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Three Fisher–Rao Identities on the Binary Statistical Manifold: Jeffreys Uniqueness, the Gudermannian Bridge, and Isotropic Retention

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Abstract. On the one-dimensional Fisher–Rao manifold of binary probability distributions parameterized by the visibility coordinate $V \in (-1, 1)$, the Fisher information equals the squared Lorentz factor: $I(V) = \gamma^2(V) = 1/(1-V^2)$. We derive three consequences of this identity concerning reparameterization-invariant priors, arclength bridges between canonical metrics, and coarse-graining of qubit measurements. First, the Jeffreys prior $\pi_J(V) = \gamma(V)/\pi$ is the unique normalized smooth probability density whose associated half-density coincides with the canonical half-density of the metric, and is therefore the unique prior covariant under every smooth reparameterization. Second, the Gudermannian function is the unique smooth bijection relating the Bures arclength coordinate $s_B = \frac{1}{2}\arcsin(V)$ to the Beltrami–Klein arclength coordinate $s_{BK} = \operatorname{arctanh}(V)$ via $2\,s_B = \operatorname{gd}(s_{BK})$; equivalently, $\arcsin(V) = \operatorname{gd}(\operatorname{arctanh}(V))$. Third, for a mixed qubit state of Bloch radius r undergoing transverse rotation, the isotropically averaged ratio of classical to quantum Fisher information has the closed form $R(r) = [2r + (r^2 - 1)\ln((1 + r)/(1 - r))]/(4r^3)$, interpolating monotonically between $R(0) = 1/3$ and $R(1) = 1/2$. Each identity is proved from scratch. These three identities are elementary consequences of $I(V) = \gamma^2(V)$; their joint geometric content positions the binary Fisher–Rao manifold at the intersection of Bayesian statistics, hyperbolic geometry, and quantum measurement theory.

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1. Introduction

The classical Fisher information of a two-outcome Bernoulli experiment with success probability $p = (1+V)/2$, expressed in the visibility coordinate, equals the squared Lorentz factor:

$$I(V) \;=\; \frac{1}{p(1-p)}\left(\frac{dp}{dV}\right)^2 \;=\; \frac{1}{1-V^2} \;=\; \gamma^2(V). \tag{1}$$

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This identity, a three-line computation reproduced in Section 2, equips the one-parameter Bernoulli family with a Riemannian metric whose coefficient is a special-relativistic γ^2. The resulting geometry is that of the open interval $(-1, 1)$ with the one-dimensional Fisher–Rao metric $g(V)\,dV^2 = \gamma^2(V)\,dV^2$.

Three questions arise naturally once this identification is made.

(i) Reparameterization-invariant priors. What is the natural reparameterization-covariant density on this manifold? The prior density proportional to $\sqrt{\det I}$, denoted the Jeffreys prior in the statistics literature [1], has the form $\gamma(V)/Z$. We prove that this prior is uniquely characterized by the property that its associated half-density is the canonical half-density of the metric, and is therefore covariant under every smooth reparameterization.

(ii) Arclength bridges between canonical metrics. The Bures metric on the qubit Bloch ball restricts to a one-dimensional metric on the equatorial diameter with coefficient $\gamma^2(V)/4$. The Beltrami–Klein model of hyperbolic geometry assigns the coefficient $\gamma^4(V)$ to the same interval [2]. In the V-chart the two are conformally related by the factor $4\gamma^2$; their arclength coordinates, however, are related by the Gudermannian function. We prove that gd is the unique smooth bijection with $\arcsin(V) = \mathrm{gd}(\operatorname{arctanh}(V))$, locating the Gudermannian as the canonical arclength bridge between the two metrics.

(iii) Retention under isotropic coarse-graining. For a mixed qubit with Bloch radius $r \in [0, 1]$ rotating about an axis perpendicular to its Bloch vector, how much Fisher information is retained, on average, when the measurement axis is drawn isotropically on S^2? We derive a closed form for the retention ratio $R(r) = \langle I_{\mathrm{cl}}\rangle_{S^2}/F_Q$ and compute its limiting values, series expansion, and derivative.

The three identities are not individually new in statistics, information geometry, or quantum measurement theory. Each is elementary once its setting is fixed, and prior treatments establish the three in separate settings. What is offered here is their derivation from the single identity (1) and the resulting observation that one binary manifold carries all three structures at once; this joint reading, rather than any one identity, is the contribution. Each section is self-contained; the proofs depend only on elementary calculus and differential geometry. Further consequences of (1) are collected by the author in two Zenodo preprints [3, 4], which are not peer-reviewed and are cited here only for context.

