $P_{8n-6} = P_{-6}$, $Q_{8n} = Q_0$, $S_{8n-4} = S_{-4}$.

(xxiii) If $R_{-1} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $S_{-3} \neq 0$ then we have $R_{8n-1} = 0$, $P_{8n-5} = P_{-5}$, $Q_{8n-7} = Q_{-7}$, $S_{8n-3} = S_{-3}$.

(xxiv) If $R_0 = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $S_{-2} \neq 0$ then we have $R_{8n} = 0$, $P_{8n-4} = P_{-4}$, $Q_{8n-6} = Q_{-6}$, $S_{8n-2} = S_{-2}$.

(xxv) If $S_{-7} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$ then we have $S_{8n-7} = 0$, $P_{8n-1} = P_{-1}$, $Q_{8n-3} = Q_{-3}$, $R_{8n-5} = R_{-5}$.

(xxvi) If $S_{-6} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$ then we have $S_{8n-6} = 0$, $P_{8n} = P_0$, $Q_{8n-2} = Q_{-2}$, $R_{8n-4} = R_{-4}$.

(xxvii) If $S_{-5} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$ then we have $S_{8n-5} = 0$, $P_{8n-7} = P_{-7}$, $Q_{8n-1} = Q_{-1}$, $R_{8n-3} = R_{-3}$.

(xxviii) If $S_{-4} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$ then we have $S_{8n-4} = 0$, $P_{8n-6} = P_{-6}$, $Q_{8n} = Q_0$, $R_{8n-2} = R_{-2}$.

(xxix) If $S_{-3} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$ then we have $S_{8n-3} = 0$, $P_{8n-5} = P_{-5}$, $Q_{8n-7} = Q_{-7}$, $R_{8n-1} = R_{-1}$.

(xxx) If $S_{-2}=0$, $P_{-4}\neq 0$, $Q_{-6}\neq 0$, $R_0\neq 0$ then we have $S_{8n-2}=0$, $P_{8n-4}=P_{-4}$, $Q_{8n-6}=Q_{-6}$, $R_{8n}=R_0$.

(xxxi) If $S_{-1}=0$, $P_{-3}\neq 0$, $Q_{-5}\neq 0$, $R_{-7}\neq 0$ then we have $S_{8n-1}=0$, $P_{8n-3}=P_{-3}$, $Q_{8n-5}=Q_{-5}$, $R_{8n-7}=R_{-7}$.

(xxxii) If $S_0=0$, $P_{-2}\neq 0$, $Q_{-4}\neq 0$, $R_{-6}\neq 0$ then we have $S_{8n}=0$, $P_{8n-2}=P_{-2}$, $Q_{8n-4}=Q_{-4}$, $R_{8n-6}=R_{-6}$.

Proof. The result follows directly from the explicit form of the solution of system (1).

3. Formulation of the Second System

In this section, we derive the solution of the following four-dimensional nonlinear system:

$$\begin{cases} P_{n+1}=\frac{P_{n-7}}{-1-Q_{n-1}R_{n-3}S_{n-5}P_{n-7}}, \\ Q_{n+1}=\frac{Q_{n-7}}{-1+R_{n-1}S_{n-3}P_{n-5}Q_{n-7}}, \\ R_{n+1}=\frac{R_{n-7}}{-1+S_{n-1}P_{n-3}Q_{n-5}R_{n-7}}, \\ S_{n+1}=\frac{S_{n-7}}{-1-P_{n-1}Q_{n-3}R_{n-5}S_{n-7}}, \end{cases} \tag{2}$$

where $n\in\mathbb{N}_0$ and the initial conditions $P_{-7},...,P_0,Q_{-7},...,Q_0,R_{-7},...,R_0,S_{-7},...,S_0$ are arbitrary real numbers such that $P_{-7}Q_{-1}R_{-3}S_{-5}\neq\pm1, P_{-6}Q_0R_{-2}S_{-4}\neq\pm1, P_{-5}Q_{-7}R_{-1}S_{-3}\neq 1, P_{-5}Q_{-7}R_{-1}S_{-3}\neq\frac{1}{2}, P_{-4}Q_{-6}R_0S_{-2}\neq 1, P_{-4}Q_{-6}R_0S_{-2}\neq\frac{1}{2}, P_{-3}Q_{-5}R_{-7}S_{-1}\neq\pm1, P_{-2}Q_{-4}R_{-6}S_0\neq\pm1, P_{-1}Q_{-3}R_{-5}S_{-7}\neq -1, P_{-1}Q_{-3}R_{-5}S_{-7}\neq -\frac{1}{2}, P_0Q_{-2}R_{-4}S_{-6}\neq -1, P_0Q_{-2}R_{-4}S_{-6}\neq -\frac{1}{2}$.

Theorem 2. *Let $\{(P_n,Q_n,R_n,S_n)\}_{n=-7}^{\infty}$ be a solution of system (2). Then, for every non-negative integer n, the solution is given by the following closed-form expressions:*

$$\begin{aligned}
P_{8n-7} &= \frac{(-1)^n P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^n}, & P_{8n-6} &= \frac{(-1)^n P_{-6}}{(P_{-6}Q_0R_{-2}S_{-4}+1)^n}, \\
P_{8n-5} &= \frac{(-1)^n P_{-5}(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^n}{(2P_{-5}Q_{-7}R_{-1}S_{-3}-1)^n}, & P_{8n-4} &= \frac{(-1)^n P_{-4}(P_{-4}Q_{-6}R_0S_{-2}-1)^n}{(2P_{-4}Q_{-6}R_0S_{-2}-1)^n}, \\
P_{8n-3} &= \frac{(-1)^n P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^n}, & P_{8n-2} &= \frac{(-1)^n P_{-2}}{(P_{-2}Q_{-4}R_{-6}S_0+1)^n}, \\
P_{8n-1} &= (-1)^n P_{-1}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n, & P_{8n} &= (-1)^n P_0(P_0Q_{-2}R_{-4}S_{-6}+1)^n, \\
Q_{8n-7} &= \frac{Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^n}, & Q_{8n-6} &= \frac{Q_{-6}}{(P_{-4}Q_{-6}R_0S_{-2}-1)^n}, \\
Q_{8n-5} &= Q_{-5}(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^n, & Q_{8n-4} &= Q_{-4}(P_{-2}Q_{-4}R_{-6}S_0-1)^n, \\
Q_{8n-3} &= \frac{(-1)^n Q_{-3}(2P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n}, & Q_{8n-2} &= \frac{(-1)^n Q_{-2}(2P_0Q_{-2}R_{-4}S_{-6}+1)^n}{(P_0Q_{-2}R_{-4}S_{-6}+1)^n},
\end{aligned}$$

$$
\begin{aligned}
Q_{8n-1} &= Q_{-1}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}, & Q_{8n} &= Q_{0}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n},\\
R_{8n-7} &= \frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n}}, & R_{8n-6} &= \frac{R_{-6}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n}},\\
R_{8n-5} &= \frac{\left(-1\right)^{n}R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n}}{\left(2P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n}}, & R_{8n-4} &= \frac{\left(-1\right)^{n}R_{-4}\left(P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n}}{\left(2P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n}},\\
R_{8n-3} &= \frac{R_{-3}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}}, & R_{8n-2} &= \frac{R_{-2}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n}},\\
R_{8n-1} &= R_{-1}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}, & R_{8n} &= R_{0}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n},\\
S_{8n-7} &= \frac{\left(-1\right)^{n}S_{-7}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n}}, & S_{8n-6} &= \frac{\left(-1\right)^{n}S_{-6}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n}},\\
S_{8n-5} &= \left(-1\right)^{n}S_{-5}\left(P_{-7}Q_{-1}R_{-3}S_{-5}+1\right)^{n}, & S_{8n-4} &= \left(-1\right)^{n}S_{-4}\left(P_{-6}Q_{0}R_{-2}S_{-4}+1\right)^{n},\\
S_{8n-3} &= \frac{\left(-1\right)^{n}S_{-3}\left(2P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}}, & S_{8n-2} &= \frac{\left(-1\right)^{n}S_{-2}\left(2P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n}},\\
S_{8n-1} &= \left(-1\right)^{n}S_{-1}\left(P_{-3}Q_{-5}R_{-7}S_{-1}+1\right)^{n}, & S_{8n} &= \left(-1\right)^{n}S_{0}\left(P_{-2}Q_{-4}R_{-6}S_{0}+1\right)^{n}.
\end{aligned}
$$

Proof. The result is immediate for $n=0$. For $n>0$, assume by induction that it holds for $n-1$ and that it can be written in the following form:

$$
\begin{aligned}
P_{8n-15} &= \frac{\left(-1\right)^{n-1}P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}+1\right)^{n-1}}, & P_{8n-14} &= \frac{\left(-1\right)^{n-1}P_{-6}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}+1\right)^{n-1}},\\
P_{8n-13} &= \frac{\left(-1\right)^{n-1}P_{-5}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}{\left(2P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}, & P_{8n-12} &= \frac{\left(-1\right)^{n-1}P_{-4}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1}}{\left(2P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1}},\\
P_{8n-11} &= \frac{\left(-1\right)^{n-1}P_{-3}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}+1\right)^{n-1}}, & P_{8n-10} &= \frac{\left(-1\right)^{n-1}P_{-2}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}+1\right)^{n-1}},\\
P_{8n-9} &= \left(-1\right)^{n-1}P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}, & P_{8n-8} &= \left(-1\right)^{n-1}P_{0}\left(P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n-1},\\
Q_{8n-15} &= \frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}, & Q_{8n-14} &= \frac{Q_{-6}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1}},\\
Q_{8n-13} &= Q_{-5}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}, & Q_{8n-12} &= Q_{-4}\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1},\\
Q_{8n-11} &= \frac{\left(-1\right)^{n-1}Q_{-3}\left(2P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}}, & Q_{8n-10} &= \frac{\left(-1\right)^{n-1}Q_{-2}\left(2P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n-1}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n-1}},\\
Q_{8n-9} &= Q_{-1}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}, & Q_{8n-8} &= Q_{0}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1},\\
R_{8n-15} &= \frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}, & R_{8n-14} &= \frac{R_{-6}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1}},\\
R_{8n-13} &= \frac{\left(-1\right)^{n-1}R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}}{\left(2P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}}, & R_{8n-12} &= \frac{\left(-1\right)^{n-1}R_{-4}\left(P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n-1}}{\left(2P_{0}Q_{-2}R_{-4}S_{-6}+1\right)^{n-1}},\\
R_{8n-11} &= \frac{R_{-3}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}, & R_{8n-10} &= \frac{R_{-2}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1}},\\
R_{8n-9} &= R_{-1}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}, & R_{8n-8} &= R_{0}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1},
\end{aligned}
$$

$$
\begin{aligned}
S_{8n-15} &= \frac{(-1)^{n-1} S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}, \qquad S_{8n-14} = \frac{(-1)^{n-1} S_{-6}}{(P_{0}Q_{-2}R_{-4}S_{-6}+1)^{n-1}},\\
S_{8n-13} &= (-1)^{n-1} S_{-5} (P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}, \qquad S_{8n-12} = (-1)^{n-1} S_{-4} (P_{-6}Q_{0}R_{-2}S_{-4}+1)^{n-1},\\
S_{8n-11} &= \frac{(-1)^{n-1} S_{-3} (2P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}, \quad S_{8n-10} = \frac{(-1)^{n-1} S_{-2} (2P_{-4}Q_{-6}R_{0}S_{-2}-1)^{n-1}}{(P_{-4}Q_{-6}R_{0}S_{-2}-1)^{n-1}},\\
S_{8n-9} &= (-1)^{n-1} S_{-1} (P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}, \qquad S_{8n-8} = (-1)^{n-1} S_{0} (P_{-2}Q_{-4}R_{-6}S_{0}+1)^{n-1}.
\end{aligned}
$$

Now, it follows from system (2) that

$$
\begin{aligned}
P_{8n-7} &= \frac{P_{8n-15}}{-1-Q_{8n-9}R_{8n-11}S_{8n-13}P_{8n-15}}\\
&= \frac{\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left(Q_{-1}(P_{-7}Q_{-1}R_{-3}S_{-5}-1)^{n-1}\right)\cdot\left(\frac{R_{-3}}{(P_{-7}Q_{-1}R_{-3}S_{-5}-1)^{n-1}}\right)\cdot\\ \left((-1)^{n-1}S_{-5}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}}{-1-P_{-7}Q_{-1}R_{-3}S_{-5}} = \frac{(-1)^{n}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n}},
\end{aligned}
$$

$$
\begin{aligned}
Q_{8n-7} &= \frac{Q_{8n-15}}{-1+R_{8n-9}S_{8n-11}P_{8n-13}Q_{8n-15}}\\
&= \frac{\frac{Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}}{-1+\left(\begin{array}{c}\left(R_{-1}(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}S_{-3}(2P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}\right)\cdot\\ \left(\frac{(-1)^{n-1}P_{-5}(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}{(2P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}\right)\cdot\left(\frac{Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n-1}}}{-1+P_{-5}Q_{-7}R_{-1}S_{-3}} = \frac{Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}-1)^{n}},
\end{aligned}
$$

$$
\begin{aligned}
R_{8n-7} &= \frac{R_{8n-15}}{-1+S_{8n-9}P_{8n-11}Q_{8n-13}R_{8n-15}}\\
&= \frac{\frac{R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^{n-1}}}{-1+\left(\begin{array}{c}\left((-1)^{n-1}S_{-1}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}\right)\cdot\\ \left(Q_{-5}(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^{n-1}\right)\cdot\left(\frac{R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^{n-1}}}{-1+P_{-3}Q_{-5}R_{-7}S_{-1}} = \frac{R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^{n}},
\end{aligned}
$$

$$\begin{aligned}
S_{8n-7} &= \frac{S_{8n-15}}{-1-P_{8n-9}Q_{8n-11}R_{8n-13}S_{8n-15}} \\
&= \frac{\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left((-1)^{n-1}P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}Q_{-3}(2P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}\right)\cdot \\ \left(\frac{(-1)^{n-1}R_{-5}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}{(2P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}\right)\cdot\left(\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}}{-1-P_{-1}Q_{-3}R_{-5}S_{-7}} = \frac{(-1)^{n}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n}}.
\end{aligned}$$

Similarly, the other relations can also be established. Therefore, by induction, the stated formulas hold for all $n \geq 0$.

Corollary 2. *Let $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ be a solution of system* (2). *Then the following statements hold:*

(i) If $P_{-7} = 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $P_{8n-7} = 0$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(ii) If $P_{-6} = 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $P_{8n-6} = 0$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(iii) If $P_{-5} = 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $P_{8n-5} = 0$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(iv) If $P_{-4} = 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $P_{8n-4} = 0$, $Q_{8n-6} = (-1)^n Q_{-6}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(v) If $P_{-3} = 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $P_{8n-3} = 0$, $Q_{8n-5} = (-1)^n Q_{-5}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(vi) If $P_{-2} = 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $P_{8n-2} = 0$, $Q_{8n-4} = (-1)^n Q_{-4}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(vii) If $P_{-1} = 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $P_{8n-1} = 0$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(viii) If $P_0 = 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $P_{8n} = 0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(ix) If $Q_{-7} = 0$, $P_{-5} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $Q_{8n-7} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(x) If $Q_{-6} = 0$, $P_{-4} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $Q_{8n-6} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(xi) If $Q_{-5} = 0$, $P_{-3} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $Q_{8n-5} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xii) If $Q_{-4} = 0$, $P_{-2} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $Q_{8n-4} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(xiii) If $Q_{-3} = 0$, $P_{-1} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $Q_{8n-3} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xiv) If $Q_{-2} = 0$, $P_0 \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $Q_{8n-2} = 0$, $P_{8n} = (-1)^n P_0$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xv) If $Q_{-1} = 0$, $P_{-7} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $Q_{8n-1} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xvi) If $Q_0 = 0$, $P_{-6} \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $Q_{8n} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(xvii) If $R_{-7} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $S_{-1} \neq 0$ then we have $R_{8n-7} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $Q_{8n-5} = (-1)^n Q_{-5}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xviii) If $R_{-6} = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $S_0 \neq 0$ then we have $R_{8n-6} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $Q_{8n-4} = (-1)^n Q_{-4}$, $S_{8n} = (-1)^n S_0$.

(xix) If $R_{-5} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $S_{-7} \neq 0$ then we have $R_{8n-5} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xx) If $R_{-4} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $S_{-6} \neq 0$ then we have $R_{8n-4} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xxi) If $R_{-3} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $S_{-5} \neq 0$ then we have $R_{8n-3} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xxii) If $R_{-2} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $S_{-4} \neq 0$ then we have $R_{8n-2} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $S_{8n-4} = (-1)^n S_{-4}$.

(xxiii) If $R_{-1} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $S_{-3} \neq 0$ then we have $R_{8n-1} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $S_{8n-3} = (-1)^n S_{-3}$.

(xxiv) If $R_0 = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $S_{-2} \neq 0$ then we have $R_{8n} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $Q_{8n-6} = (-1)^n Q_{-6}$, $S_{8n-2} = (-1)^n S_{-2}$.

(xxv) If $S_{-7} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$ then we have $S_{8n-7} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$.

(xxvi) If $S_{-6} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$ then we have $S_{8n-6} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$.

(xxvii) If $S_{-5} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$ then we have $S_{8n-5} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$.

(xxviii) If $S_{-4} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$ then we have $S_{8n-4} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$.

(xxix) If $S_{-3} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$ then we have $S_{8n-3} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$.

(xxx) If $S_{-2} = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$ then we have $S_{8n-2} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $Q_{8n-6} = (-1)^n Q_{-6}$, $R_{8n} = (-1)^n R_0$.

(xxxi) If $S_{-1} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$ then we have $S_{8n-1} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $Q_{8n-5} = (-1)^n Q_{-5}$, $R_{8n-7} = (-1)^n R_{-7}$.

(xxxii) If $S_0 = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$ then we have $S_{8n} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $Q_{8n-4} = (-1)^n Q_{-4}$, $R_{8n-6} = (-1)^n R_{-6}$.

Proof. The result follows directly from the explicit form of the solution of system (2).

4. Formulation of the Third System

This section focuses on the investigation of the solution and dynamic properties of the following four-dimensional nonlinear system:

$$\begin{cases} P_{n+1} = \frac{P_{n-7}}{-1+Q_{n-1}R_{n-3}S_{n-5}P_{n-7}}, \\ Q_{n+1} = \frac{Q_{n-7}}{-1+R_{n-1}S_{n-3}P_{n-5}Q_{n-7}}, \\ R_{n+1} = \frac{R_{n-7}}{-1+S_{n-1}P_{n-3}Q_{n-5}R_{n-7}}, \\ S_{n+1} = \frac{S_{n-7}}{-1+P_{n-1}Q_{n-3}R_{n-5}S_{n-7}}, \end{cases} \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions $P_{-7}, ..., P_0, Q_{-7}, ..., Q_0, R_{-7}, ..., R_0, S_{-7}, ..., S_0$ are arbitrary real numbers such that $P_{-7}Q_{-1}R_{-3}S_{-5} \neq 1, P_{-6}Q_0R_{-2}S_{-4} \neq 1, P_{-5}Q_{-7}R_{-1}S_{-3} \neq 1, P_{-4}Q_{-6}R_0S_{-2} \neq 1, P_{-3}Q_{-5}R_{-7}S_{-1} \neq 1, P_{-2}Q_{-4}R_{-6}S_0 \neq 1, P_{-1}Q_{-3}R_{-5}S_{-7} \neq 1, P_0Q_{-2}R_{-4}S_{-6} \neq 1$.

Theorem 3. *Let $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ be a solution of system* (3)*. Then, for every non-negative integer n, the solution is given by the following closed-form expressions:*

$$P_{8n-7} = \frac{P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}-1)^n}, \qquad P_{8n-6} = \frac{P_{-6}}{(P_{-6}Q_0R_{-2}S_{-4}-1)^n},$$

$$P_{8n-5} = P_{-5}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^n, \qquad P_{8n-4} = P_{-4}\left(P_{-4}Q_{-6}R_0S_{-2}-1\right)^n,$$

$$P_{8n-3} = \frac{P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}-1)^n}, \qquad P_{8n-2} = \frac{P_{-2}}{(P_{-2}Q_{-4}R_{-6}S_0-1)^n},$$

$$
\begin{aligned}
P_{8n-1} &= P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n}, & P_{8n} &= P_{0}\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n},\\
Q_{8n-7} &= \frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}}, & Q_{8n-6} &= \frac{Q_{-6}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n}},\\
Q_{8n-5} &= Q_{-5}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n}, & Q_{8n-4} &= Q_{-4}\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n},\\
Q_{8n-3} &= \frac{Q_{-3}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n}}, & Q_{8n-2} &= \frac{Q_{-2}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n}},\\
Q_{8n-1} &= Q_{-1}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}, & Q_{8n} &= Q_{0}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n},\\
R_{8n-7} &= \frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n}}, & R_{8n-6} &= \frac{R_{-6}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n}},\\
R_{8n-5} &= R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n}, & R_{8n-4} &= R_{-4}\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n},\\
R_{8n-3} &= \frac{R_{-3}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}}, & R_{8n-2} &= \frac{R_{-2}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n}},\\
R_{8n-1} &= R_{-1}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}, & R_{8n} &= R_{0}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n},\\
S_{8n-7} &= \frac{S_{-7}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n}}, & S_{8n-6} &= \frac{S_{-6}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n}},\\
S_{8n-5} &= S_{-5}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}, & S_{8n-4} &= S_{-4}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n},\\
S_{8n-3} &= \frac{S_{-3}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}}, & S_{8n-2} &= \frac{S_{-2}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n}},\\
S_{8n-1} &= S_{-1}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n}, & S_{8n} &= S_{0}\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n}.
\end{aligned}
$$