Organization. Section 2 fixes notation and proves equation (1). Section 3 proves the Jeffreys-prior uniqueness theorem. Section 4 proves the Gudermannian uniqueness theorem. Section 5 proves the closed-form retention theorem. Section 6 discusses applications and open directions.

2. Setup and the Fisher–Lorentz identity

Consider a two-outcome projective measurement on a qubit with outcome probabilities

$$p = \tfrac{1+V}{2}, \qquad 1-p = \tfrac{1-V}{2}, \qquad V \in (-1, 1).$$

Here V is the measurement visibility: $V = \pm 1$ corresponds to a deterministic outcome, $V = 0$ to a fair coin. The classical Fisher information of this family with respect to V is

$$I(V) \;=\; \sum_{k\in\{0,1\}} \frac{1}{p_k}\left(\frac{dp_k}{dV}\right)^2 \;=\; \frac{(1/2)^2}{(1+V)/2} + \frac{(-1/2)^2}{(1-V)/2} \;=\; \frac{1}{1-V^2}.$$

Writing $\gamma(V) = 1/\sqrt{1-V^2}$ for the Lorentz factor of rapidity $\eta = \operatorname{arctanh}(V)$ recovers the identity $I(V) = \gamma^2(V)$. The *binary Fisher–Rao manifold* (M, g) is $M = (-1, 1)$ equipped with the Riemannian metric $g(V)\,dV^2 = \gamma^2(V)\,dV^2$. Throughout, we use rapidity $\eta = \operatorname{arctanh}(V)$ interchangeably with visibility V; under this change of coordinates $\gamma^2(V) = \cosh^2(\eta)$ and the metric $g(V)\,dV^2$ becomes the flat metric $d\eta^2$.

The following Riemannian volume computation is used below.

Lemma 1. *On the* 1*D Riemannian manifold* (M, g) *with* $g(V) = \gamma^2(V)$*, the Riemannian volume form is* $\omega_g = \gamma(V)\,dV$*. The total Riemannian volume is*

$$\operatorname{vol}(M, g) \;=\; \int_{-1}^{1} \gamma(V)\,dV \;=\; \int_{-1}^{1} \frac{dV}{\sqrt{1-V^2}} \;=\; \pi.$$

General background on the Fisher–Rao metric and its invariance properties may be found in Amari [5] and Nielsen [6]; the classical invariance theorem underlying Definition 1 below is due to Chentsov [7].

3. The Jeffreys prior as a unique invariant half-density

Let $\pi(V)\,dV$ be a probability measure on M with smooth positive density π. Under an orientation-preserving diffeomorphism $\varphi\colon M \to M$, the pushforward $\varphi_*\pi$ has density

$$(\varphi_*\pi)(V) \;=\; \pi(\varphi^{-1}(V)) \cdot (\varphi^{-1})'(V).$$

We compare this transformation with the way half-densities transform on (M, g).

Definition 1. *A* half-density *on* (M, g) *is a section of the bundle* $|\Lambda|^{1/2}$ *of half-densities, with transformation rule* $\sigma \mapsto |\varphi'|^{1/2}\sigma$ *under* $\varphi \in \operatorname{Diff}_+(M)$*. On* (M, g) *with metric coefficient* $g(V) = \gamma^2(V)$*, the canonical half-density attached to* g *is* $\omega_g^{1/2} = \sqrt{\gamma(V)}(dV)^{1/2}$*, while the density of the volume form itself is* $\gamma(V)\,dV$*.*

Theorem 1 (Jeffreys Prior Uniqueness)**.** *On* $(M, g) = ((-1, 1), \gamma^2(V)\,dV^2)$*, the Jeffreys prior*

$$\pi_J(V) \;=\; \frac{\gamma(V)}{\pi}$$

is the unique normalized smooth probability density whose associated half-density $\pi_J^{1/2}(dV)^{1/2}$ *coincides with the canonical half-density* $\omega_g^{1/2}$ *of* (M, g)*, and is therefore the unique probability density on* M *covariant under every smooth reparameterization* $\varphi \in \operatorname{Diff}_+(M)$*.*

Proof. A smooth positive probability density π determines a half-density $\pi^{1/2}(dV)^{1/2}$. This half-density coincides with the canonical half-density $\omega_g^{1/2} = \sqrt{\gamma(V)}(dV)^{1/2}$ of Definition 1 if and only if $\pi^{1/2}(V) = \sqrt{\gamma(V)}$, i.e. $\pi(V) = \gamma(V)$. Normalization $\int_{-1}^{1} \gamma(V)\, dV = \pi$ (Lemma 1) gives $\pi_J(V) = \gamma(V)/\pi$.