Proof. The result is immediate for $n = 0$. For $n > 0$, assume by induction that it holds for $n-1$ and that it can be written in the following form:

$$
\begin{aligned}
P_{8n-15} &= \frac{P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}, & P_{8n-14} &= \frac{P_{-6}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1}},\\
P_{8n-13} &= P_{-5}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}, & P_{8n-12} &= P_{-4}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1},\\
P_{8n-11} &= \frac{P_{-3}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}, & P_{8n-10} &= \frac{P_{-2}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1}},\\
P_{8n-9} &= P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}, & P_{8n-8} &= P_{0}\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n-1},\\
Q_{8n-15} &= \frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}, & Q_{8n-14} &= \frac{Q_{-6}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1}},\\
Q_{8n-13} &= Q_{-5}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}, & Q_{8n-12} &= Q_{-4}\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1},\\
Q_{8n-11} &= \frac{Q_{-3}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}}, & Q_{8n-10} &= \frac{Q_{-2}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n-1}},\\
Q_{8n-9} &= Q_{-1}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}, & Q_{8n-8} &= Q_{0}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1},\\
R_{8n-15} &= \frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}, & R_{8n-14} &= \frac{R_{-6}}{\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1}},\\
R_{8n-13} &= R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}, & R_{8n-12} &= R_{-4}\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n-1},\\
R_{8n-11} &= \frac{R_{-3}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}, & R_{8n-10} &= \frac{R_{-2}}{\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1}},
\end{aligned}
$$

$$\begin{aligned}
R_{8n-9} &= R_{-1}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}, & R_{8n-8} &= R_{0}\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1},\\
S_{8n-15} &= \frac{S_{-7}}{\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}}, & S_{8n-14} &= \frac{S_{-6}}{\left(P_{0}Q_{-2}R_{-4}S_{-6}-1\right)^{n-1}},\\
S_{8n-13} &= S_{-5}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}, & S_{8n-12} &= S_{-4}\left(P_{-6}Q_{0}R_{-2}S_{-4}-1\right)^{n-1},\\
S_{8n-11} &= \frac{S_{-3}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}, & S_{8n-10} &= \frac{S_{-2}}{\left(P_{-4}Q_{-6}R_{0}S_{-2}-1\right)^{n-1}},\\
S_{8n-9} &= S_{-1}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}, & S_{8n-8} &= S_{0}\left(P_{-2}Q_{-4}R_{-6}S_{0}-1\right)^{n-1}.
\end{aligned}$$

Now, it follows from system (3) that

$$\begin{aligned}
P_{8n-7} &= \frac{P_{8n-15}}{-1+Q_{8n-9}R_{8n-11}S_{8n-13}P_{8n-15}}\\
&= \frac{\frac{P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}}{-1+\left(\begin{array}{c}\left(Q_{-1}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}\right)\cdot\left(\frac{R_{-3}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}\right)\cdot\\ \left(S_{-5}\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}\right)\cdot\left(\frac{P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n-1}}}{-1+P_{-7}Q_{-1}R_{-3}S_{-5}} = \frac{P_{-7}}{\left(P_{-7}Q_{-1}R_{-3}S_{-5}-1\right)^{n}},
\end{aligned}$$

$$\begin{aligned}
Q_{8n-7} &= \frac{Q_{8n-15}}{-1+R_{8n-9}S_{8n-11}P_{8n-13}Q_{8n-15}}\\
&= \frac{\frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}}{-1+\left(\begin{array}{c}\left(R_{-1}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}\right)\cdot\left(\frac{S_{-3}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}\right)\cdot\\ \left(P_{-5}\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}\right)\cdot\left(\frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n-1}}}{-1+P_{-5}Q_{-7}R_{-1}S_{-3}} = \frac{Q_{-7}}{\left(P_{-5}Q_{-7}R_{-1}S_{-3}-1\right)^{n}},
\end{aligned}$$

$$\begin{aligned}
R_{8n-7} &= \frac{R_{8n-15}}{-1+S_{8n-9}P_{8n-11}Q_{8n-13}R_{8n-15}}\\
&= \frac{\frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}}{-1+\left(\begin{array}{c}\left(S_{-1}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}\right)\cdot\left(\frac{P_{-3}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}\right)\cdot\\ \left(Q_{-5}\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}\right)\cdot\left(\frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}\right)\end{array}\right)}\\
&= \frac{\frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n-1}}}{-1+P_{-3}Q_{-5}R_{-7}S_{-1}} = \frac{R_{-7}}{\left(P_{-3}Q_{-5}R_{-7}S_{-1}-1\right)^{n}},
\end{aligned}$$

$$
\begin{aligned}
S_{8n-7} &= \frac{S_{8n-15}}{-1+P_{8n-9}Q_{8n-11}R_{8n-13}S_{8n-15}} \\
&= \frac{\frac{S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}-1)^{n-1}}}{-1+\left(\begin{array}{c}\left(P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}\right)\cdot\left(\frac{Q_{-3}}{(P_{-1}Q_{-3}R_{-5}S_{-7}-1)^{n-1}}\right)\cdot \\ \left(R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}-1\right)^{n-1}\right)\cdot\left(\frac{S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}-1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}-1)^{n-1}}}{-1+P_{-1}Q_{-3}R_{-5}S_{-7}} = \frac{S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}-1)^{n}}.
\end{aligned}
$$

Similarly, the other relations can also be established. Therefore, by induction, the stated formulas hold for all $n \geq 0$.

Corollary 3. *The system* (3) *admits a periodic solution of period eight if and only if the following conditions are satisfied:*

$$
\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5} = P_{-6}Q_{0}R_{-2}S_{-4} = P_{-5}Q_{-7}R_{-1}S_{-3} = P_{-4}Q_{-6}R_{0}S_{-2} = \\
P_{-3}Q_{-5}R_{-7}S_{-1} = P_{-2}Q_{-4}R_{-6}S_{0} = P_{-1}Q_{-3}R_{-5}S_{-7} = P_{0}Q_{-2}R_{-4}S_{-6} = 2.
\end{gathered}
$$

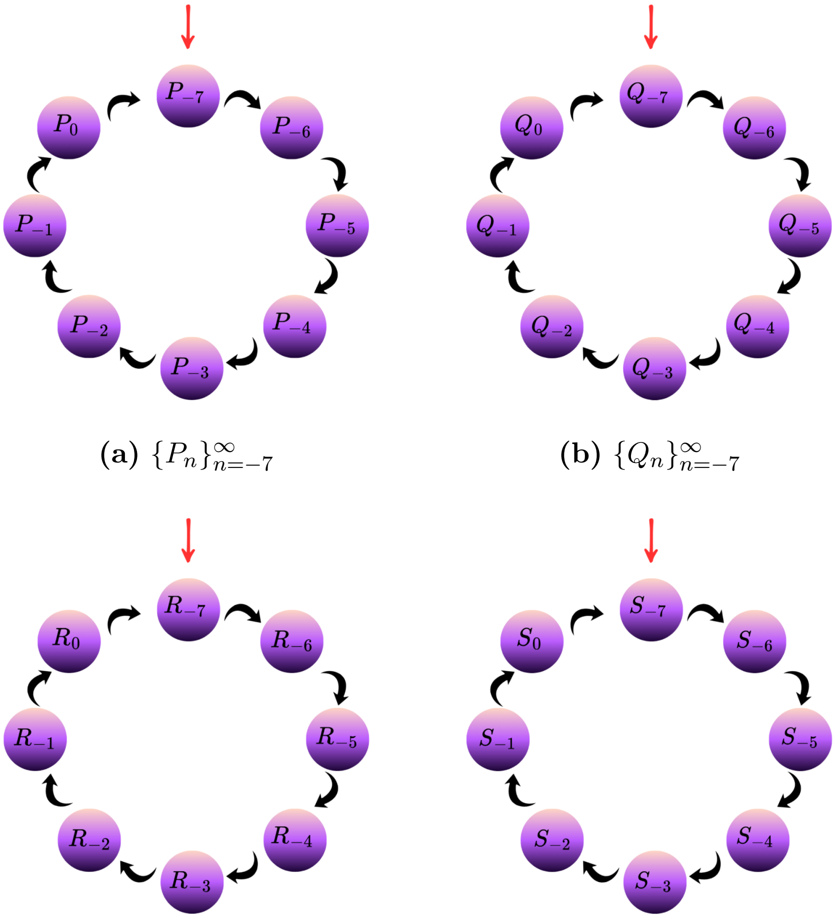
In this case, the solution has the following form (see Figure 1)

$$\{P_n\}_{n=-7}^{\infty} = \{P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, \ldots\},$$

$$\{Q_n\}_{n=-7}^{\infty} = \{Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, \ldots\},$$

$$\{R_n\}_{n=-7}^{\infty} = \{R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, \ldots\},$$

$$\{S_n\}_{n=-7}^{\infty} = \{S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, \ldots\}.$$



(a) $\{P_n\}_{n=-7}^{\infty}$ **(b)** $\{Q_n\}_{n=-7}^{\infty}$

(c) $\{R_n\}_{n=-7}^{\infty}$ **(d)** $\{S_n\}_{n=-7}^{\infty}$

Figure 1: Illustration of a periodic solution of period eight for system (3).

Proof. First, suppose that there exists a solution of period eight

$$\{P_n\}_{n=-7}^{\infty} = \{P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, \dots\},$$

$$\{Q_n\}_{n=-7}^{\infty} = \{Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, \dots\},$$

$$\{R_n\}_{n=-7}^{\infty} = \{R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, \dots\},$$

$$\{S_n\}_{n=-7}^{\infty} = \{S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, \dots\}.$$

From the system (3), we see that

$$P_{-7} = \frac{P_{-7}}{-1 + Q_{-1}R_{-3}S_{-5}P_{-7}}, P_{-6} = \frac{P_{-6}}{-1 + Q_0R_{-2}S_{-4}P_{-6}}, P_{-5} = \frac{P_{-5}}{-1 + Q_{-7}R_{-1}S_{-3}P_{-5}},$$

$$\begin{aligned}
P_{-4} &= \frac{P_{-4}}{-1+Q_{-6}R_0S_{-2}P_{-4}}, P_{-3}=\frac{P_{-3}}{-1+Q_{-5}R_{-7}S_{-1}P_{-3}}, P_{-2}=\frac{P_{-2}}{-1+Q_{-4}R_{-6}S_0P_{-2}},\\
P_{-1} &= \frac{P_{-1}}{-1+Q_{-3}R_{-5}S_{-7}P_{-1}}, P_0=\frac{P_0}{-1+Q_{-2}R_{-4}S_{-6}P_0}, Q_{-7}=\frac{Q_{-7}}{-1+R_{-1}S_{-3}P_{-5}Q_{-7}},\\
Q_{-6} &= \frac{Q_{-6}}{-1+R_0S_{-2}P_{-4}Q_{-6}}, Q_{-5}=\frac{Q_{-5}}{-1+R_{-7}S_{-1}P_{-3}Q_{-5}}, Q_{-4}=\frac{Q_{-4}}{-1+R_{-6}S_0P_{-2}Q_{-4}},\\
Q_{-3} &= \frac{Q_{-3}}{-1+R_{-5}S_{-7}P_{-1}Q_{-3}}, Q_{-2}=\frac{Q_{-2}}{-1+R_{-4}S_{-6}P_0Q_{-2}}, Q_{-1}=\frac{Q_{-1}}{-1+R_{-3}S_{-5}P_{-7}Q_{-1}},\\
Q_0 &= \frac{Q_0}{-1+R_{-2}S_{-4}P_{-6}Q_0}, R_{-7}=\frac{R_{-7}}{-1+S_{-1}P_{-3}Q_{-5}R_{-7}}, R_{-6}=\frac{R_{-6}}{-1+S_0P_{-2}Q_{-4}R_{-6}},\\
R_{-5} &= \frac{R_{-5}}{-1+S_{-7}P_{-1}Q_{-3}R_{-5}}, R_{-4}=\frac{R_{-4}}{-1+S_{-6}P_0Q_{-2}R_{-4}}, R_{-3}=\frac{R_{-3}}{-1+S_{-5}P_{-7}Q_{-1}R_{-3}},\\
R_{-2} &= \frac{R_{-2}}{-1+S_{-4}P_{-6}Q_0R_{-2}}, R_{-1}=\frac{R_{-1}}{-1+S_{-3}P_{-5}Q_{-7}R_{-1}}, R_0=\frac{R_0}{-1+S_{-2}P_{-4}Q_{-6}R_0},\\
S_{-7} &= \frac{S_{-7}}{-1+P_{-1}Q_{-3}R_{-5}S_{-7}}, S_{-6}=\frac{S_{-6}}{-1+P_0Q_{-2}R_{-4}S_{-6}}, S_{-5}=\frac{S_{-5}}{-1+P_{-7}Q_{-1}R_{-3}S_{-5}},\\
S_{-4} &= \frac{S_{-4}}{-1+P_{-6}Q_0R_{-2}S_{-4}}, S_{-3}=\frac{S_{-3}}{-1+P_{-5}Q_{-7}R_{-1}S_{-3}}, S_{-2}=\frac{S_{-2}}{-1+P_{-4}Q_{-6}R_0S_{-2}},\\
S_{-1} &= \frac{S_{-1}}{-1+P_{-3}Q_{-5}R_{-7}S_{-1}}, S_0=\frac{S_0}{-1+P_{-2}Q_{-4}R_{-6}S_0}.
\end{aligned}$$

Then

$$\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5}=P_{-6}Q_0R_{-2}S_{-4}=P_{-5}Q_{-7}R_{-1}S_{-3}=P_{-4}Q_{-6}R_0S_{-2}=\\
P_{-3}Q_{-5}R_{-7}S_{-1}=P_{-2}Q_{-4}R_{-6}S_0=P_{-1}Q_{-3}R_{-5}S_{-7}=P_0Q_{-2}R_{-4}S_{-6}=2.
\end{gathered}$$

Second, suppose the converse:

$$\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5}=P_{-6}Q_0R_{-2}S_{-4}=P_{-5}Q_{-7}R_{-1}S_{-3}=P_{-4}Q_{-6}R_0S_{-2}=\\
P_{-3}Q_{-5}R_{-7}S_{-1}=P_{-2}Q_{-4}R_{-6}S_0=P_{-1}Q_{-3}R_{-5}S_{-7}=P_0Q_{-2}R_{-4}S_{-6}=2.
\end{gathered}$$

Then, we see from the solution of system (3) that

$$\begin{array}{ccc}
P_{8n-7}=P_{-7}, & P_{8n-6}=P_{-6}, & P_{8n-5}=P_{-5},\\
P_{8n-4}=P_{-4}, & P_{8n-3}=P_{-3}, & P_{8n-2}=P_{-2},\\
P_{8n-1}=P_{-1}, & P_{8n}=P_0, & Q_{8n-7}=Q_{-7},\\
Q_{8n-6}=Q_{-6}, & Q_{8n-5}=Q_{-5}, & Q_{8n-4}=Q_{-4},\\
Q_{8n-3}=Q_{-3}, & Q_{8n-2}=Q_{-2}, & Q_{8n-1}=Q_{-1},\\
Q_{8n}=Q_0, & R_{8n-7}=R_{-7}, & R_{8n-6}=R_{-6},\\
R_{8n-5}=R_{-5}, & R_{8n-4}=R_{-4}, & R_{8n-3}=R_{-3},\\
R_{8n-2}=R_{-2}, & R_{8n-1}=R_{-1}, & R_{8n}=R_0,\\
S_{8n-7}=S_{-7}, & S_{8n-6}=S_{-6}, & S_{8n-5}=S_{-5},
\end{array}$$

$$S_{8n-4} = S_{-4}, \qquad S_{8n-3} = S_{-3}, \qquad S_{8n-2} = S_{-2},$$
$$S_{8n-1} = S_{-1}, \qquad S_{8n=} = S_0.$$

Thus, we obtain a solution of period eight, and the proof is complete.

Corollary 4. *Let* $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ *be a solution of system* (3). *Then the following statements hold:*

(i) If $P_{-7} = 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $P_{8n-7} = 0$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(ii) If $P_{-6} = 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $P_{8n-6} = 0$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(iii) If $P_{-5} = 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $P_{8n-5} = 0$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(iv) If $P_{-4} = 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $P_{8n-4} = 0$, $Q_{8n-6} = (-1)^n Q_{-6}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(v) If $P_{-3} = 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $P_{8n-3} = 0$, $Q_{8n-5} = (-1)^n Q_{-5}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(vi) If $P_{-2} = 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $P_{8n-2} = 0$, $Q_{8n-4} = (-1)^n Q_{-4}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(vii) If $P_{-1} = 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $P_{8n-1} = 0$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(viii) If $P_0 = 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $P_{8n} = 0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(ix) If $Q_{-7} = 0$, $P_{-5} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $Q_{8n-7} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(x) If $Q_{-6} = 0$, $P_{-4} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $Q_{8n-6} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(xi) If $Q_{-5} = 0$, $P_{-3} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $Q_{8n-5} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xii) If $Q_{-4} = 0$, $P_{-2} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $Q_{8n-4} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(xiii) If $Q_{-3} = 0$, $P_{-1} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $Q_{8n-3} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xiv) If $Q_{-2} = 0$, $P_0 \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $Q_{8n-2} = 0$, $P_{8n} = (-1)^n P_0$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xv) If $Q_{-1} = 0$, $P_{-7} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $Q_{8n-1} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xvi) If $Q_0 = 0$, $P_{-6} \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $Q_{8n} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(xvii) If $R_{-7} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $S_{-1} \neq 0$ then we have $R_{8n-7} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $Q_{8n-5} = (-1)^n Q_{-5}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xviii) If $R_{-6} = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $S_0 \neq 0$ then we have $R_{8n-6} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $Q_{8n-4} = (-1)^n Q_{-4}$, $S_{8n} = (-1)^n S_0$.

(xix) If $R_{-5} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $S_{-7} \neq 0$ then we have $R_{8n-5} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xx) If $R_{-4} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $S_{-6} \neq 0$ then we have $R_{8n-4} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xxi) If $R_{-3} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $S_{-5} \neq 0$ then we have $R_{8n-3} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xxii) If $R_{-2} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $S_{-4} \neq 0$ then we have $R_{8n-2} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $S_{8n-4} = (-1)^n S_{-4}$.

(xxiii) If $R_{-1} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $S_{-3} \neq 0$ then we have $R_{8n-1} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $S_{8n-3} = (-1)^n S_{-3}$.

(xxiv) If $R_0 = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $S_{-2} \neq 0$ then we have $R_{8n} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $Q_{8n-6} = (-1)^n Q_{-6}$, $S_{8n-2} = (-1)^n S_{-2}$.

(xxv) If $S_{-7} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$ then we have $S_{8n-7} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$.

(xxvi) If $S_{-6} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$ then we have $S_{8n-6} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$.

(xxvii) If $S_{-5} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$ then we have $S_{8n-5} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$.

(xxviii) If $S_{-4} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$ then we have $S_{8n-4} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$.

(xxix) If $S_{-3} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$ then we have $S_{8n-3} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$.

(xxx) If $S_{-2}=0$, $P_{-4}\neq 0$, $Q_{-6}\neq 0$, $R_0\neq 0$ then we have $S_{8n-2}=0$, $P_{8n-4}=(-1)^nP_{-4}$, $Q_{8n-6}=(-1)^nQ_{-6}$, $R_{8n}=(-1)^nR_0$.

(xxxi) If $S_{-1}=0$, $P_{-3}\neq 0$, $Q_{-5}\neq 0$, $R_{-7}\neq 0$ then we have $S_{8n-1}=0$, $P_{8n-3}=(-1)^nP_{-3}$, $Q_{8n-5}=(-1)^nQ_{-5}$, $R_{8n-7}=(-1)^nR_{-7}$.

(xxxii) If $S_0=0$, $P_{-2}\neq 0$, $Q_{-4}\neq 0$, $R_{-6}\neq 0$ then we have $S_{8n}=0$, $P_{8n-2}=(-1)^nP_{-2}$, $Q_{8n-4}=(-1)^nQ_{-4}$, $R_{8n-6}=(-1)^nR_{-6}$.

Proof. The result follows directly from the explicit form of the solution of system (3).

5. Formulation of the Fourth System

This section is devoted to studying the solutions and properties of the following four-dimensional nonlinear system:

$$\begin{cases} P_{n+1}=\frac{P_{n-7}}{-1-Q_{n-1}R_{n-3}S_{n-5}P_{n-7}}, \\ Q_{n+1}=\frac{Q_{n-7}}{-1-R_{n-1}S_{n-3}P_{n-5}Q_{n-7}}, \\ R_{n+1}=\frac{R_{n-7}}{-1-S_{n-1}P_{n-3}Q_{n-5}R_{n-7}}, \\ S_{n+1}=\frac{S_{n-7}}{-1-P_{n-1}Q_{n-3}R_{n-5}S_{n-7}}, \end{cases} \tag{4}$$

where $n\in\mathbb{N}_0$ and the initial conditions $P_{-7},...,P_0,Q_{-7},...,Q_0,R_{-7},...,R_0,S_{-7},...,S_0$ are arbitrary real numbers such that $P_{-7}Q_{-1}R_{-3}S_{-5}\neq -1, P_{-6}Q_0R_{-2}S_{-4}\neq -1, P_{-5}Q_{-7}R_{-1}S_{-3}\neq -1, P_{-4}Q_{-6}R_0S_{-2}\neq -1, P_{-3}Q_{-5}R_{-7}S_{-1}\neq -1, P_{-2}Q_{-4}R_{-6}S_0\neq -1, P_{-1}Q_{-3}R_{-5}S_{-7}\neq -1, P_0Q_{-2}R_{-4}S_{-6}\neq -1$.