It remains to see why this characterization is reparameterization-invariant. Every smooth positive density π yields a half-density $\pi^{1/2}(dV)^{1/2}$, and under a change of coordinate $V \mapsto \varphi(V)$ each transforms by the same rule $\sigma \mapsto |\varphi'|^{1/2}\sigma$; this common transformation law does not by itself distinguish any density from the others. What singles out π_J is that its half-density equals $\omega_g^{1/2}$, the half-density built intrinsically from the metric g. Because $\omega_g^{1/2}$ is a coordinate-independent object, the identity $\pi^{1/2}(dV)^{1/2} = \omega_g^{1/2}$ is chart-independent: it holds in one coordinate if and only if it holds in all. The density it determines is therefore the same in every parameterization, and by the first paragraph that density is $\pi_J(V) = \gamma(V)/\pi$, unique up to the constant fixed by normalization [7–9].

Remark 1. *The theorem identifies the invariant prior on the qubit equator with the Lorentz factor [1]. Operationally, $\gamma(V)/\pi$ pushes forward to uniform measure under the change of variable $\theta = \arcsin(V)$: if $V \sim \pi_J$ then $\theta \sim \mathrm{Uniform}(-\pi/2, \pi/2)$. The reparameterization-invariant density on qubit visibility, written in the natural geodesic coordinate θ, is flat.*

4. The Gudermannian as a unique arclength bridge

Consider two Riemannian structures on $I = (-1, 1)$:

$$
\begin{aligned}
g_B(V)\, dV^2 &= \frac{\gamma^2(V)}{4}\, dV^2 && \text{(Bures, restricted 1D coefficient)},\\
g_{BK}(V)\, dV^2 &= \gamma^4(V)\, dV^2 && \text{(Beltrami–Klein, 1D coefficient)}.
\end{aligned}
$$

The first is the radial-direction coefficient of the Bures metric on the qubit Bloch ball evaluated on the equatorial diameter [10]. The second is the standard Beltrami–Klein metric on $(-1, 1)$ viewed as a chart of the hyperbolic line [2].

The two metrics are conformally related in the V-chart: $g_{BK}(V) = 4\gamma^2(V)\, g_B(V)$, so the identity map $(I, g_{BK}) \to (I, g_B)$ is conformal with conformal factor $4\gamma^2$. The content of the Gudermannian-bridge theorem is not about the V-chart itself but about the *arclength coordinates* of the two metrics.

Definition 2. *The Bures arclength from* 0 *to* V *is*

$$
s_B(V) \;=\; \int_0^V \tfrac{1}{2}\gamma(V')\, dV' \;=\; \tfrac{1}{2}\arcsin(V).
$$

The Beltrami–Klein arclength from 0 *to* V *is*

$$
s_{BK}(V) \;=\; \int_0^V \gamma^2(V')\, dV' \;=\; \operatorname{arctanh}(V).
$$

Theorem 2 (Gudermannian Arclength Bridge)**.** *The Bures and Beltrami–Klein arclength coordinates of the interval* $(-1, 1)$ *are related by*

$$2\, s_B(V) \;=\; \operatorname{gd}\bigl(s_{BK}(V)\bigr), \qquad \textit{i.e.,} \qquad \arcsin(V) \;=\; \operatorname{gd}\bigl(\operatorname{arctanh}(V)\bigr), \tag{2}$$

where $\operatorname{gd}\colon \mathbb{R} \to (-\pi/2, \pi/2)$ *is the Gudermannian function*

$$\operatorname{gd}(\eta) \;=\; \arctan(\sinh\eta) \;=\; \arcsin(\tanh\eta) \;=\; 2\arctan(\tanh(\eta/2)).$$

The function gd *is the unique orientation-preserving diffeomorphism* $\mathbb{R} \to (-\pi/2, \pi/2)$ *with* $\operatorname{gd}'(\eta) = \operatorname{sech}(\eta)$ *and* $\operatorname{gd}(0) = 0$*; no other smooth bijection relates the Bures and Beltrami–Klein arclengths of* $(-1, 1)$.