Theorem 4. *Let $\{(P_n,Q_n,R_n,S_n)\}_{n=-7}^{\infty}$ be a solution of system (4). Then, for every non-negative integer n, the solution is given by the following closed-form expressions:*

$$\begin{aligned}
P_{8n-7} &= \frac{(-1)^n P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^n}, & P_{8n-6} &= \frac{(-1)^n P_{-6}}{(P_{-6}Q_0R_{-2}S_{-4}+1)^n}, \\
P_{8n-5} &= (-1)^n P_{-5}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^n, & P_{8n-4} &= (-1)^n P_{-4}(P_{-4}Q_{-6}R_0S_{-2}+1)^n, \\
P_{8n-3} &= \frac{(-1)^n P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^n}, & P_{8n-2} &= \frac{(-1)^n P_{-2}}{(P_{-2}Q_{-4}R_{-6}S_0+1)^n}, \\
P_{8n-1} &= (-1)^n P_{-1}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n, & P_{8n} &= (-1)^n P_0(P_0Q_{-2}R_{-4}S_{-6}+1)^n, \\
Q_{8n-7} &= \frac{(-1)^n Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^n}, & Q_{8n-6} &= \frac{(-1)^n Q_{-6}}{(P_{-4}Q_{-6}R_0S_{-2}+1)^n}, \\
Q_{8n-5} &= (-1)^n Q_{-5}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^n, & Q_{8n-4} &= (-1)^n Q_{-4}(P_{-2}Q_{-4}R_{-6}S_0+1)^n, \\
Q_{8n-3} &= \frac{(-1)^n Q_{-3}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n}, & Q_{8n-2} &= \frac{(-1)^n Q_{-2}}{(P_0Q_{-2}R_{-4}S_{-6}+1)^n}, \\
Q_{8n-1} &= (-1)^n Q_{-1}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^n, & Q_{8n} &= (-1)^n Q_0(P_{-6}Q_0R_{-2}S_{-4}+1)^n,
\end{aligned}$$

$$\begin{aligned}
R_{8n-7} &= \frac{(-1)^n R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^n}, & R_{8n-6} &= \frac{(-1)^n R_{-6}}{(P_{-2}Q_{-4}R_{-6}S_0+1)^n},\\
R_{8n-5} &= (-1)^n R_{-5}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n, & R_{8n-4} &= (-1)^n R_{-4}(P_0Q_{-2}R_{-4}S_{-6}+1)^n,\\
R_{8n-3} &= \frac{(-1)^n R_{-3}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^n}, & R_{8n-2} &= \frac{(-1)^n R_{-2}}{(P_{-6}Q_0R_{-2}S_{-4}+1)^n},\\
R_{8n-1} &= (-1)^n R_{-1}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^n, & R_{8n} &= (-1)^n R_0(P_{-4}Q_{-6}R_0S_{-2}+1)^n,\\
S_{8n-7} &= \frac{(-1)^n S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^n}, & S_{8n-6} &= \frac{(-1)^n S_{-6}}{(P_0Q_{-2}R_{-4}S_{-6}+1)^n},\\
S_{8n-5} &= (-1)^n S_{-5}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^n, & S_{8n-4} &= (-1)^n S_{-4}(P_{-6}Q_0R_{-2}S_{-4}+1)^n,\\
S_{8n-3} &= \frac{(-1)^n S_{-3}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^n}, & S_{8n-2} &= \frac{(-1)^n S_{-2}}{(P_{-4}Q_{-6}R_0S_{-2}+1)^n},\\
S_{8n-1} &= (-1)^n S_{-1}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^n, & S_{8n} &= (-1)^n S_0(P_{-2}Q_{-4}R_{-6}S_0+1)^n,
\end{aligned}$$

Proof. The result is immediate for $n = 0$. For $n > 0$, assume by induction that it holds for $n-1$ and that it can be written in the following form:

$$\begin{aligned}
P_{8n-15} &= \frac{(-1)^{n-1} P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}, & P_{8n-14} &= \frac{(-1)^{n-1} P_{-6}}{(P_{-6}Q_0R_{-2}S_{-4}+1)^{n-1}},\\
P_{8n-13} &= (-1)^{n-1} P_{-5}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}, & P_{8n-12} &= (-1)^{n-1} P_{-4}(P_{-4}Q_{-6}R_0S_{-2}+1)^{n-1},\\
P_{8n-11} &= \frac{(-1)^{n-1} P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}, & P_{8n-10} &= \frac{(-1)^{n-1} P_{-2}}{(P_{-2}Q_{-4}R_{-6}S_0+1)^{n-1}},\\
P_{8n-9} &= (-1)^{n-1} P_{-1}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}, & P_{8n-8} &= (-1)^{n-1} P_0(P_0Q_{-2}R_{-4}S_{-6}+1)^{n-1},\\
Q_{8n-15} &= \frac{(-1)^{n-1} Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}, & Q_{8n-14} &= \frac{(-1)^{n-1} Q_{-6}}{(P_{-4}Q_{-6}R_0S_{-2}+1)^{n-1}},\\
Q_{8n-13} &= (-1)^{n-1} Q_{-5}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}, & Q_{8n-12} &= (-1)^{n-1} Q_{-4}(P_{-2}Q_{-4}R_{-6}S_0+1)^{n-1},\\
Q_{8n-11} &= \frac{(-1)^{n-1} Q_{-3}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}, & Q_{8n-10} &= \frac{(-1)^{n-1} Q_{-2}}{(P_0Q_{-2}R_{-4}S_{-6}+1)^{n-1}},\\
Q_{8n-9} &= (-1)^{n-1} Q_{-1}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}, & Q_{8n-8} &= (-1)^{n-1} Q_0(P_{-6}Q_0R_{-2}S_{-4}+1)^{n-1},\\
R_{8n-15} &= \frac{(-1)^{n-1} R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}, & R_{8n-14} &= \frac{(-1)^{n-1} R_{-6}}{(P_{-2}Q_{-4}R_{-6}S_0+1)^{n-1}},\\
R_{8n-13} &= (-1)^{n-1} R_{-5}(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}, & R_{8n-12} &= (-1)^{n-1} R_{-4}(P_0Q_{-2}R_{-4}S_{-6}+1)^{n-1},\\
R_{8n-11} &= \frac{(-1)^{n-1} R_{-3}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}, & R_{8n-10} &= \frac{(-1)^{n-1} R_{-2}}{(P_{-6}Q_0R_{-2}S_{-4}+1)^{n-1}},\\
R_{8n-9} &= (-1)^{n-1} R_{-1}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}, & R_{8n-8} &= (-1)^{n-1} R_0(P_{-4}Q_{-6}R_0S_{-2}+1)^{n-1},\\
S_{8n-15} &= \frac{(-1)^{n-1} S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}, & S_{8n-14} &= \frac{(-1)^{n-1} S_{-6}}{(P_0Q_{-2}R_{-4}S_{-6}+1)^{n-1}},\\
S_{8n-13} &= (-1)^{n-1} S_{-5}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}, & S_{8n-12} &= (-1)^{n-1} S_{-4}(P_{-6}Q_0R_{-2}S_{-4}+1)^{n-1},
\end{aligned}$$

$$S_{8n-11} = \frac{(-1)^{n-1} S_{-3}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}, \qquad S_{8n-10} = \frac{(-1)^{n-1} S_{-2}}{(P_{-4}Q_{-6}R_{0}S_{-2}+1)^{n-1}},$$
$$S_{8n-9} = (-1)^{n-1} S_{-1}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}, \qquad S_{8n-8} = (-1)^{n-1} S_{0}(P_{-2}Q_{-4}R_{-6}S_{0}+1)^{n-1},$$

Now, it follows from system (4) that

$$\begin{aligned}
P_{8n-7} &= \frac{P_{8n-15}}{-1-Q_{8n-9}R_{8n-11}S_{8n-13}P_{8n-15}} \\
&= \frac{\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left((-1)^{n-1}Q_{-1}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}R_{-3}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}\right)\cdot \\ \left((-1)^{n-1}S_{-5}(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{(-1)^{n-1}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n-1}}}{-1-P_{-7}Q_{-1}R_{-3}S_{-5}} = \frac{(-1)^{n}P_{-7}}{(P_{-7}Q_{-1}R_{-3}S_{-5}+1)^{n}},
\end{aligned}$$

$$\begin{aligned}
Q_{8n-7} &= \frac{Q_{8n-15}}{-1-R_{8n-9}S_{8n-11}P_{8n-13}Q_{8n-15}} \\
&= \frac{\frac{(-1)^{n-1}Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left((-1)^{n-1}R_{-1}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}S_{-3}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}\right)\cdot \\ \left((-1)^{n-1}P_{-5}(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{(-1)^{n-1}Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n-1}}}{-1-P_{-5}Q_{-7}R_{-1}S_{-3}} = \frac{(-1)^{n}Q_{-7}}{(P_{-5}Q_{-7}R_{-1}S_{-3}+1)^{n}},
\end{aligned}$$

$$\begin{aligned}
R_{8n-7} &= \frac{R_{8n-15}}{-1-S_{8n-9}P_{8n-11}Q_{8n-13}R_{8n-15}} \\
&= \frac{\frac{(-1)^{n-1}R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left((-1)^{n-1}S_{-1}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}P_{-3}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}\right)\cdot \\ \left((-1)^{n-1}Q_{-5}(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{(-1)^{n-1}R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n-1}}}{-1-P_{-3}Q_{-5}R_{-7}S_{-1}} = \frac{(-1)^{n}R_{-7}}{(P_{-3}Q_{-5}R_{-7}S_{-1}+1)^{n}},
\end{aligned}$$

$$\begin{aligned}
S_{8n-7} &= \frac{S_{8n-15}}{-1-P_{8n-9}Q_{8n-11}R_{8n-13}S_{8n-15}} \\
&= \frac{\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}}{-1-\left(\begin{array}{c}\left((-1)^{n-1}P_{-1}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}Q_{-3}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}\right)\cdot \\ \left((-1)^{n-1}R_{-5}\left(P_{-1}Q_{-3}R_{-5}S_{-7}+1\right)^{n-1}\right)\cdot\left(\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}\right)\end{array}\right)} \\
&= \frac{\frac{(-1)^{n-1}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n-1}}}{-1-P_{-1}Q_{-3}R_{-5}S_{-7}} = \frac{(-1)^{n}S_{-7}}{(P_{-1}Q_{-3}R_{-5}S_{-7}+1)^{n}}.
\end{aligned}$$

Similarly, the other relations can also be established. Therefore, by induction, the stated formulas hold for all $n \geq 0$.

Corollary 5. *The system* (4) *admits a periodic solution of period eight if and only if the following conditions are satisfied:*

$$\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5} = P_{-6}Q_{0}R_{-2}S_{-4} = P_{-5}Q_{-7}R_{-1}S_{-3} = P_{-4}Q_{-6}R_{0}S_{-2} = \\
P_{-3}Q_{-5}R_{-7}S_{-1} = P_{-2}Q_{-4}R_{-6}S_{0} = P_{-1}Q_{-3}R_{-5}S_{-7} = P_{0}Q_{-2}R_{-4}S_{-6} = -2.
\end{gathered}$$

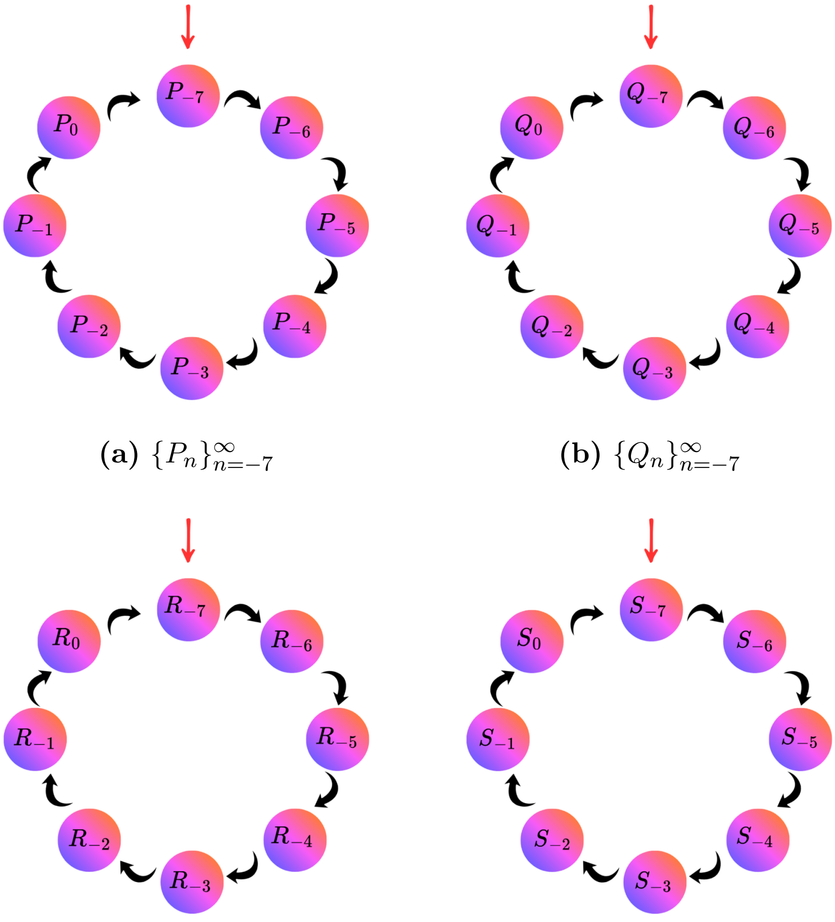
In this case, the solution has the following form (see Figure 2)

$$\{P_n\}_{n=-7}^{\infty} = \{P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_{0}, P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_{0}, \dots\},$$

$$\{Q_n\}_{n=-7}^{\infty} = \{Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_{0}, Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_{0}, \dots\},$$

$$\{R_n\}_{n=-7}^{\infty} = \{R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_{0}, R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_{0}, \dots\},$$

$$\{S_n\}_{n=-7}^{\infty} = \{S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_{0}, S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_{0}, \dots\}.$$



(c) $\{R_n\}_{n=-7}^{\infty}$ **(d)** $\{S_n\}_{n=-7}^{\infty}$

Figure 2: Illustration of a periodic solution of period eight for system (4).

Proof. First, suppose that there exists a solution of period eight

$$\{P_n\}_{n=-7}^{\infty} = \{P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, P_{-7}, P_{-6}, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, \dots\},$$

$$\{Q_n\}_{n=-7}^{\infty} = \{Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, Q_{-7}, Q_{-6}, Q_{-5}, Q_{-4}, Q_{-3}, Q_{-2}, Q_{-1}, Q_0, \dots\},$$

$$\{R_n\}_{n=-7}^{\infty} = \{R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, R_{-7}, R_{-6}, R_{-5}, R_{-4}, R_{-3}, R_{-2}, R_{-1}, R_0, \dots\},$$

$$\{S_n\}_{n=-7}^{\infty} = \{S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, S_{-7}, S_{-6}, S_{-5}, S_{-4}, S_{-3}, S_{-2}, S_{-1}, S_0, \dots\}.$$

From the system (4), we see that

$$P_{-7} = \frac{P_{-7}}{-1 - Q_{-1}R_{-3}S_{-5}P_{-7}}, P_{-6} = \frac{P_{-6}}{-1 - Q_0R_{-2}S_{-4}P_{-6}}, P_{-5} = \frac{P_{-5}}{-1 - Q_{-7}R_{-1}S_{-3}P_{-5}},$$

$$\begin{aligned}
P_{-4} &= \frac{P_{-4}}{-1-Q_{-6}R_0S_{-2}P_{-4}}, P_{-3}=\frac{P_{-3}}{-1-Q_{-5}R_{-7}S_{-1}P_{-3}}, P_{-2}=\frac{P_{-2}}{-1-Q_{-4}R_{-6}S_0P_{-2}},\\
P_{-1} &= \frac{P_{-1}}{-1-Q_{-3}R_{-5}S_{-7}P_{-1}}, P_0=\frac{P_0}{-1-Q_{-2}R_{-4}S_{-6}P_0}, Q_{-7}=\frac{Q_{-7}}{-1-R_{-1}S_{-3}P_{-5}Q_{-7}},\\
Q_{-6} &= \frac{Q_{-6}}{-1-R_0S_{-2}P_{-4}Q_{-6}}, Q_{-5}=\frac{Q_{-5}}{-1-R_{-7}S_{-1}P_{-3}Q_{-5}}, Q_{-4}=\frac{Q_{-4}}{-1-R_{-6}S_0P_{-2}Q_{-4}},\\
Q_{-3} &= \frac{Q_{-3}}{-1-R_{-5}S_{-7}P_{-1}Q_{-3}}, Q_{-2}=\frac{Q_{-2}}{-1-R_{-4}S_{-6}P_0Q_{-2}}, Q_{-1}=\frac{Q_{-1}}{-1-R_{-3}S_{-5}P_{-7}Q_{-1}},\\
Q_0 &= \frac{Q_0}{-1-R_{-2}S_{-4}P_{-6}Q_0}, R_{-7}=\frac{R_{-7}}{-1-S_{-1}P_{-3}Q_{-5}R_{-7}}, R_{-6}=\frac{R_{-6}}{-1-S_0P_{-2}Q_{-4}R_{-6}},\\
R_{-5} &= \frac{R_{-5}}{-1-S_{-7}P_{-1}Q_{-3}R_{-5}}, R_{-4}=\frac{R_{-4}}{-1-S_{-6}P_0Q_{-2}R_{-4}}, R_{-3}=\frac{R_{-3}}{-1-S_{-5}P_{-7}Q_{-1}R_{-3}},\\
R_{-2} &= \frac{R_{-2}}{-1-S_{-4}P_{-6}Q_0R_{-2}}, R_{-1}=\frac{R_{-1}}{-1-S_{-3}P_{-5}Q_{-7}R_{-1}}, R_0=\frac{R_0}{-1-S_{-2}P_{-4}Q_{-6}R_0},\\
S_{-7} &= \frac{S_{-7}}{-1-P_{-1}Q_{-3}R_{-5}S_{-7}}, S_{-6}=\frac{S_{-6}}{-1-P_0Q_{-2}R_{-4}S_{-6}}, S_{-5}=\frac{S_{-5}}{-1-P_{-7}Q_{-1}R_{-3}S_{-5}},\\
S_{-4} &= \frac{S_{-4}}{-1-P_{-6}Q_0R_{-2}S_{-4}}, S_{-3}=\frac{S_{-3}}{-1-P_{-5}Q_{-7}R_{-1}S_{-3}}, S_{-2}=\frac{S_{-2}}{-1-P_{-4}Q_{-6}R_0S_{-2}},\\
S_{-1} &= \frac{S_{-1}}{-1-P_{-3}Q_{-5}R_{-7}S_{-1}}, S_0=\frac{S_0}{-1-P_{-2}Q_{-4}R_{-6}S_0}.
\end{aligned}$$

Then

$$\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5}=P_{-6}Q_0R_{-2}S_{-4}=P_{-5}Q_{-7}R_{-1}S_{-3}=P_{-4}Q_{-6}R_0S_{-2}=\\
P_{-3}Q_{-5}R_{-7}S_{-1}=P_{-2}Q_{-4}R_{-6}S_0=P_{-1}Q_{-3}R_{-5}S_{-7}=P_0Q_{-2}R_{-4}S_{-6}=-2.
\end{gathered}$$

Second, suppose the converse:

$$\begin{gathered}
P_{-7}Q_{-1}R_{-3}S_{-5}=P_{-6}Q_0R_{-2}S_{-4}=P_{-5}Q_{-7}R_{-1}S_{-3}=P_{-4}Q_{-6}R_0S_{-2}=\\
P_{-3}Q_{-5}R_{-7}S_{-1}=P_{-2}Q_{-4}R_{-6}S_0=P_{-1}Q_{-3}R_{-5}S_{-7}=P_0Q_{-2}R_{-4}S_{-6}=-2.
\end{gathered}$$

Then, we see from the solution of system (4) that

$$\begin{array}{ccc}
P_{8n-7}=P_{-7}, & P_{8n-6}=P_{-6}, & P_{8n-5}=P_{-5},\\
P_{8n-4}=P_{-4}, & P_{8n-3}=P_{-3}, & P_{8n-2}=P_{-2},\\
P_{8n-1}=P_{-1}, & P_{8n}=P_0, & Q_{8n-7}=Q_{-7},\\
Q_{8n-6}=Q_{-6}, & Q_{8n-5}=Q_{-5}, & Q_{8n-4}=Q_{-4},\\
Q_{8n-3}=Q_{-3}, & Q_{8n-2}=Q_{-2}, & Q_{8n-1}=Q_{-1},\\
Q_{8n}=Q_0, & R_{8n-7}=R_{-7}, & R_{8n-6}=R_{-6},\\
R_{8n-5}=R_{-5}, & R_{8n-4}=R_{-4}, & R_{8n-3}=R_{-3},\\
R_{8n-2}=R_{-2}, & R_{8n-1}=R_{-1}, & R_{8n}=R_0,\\
S_{8n-7}=S_{-7}, & S_{8n-6}=S_{-6}, & S_{8n-5}=S_{-5},
\end{array}$$

$$S_{8n-4} = S_{-4}, \qquad S_{8n-3} = S_{-3}, \qquad S_{8n-2} = S_{-2},$$
$$S_{8n-1} = S_{-1}, \qquad S_{8n=} = S_0.$$

Thus, we obtain a solution of period eight, and the proof is complete.