Proof. From the definitions, $s_B(V) = \frac{1}{2}\arcsin(V)$ and $s_{BK}(V) = \operatorname{arctanh}(V)$. Using $V = \tanh(s_{BK})$ and the classical identity $\arcsin(\tanh\eta) = \operatorname{gd}(\eta)$ [11]:

$$2\, s_B(V) \;=\; \arcsin(V) \;=\; \arcsin(\tanh(s_{BK}(V))) \;=\; \operatorname{gd}(s_{BK}(V)).$$

For uniqueness, suppose $f\colon \mathbb{R} \to (-\pi/2, \pi/2)$ is another smooth bijection satisfying $2\, s_B(V) = f(s_{BK}(V))$ for all $V \in (-1, 1)$. Then $f(s_{BK}(V)) = \arcsin(V)$. Since $s_{BK}\colon (-1, 1) \to \mathbb{R}$ is a smooth bijection with inverse tanh, substituting $\eta = s_{BK}(V)$ gives $f(\eta) = \arcsin(\tanh\eta) = \operatorname{gd}(\eta)$. Hence $f = \operatorname{gd}$.

Corollary 1 (Conformal factor in the V-chart)**.** *The Bures and Beltrami–Klein metrics on the* V*-chart of* $(-1, 1)$ *are conformally related:*

$$g_{BK}(V) \;=\; 4\gamma^2(V)\, g_B(V).$$

The conformal factor $4\gamma^2(V)$ *is four times the squared Lorentz factor; it diverges at the boundary* $V \to \pm 1$.

Proof. Direct from the metric coefficients: $g_{BK}/g_B = \gamma^4/(\gamma^2/4) = 4\gamma^2$.

Remark 2. *The Gudermannian function originated in cartography (relating the Mercator projection to the equirectangular projection) and appears in special relativity as the conversion between rapidity* η *and velocity-angle* $\operatorname{gd}(\eta) \in (-\pi/2, \pi/2)$ *[12, 13]. Theorem 2 locates it as the unique arclength-transfer map between the Bures metric on the qubit equator and the Beltrami–Klein metric on the hyperbolic line. Equivalently, a uniform distribution on the Bures angle* $2\, s_B$ *pushes forward to the* gd*-image of a uniform distribution on Beltrami–Klein rapidity.*

5. Isotropic measurement retention

Let $\rho(\theta) = \frac{1}{2}(I + r\,\hat{\mathbf{r}}(\theta)\cdot\boldsymbol{\sigma})$ be a mixed qubit state with Bloch radius $r \in [0, 1]$ and unit Bloch direction $\hat{\mathbf{r}}(\theta)$ undergoing transverse rotation of unit angular speed about an axis $\hat{\mathbf{a}} \perp \hat{\mathbf{r}}$. The quantum Fisher information with respect to θ is $F_Q = r^2$; this identification follows from the monotone metric uniqueness theorem of Petz [14] together with the symmetric-logarithmic-derivative formula of Braunstein and Caves [10].

Fix $\hat{\mathbf{r}}(0) = \hat{\mathbf{z}}$, $\hat{\mathbf{a}} = \hat{\mathbf{y}}$, so $\partial_\theta \hat{\mathbf{r}}|_{\theta=0} = \hat{\mathbf{x}}$. For a projective measurement along a unit vector $\hat{\mathbf{n}} \in S^2$, the outcome probability at parameter θ is

$$p(\hat{\mathbf{n}};\theta) \;=\; \tfrac{1}{2}\big(1 + r\,\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}(\theta)\big),$$

and the classical Fisher information with respect to θ at $\theta = 0$ is

$$I_{\mathrm{cl}}(\hat{\mathbf{n}}) \;=\; \frac{(r\,\hat{\mathbf{n}}\cdot\hat{\mathbf{x}})^2}{1-(r\,\hat{\mathbf{n}}\cdot\hat{\mathbf{z}})^2} \;=\; \frac{r^2 n_x^2}{1-r^2 n_z^2}.$$

Theorem 3 (Isotropic Retention)**.** *The ratio of isotropically averaged classical Fisher information to quantum Fisher information is*

$$R(r) \;=\; \frac{\langle I_{\mathrm{cl}}\rangle_{S^2}}{F_Q} \;=\; \frac{1}{4r^3}\left[2r + (r^2-1)\ln\frac{1+r}{1-r}\right]. \tag{3}$$

The function $R\colon [0,1] \to \mathbb{R}$ is smooth, monotonically increasing, and satisfies

$$R(0) \;=\; \tfrac{1}{3}, \qquad R(1) \;=\; \tfrac{1}{2}, \qquad R(r) \;=\; \tfrac{1}{3} + \tfrac{r^2}{15} + O(r^4) \ \textit{as } r \to 0.$$

Proof. Parameterize $\hat{\mathbf{n}} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$. Then

$$I_{\mathrm{cl}}(\hat{\mathbf{n}}) \;=\; \frac{r^2\sin^2\alpha\cos^2\beta}{1-r^2\cos^2\alpha}.$$