Corollary 6. *Let* $\{(P_n, Q_n, R_n, S_n)\}_{n=-7}^{\infty}$ *be a solution of system* (4)*. Then the following statements hold:*

(i) If $P_{-7} = 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $P_{8n-7} = 0$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(ii) If $P_{-6} = 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $P_{8n-6} = 0$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(iii) If $P_{-5} = 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $P_{8n-5} = 0$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(iv) If $P_{-4} = 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $P_{8n-4} = 0$, $Q_{8n-6} = (-1)^n Q_{-6}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(v) If $P_{-3} = 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $P_{8n-3} = 0$, $Q_{8n-5} = (-1)^n Q_{-5}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(vi) If $P_{-2} = 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $P_{8n-2} = 0$, $Q_{8n-4} = (-1)^n Q_{-4}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(vii) If $P_{-1} = 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $P_{8n-1} = 0$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(viii) If $P_0 = 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $P_{8n} = 0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(ix) If $Q_{-7} = 0$, $P_{-5} \neq 0$, $R_{-1} \neq 0$, $S_{-3} \neq 0$ then we have $Q_{8n-7} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $R_{8n-1} = (-1)^n R_{-1}$, $S_{8n-3} = (-1)^n S_{-3}$.

(x) If $Q_{-6} = 0$, $P_{-4} \neq 0$, $R_0 \neq 0$, $S_{-2} \neq 0$ then we have $Q_{8n-6} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $R_{8n} = (-1)^n R_0$, $S_{8n-2} = (-1)^n S_{-2}$.

(xi) If $Q_{-5} = 0$, $P_{-3} \neq 0$, $R_{-7} \neq 0$, $S_{-1} \neq 0$ then we have $Q_{8n-5} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $R_{8n-7} = (-1)^n R_{-7}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xii) If $Q_{-4} = 0$, $P_{-2} \neq 0$, $R_{-6} \neq 0$, $S_0 \neq 0$ then we have $Q_{8n-4} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $R_{8n-6} = (-1)^n R_{-6}$, $S_{8n} = (-1)^n S_0$.

(xiii) If $Q_{-3} = 0$, $P_{-1} \neq 0$, $R_{-5} \neq 0$, $S_{-7} \neq 0$ then we have $Q_{8n-3} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $R_{8n-5} = (-1)^n R_{-5}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xiv) If $Q_{-2} = 0$, $P_0 \neq 0$, $R_{-4} \neq 0$, $S_{-6} \neq 0$ then we have $Q_{8n-2} = 0$, $P_{8n} = (-1)^n P_0$, $R_{8n-4} = (-1)^n R_{-4}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xv) If $Q_{-1} = 0$, $P_{-7} \neq 0$, $R_{-3} \neq 0$, $S_{-5} \neq 0$ then we have $Q_{8n-1} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $R_{8n-3} = (-1)^n R_{-3}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xvi) If $Q_0 = 0$, $P_{-6} \neq 0$, $R_{-2} \neq 0$, $S_{-4} \neq 0$ then we have $Q_{8n} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $R_{8n-2} = (-1)^n R_{-2}$, $S_{8n-4} = (-1)^n S_{-4}$.

(xvii) If $R_{-7} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $S_{-1} \neq 0$ then we have $R_{8n-7} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $Q_{8n-5} = (-1)^n Q_{-5}$, $S_{8n-1} = (-1)^n S_{-1}$.

(xviii) If $R_{-6} = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $S_0 \neq 0$ then we have $R_{8n-6} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $Q_{8n-4} = (-1)^n Q_{-4}$, $S_{8n} = (-1)^n S_0$.

(xix) If $R_{-5} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $S_{-7} \neq 0$ then we have $R_{8n-5} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $S_{8n-7} = (-1)^n S_{-7}$.

(xx) If $R_{-4} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $S_{-6} \neq 0$ then we have $R_{8n-4} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $S_{8n-6} = (-1)^n S_{-6}$.

(xxi) If $R_{-3} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $S_{-5} \neq 0$ then we have $R_{8n-3} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $S_{8n-5} = (-1)^n S_{-5}$.

(xxii) If $R_{-2} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $S_{-4} \neq 0$ then we have $R_{8n-2} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $S_{8n-4} = (-1)^n S_{-4}$.

(xxiii) If $R_{-1} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $S_{-3} \neq 0$ then we have $R_{8n-1} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $S_{8n-3} = (-1)^n S_{-3}$.

(xxiv) If $R_0 = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $S_{-2} \neq 0$ then we have $R_{8n} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $Q_{8n-6} = (-1)^n Q_{-6}$, $S_{8n-2} = (-1)^n S_{-2}$.

(xxv) If $S_{-7} = 0$, $P_{-1} \neq 0$, $Q_{-3} \neq 0$, $R_{-5} \neq 0$ then we have $S_{8n-7} = 0$, $P_{8n-1} = (-1)^n P_{-1}$, $Q_{8n-3} = (-1)^n Q_{-3}$, $R_{8n-5} = (-1)^n R_{-5}$.

(xxvi) If $S_{-6} = 0$, $P_0 \neq 0$, $Q_{-2} \neq 0$, $R_{-4} \neq 0$ then we have $S_{8n-6} = 0$, $P_{8n} = (-1)^n P_0$, $Q_{8n-2} = (-1)^n Q_{-2}$, $R_{8n-4} = (-1)^n R_{-4}$.

(xxvii) If $S_{-5} = 0$, $P_{-7} \neq 0$, $Q_{-1} \neq 0$, $R_{-3} \neq 0$ then we have $S_{8n-5} = 0$, $P_{8n-7} = (-1)^n P_{-7}$, $Q_{8n-1} = (-1)^n Q_{-1}$, $R_{8n-3} = (-1)^n R_{-3}$.

(xxviii) If $S_{-4} = 0$, $P_{-6} \neq 0$, $Q_0 \neq 0$, $R_{-2} \neq 0$ then we have $S_{8n-4} = 0$, $P_{8n-6} = (-1)^n P_{-6}$, $Q_{8n} = (-1)^n Q_0$, $R_{8n-2} = (-1)^n R_{-2}$.

(xxix) If $S_{-3} = 0$, $P_{-5} \neq 0$, $Q_{-7} \neq 0$, $R_{-1} \neq 0$ then we have $S_{8n-3} = 0$, $P_{8n-5} = (-1)^n P_{-5}$, $Q_{8n-7} = (-1)^n Q_{-7}$, $R_{8n-1} = (-1)^n R_{-1}$.

(xxx) If $S_{-2} = 0$, $P_{-4} \neq 0$, $Q_{-6} \neq 0$, $R_0 \neq 0$ then we have $S_{8n-2} = 0$, $P_{8n-4} = (-1)^n P_{-4}$, $Q_{8n-6} = (-1)^n Q_{-6}$, $R_{8n} = (-1)^n R_0$.

(xxxi) If $S_{-1} = 0$, $P_{-3} \neq 0$, $Q_{-5} \neq 0$, $R_{-7} \neq 0$ then we have $S_{8n-1} = 0$, $P_{8n-3} = (-1)^n P_{-3}$, $Q_{8n-5} = (-1)^n Q_{-5}$, $R_{8n-7} = (-1)^n R_{-7}$.

(xxxii) If $S_0 = 0$, $P_{-2} \neq 0$, $Q_{-4} \neq 0$, $R_{-6} \neq 0$ then we have $S_{8n} = 0$, $P_{8n-2} = (-1)^n P_{-2}$, $Q_{8n-4} = (-1)^n Q_{-4}$, $R_{8n-6} = (-1)^n R_{-6}$.

Proof. The result follows directly from the explicit form of the solution of system (4).

6. Numerical Simulations

The numerical section is used to support and validate the theoretical results obtained in the previous sections. Through numerical simulations, we illustrate how the solutions of the systems evolve over time under different initial conditions (see Table 1). The results also highlight the role of initial conditions in the periodicity and boundedness of the solutions. In addition, the figures demonstrate the sensitivity of the systems to changes in the initial conditions. The simulations are carried out using five different sets of initial conditions, each corresponding to a distinct dynamical behavior.

Table 1: Initial conditions used in the numerical simulations for different cases.

Initial Conditions	**IC1**	**IC2**	**IC3**	**IC4**	**IC5**
P_{-7}	3	0.3	0	2	-2
P_{-6}	2.5	0.5	0.7	0.5	0.5
P_{-5}	2	0.2	0.3	3	3
P_{-4}	1.5	0.1	0.1	1	1
P_{-3}	0.5	0.6	0.6	4	4
P_{-2}	1	1.9	0.2	8	-8
P_{-1}	6.4	0.4	0.4	0.5	0.5
P_0	4.1	0.7	0.5	1	1
Q_{-7}	1	1	0.5	2	2
Q_{-6}	0.5	0.5	0.3	0.5	-0.5
Q_{-5}	1.6	1.2	0.2	1/4	1/4
Q_{-4}	2.8	0.8	0.8	2	2
Q_{-3}	4.2	0.1	0.7	1	-1
Q_{-2}	5.7	1.7	0.5	8	8
Q_{-1}	3.3	0.3	0.4	0.5	0.5
Q_0	1.8	0.2	0.1	4	-4
R_{-7}	0.8	0.8	0.5	1	-1
R_{-6}	2	0.2	0.2	0.5	0.5
R_{-5}	1.5	1.5	0.1	0.5	0.5
R_{-4}	0.3	0.3	0.3	1/4	-1/4
R_{-3}	4.2	0.7	0.7	6	6
R_{-2}	5	1.1	0.9	2	2
R_{-1}	3.8	0.8	0.4	1/3	1/3
R_0	2.4	1.2	0.6	8	8
S_{-7}	1	1	0.1	8	8
S_{-6}	3.8	0.8	0.4	1	1
S_{-5}	0.5	0.5	0.5	1/3	1/3
S_{-4}	2.5	2.1	0.3	0.5	0.5
S_{-3}	4.5	0.3	0.6	1	-1
S_{-2}	1.8	0.8	0.1	0.5	0.5
S_{-1}	3	0.3	0.2	2	2
S_0	1.9	0.4	0.1	1/4	1/4

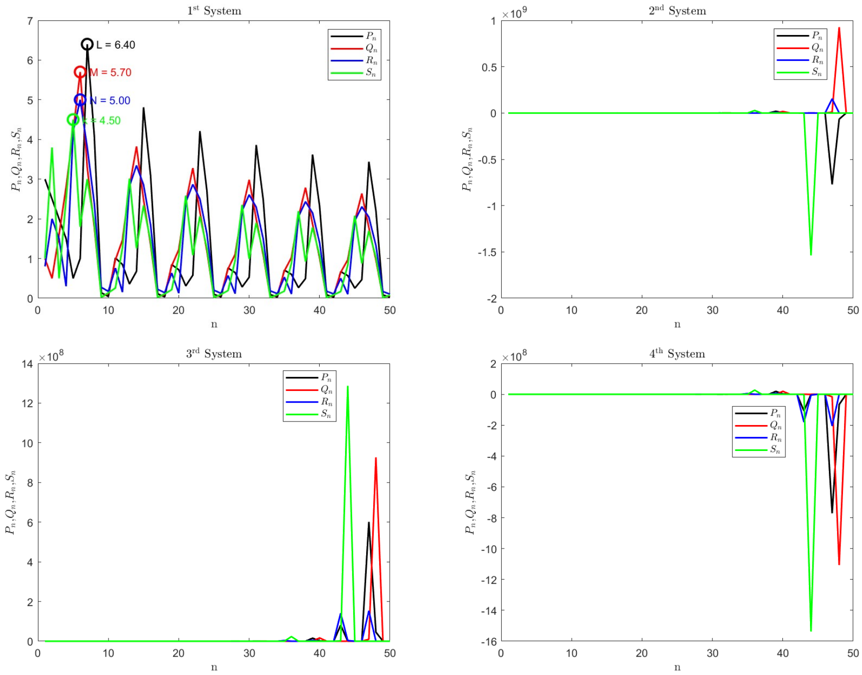


Figure 3: Solution behavior of systems (1)-(4) corresponding to IC1.

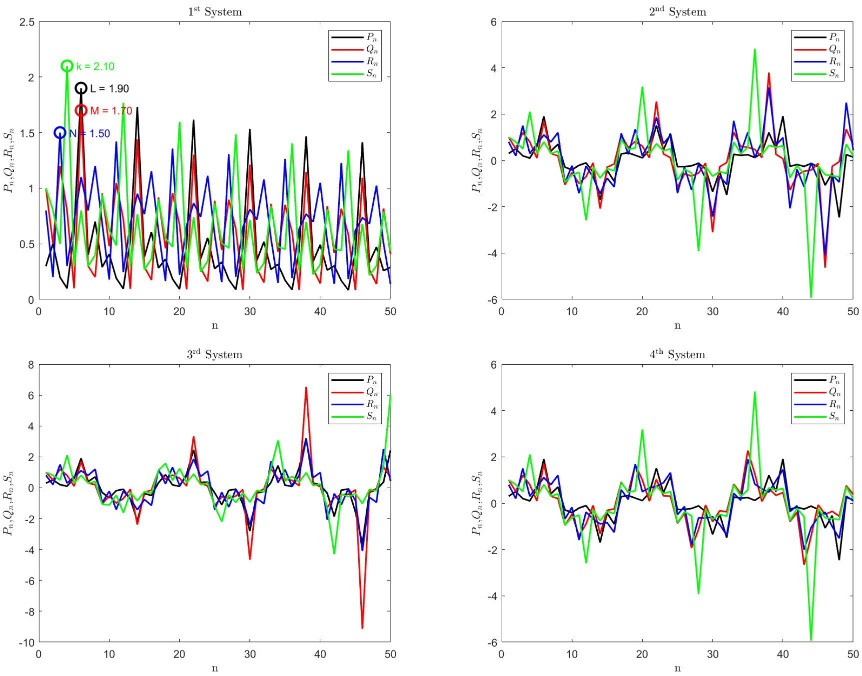


Figure 4: Solution behavior of systems (1)-(4) corresponding to IC2.

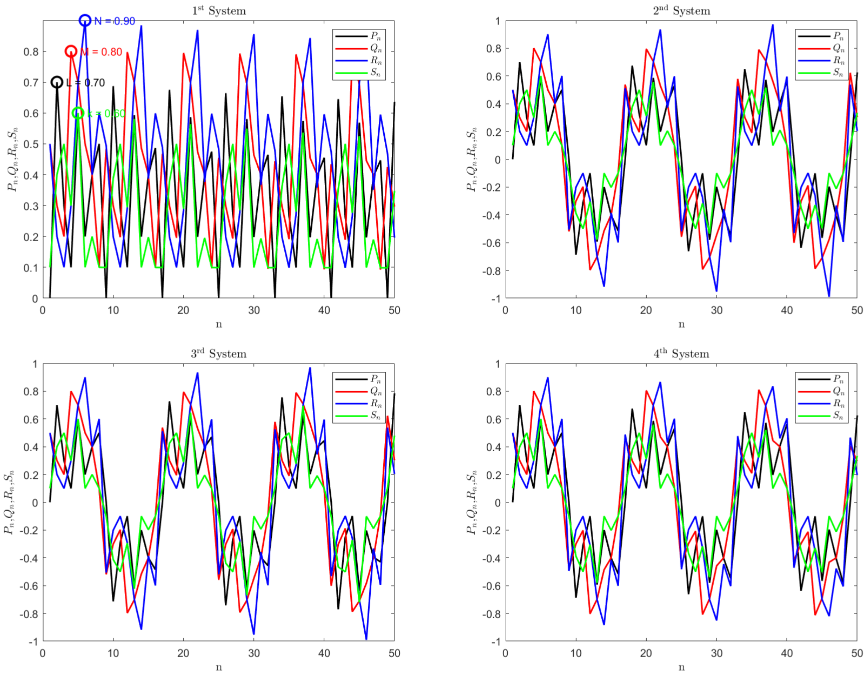


Figure 5: Solution behavior of systems (1)-(4) corresponding to IC3.

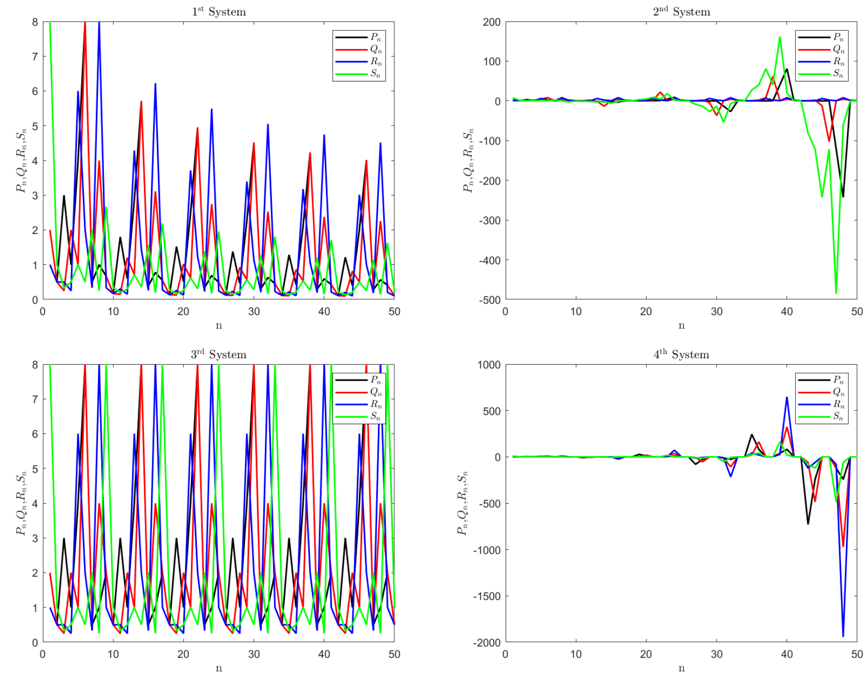


Figure 6: Solution behavior of systems (1)-(4) corresponding to IC4.

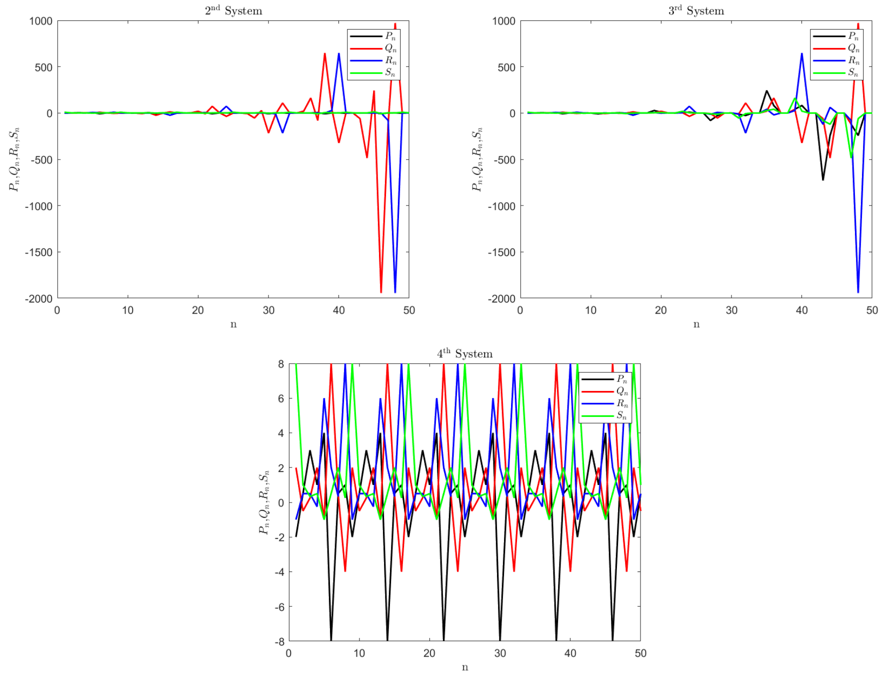


Figure 7: Solution behavior of systems (2)-(4) corresponding to IC5.

Figures 3 and 4 illustrate the behavior of the solutions under slight variations in the initial values. The differences in the solution trajectories highlight the sensitivity of the systems to changes in the initial conditions. Figure 5 presents the behavior of the solutions when one of the initial values is zero. Figure 6 demonstrates that the third system admits periodic solutions of period eight when the conditions stated in Corollary 3 are satisfied. Furthermore, Figure 7 shows that the fourth system exhibits periodic solutions of period eight provided that the conditions given in Corollary 5 hold.

7. Conclusion

In this paper, we examine the qualitative behavior and derive closed-form solutions for a four-dimensional eighth-order system of difference equations under four different cases. The results show that the first system has bounded solutions with non-increasing subsequences, while the third and fourth systems exhibit periodic solutions of period eight under certain conditions. The behavior of the solutions is also analyzed when one of the initial values is zero. Furthermore, numerical simulations confirm and illustrate the theoretical results.

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