The isotropic average over S^2 is

$$\langle I_{\mathrm{cl}}\rangle_{S^2} \;=\; \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{\pi} \frac{r^2\sin^2\alpha\cos^2\beta}{1-r^2\cos^2\alpha}\sin\alpha\,d\alpha\,d\beta.$$

The β-integral contributes $\int_0^{2\pi}\cos^2\beta\,d\beta = \pi$. What remains is

$$\langle I_{\mathrm{cl}}\rangle_{S^2} \;=\; \frac{r^2}{4}\int_0^{\pi}\frac{\sin^3\alpha}{1-r^2\cos^2\alpha}\,d\alpha \;=\; \frac{r^2}{2}\int_0^1 \frac{1-u^2}{1-r^2u^2}\,du,$$

where $u = \cos\alpha$. Decompose

$$\frac{1-u^2}{1-r^2u^2} \;=\; \frac{1}{r^2} + \frac{r^2-1}{r^2(1-r^2u^2)}.$$

Then

$$\int_0^1 \frac{1-u^2}{1-r^2u^2}\,du \;=\; \frac{1}{r^2} - \frac{1-r^2}{r^2}\int_0^1\frac{du}{1-r^2u^2}.$$

The remaining integral is $\int_0^1 du/(1-r^2u^2) = (1/(2r))\ln((1+r)/(1-r))$. Hence

$$\int_0^1 \frac{1-u^2}{1-r^2u^2}\,du \;=\; \frac{1}{r^2} - \frac{1-r^2}{2r^3}\ln\frac{1+r}{1-r}.$$

Multiplying by $r^2/2$:

$$\langle I_{\mathrm{cl}}\rangle_{S^2} \;=\; \frac{1}{2} - \frac{1-r^2}{4r}\ln\frac{1+r}{1-r}.$$

Dividing by $F_Q = r^2$:

$$R(r) \;=\; \frac{1}{2r^2} - \frac{1-r^2}{4r^3}\ln\frac{1+r}{1-r},$$

which rearranges to (3).

Limits. At $r=1$: $\ln((1+r)/(1-r)) \to \infty$ while $(1-r^2)\to 0$. Using $\ln((1+r)/(1-r)) = -\ln(1-r^2)+2\ln(1+r)-\ln 2+O(1)$ and $(1-r^2)\ln(1-r^2)\to 0$, the second term vanishes and $R(1)=1/2$. At $r=0$: expand

$$\ln\frac{1+r}{1-r} \;=\; 2r + \tfrac{2r^3}{3} + \tfrac{2r^5}{5} + O(r^7),$$

so

$$R(r) \;=\; \frac{1}{4r^3}\Big[2r + (r^2-1)\Big(2r + \tfrac{2r^3}{3} + \tfrac{2r^5}{5} + \dots\Big)\Big].$$

Expanding $(r^2-1)(2r+2r^3/3+2r^5/5+\dots) = -2r+2r^3(1-1/3)+2r^5(1/3-1/5)+O(r^7) = -2r+4r^3/3+4r^5/15+O(r^7)$. Adding $2r$ and dividing by $4r^3$:

$$R(r) \;=\; \frac{4r^3/3+4r^5/15+O(r^7)}{4r^3} \;=\; \tfrac{1}{3} + \tfrac{r^2}{15} + O(r^4).$$

Monotonicity. Differentiating $R(r) = [2r+(r^2-1)L(r)]/(4r^3)$ with $L(r)=\ln((1+r)/(1-r))$ and $L'(r) = 2/(1-r^2)$, we obtain after simplification

$$R'(r) \;=\; \frac{(3-r^2)L(r)-6r}{4r^4}, \qquad r\in(0,1).$$

The two series

$$L(r) \;=\; 2r + \tfrac{2r^3}{3} + \tfrac{2r^5}{5} + \tfrac{2r^7}{7} + O(r^9), \qquad \frac{6r}{3-r^2} \;=\; 2r + \tfrac{2r^3}{3} + \tfrac{2r^5}{9} + \tfrac{2r^7}{27} + O(r^9),$$

agree at orders r and r^3; from order r^5 onward every coefficient of $L(r)$ strictly exceeds that of $6r/(3-r^2)$, since $1/(2k+1) > 1/3^k$ for all $k\ge 2$. Hence $L(r) > 6r/(3-r^2)$ on $(0,1)$, and therefore $R'(r)>0$ on $(0,1)$. Equivalently, $(3-r^2)L(r)-6r = 8r^5/15+O(r^7) > 0$ for small r, and the inequality propagates by monotonicity of the remainder.

Numerical values to four digits are reported in Table 1.

6. Discussion and applications

Reparameterization-invariant prior for qubit visibility estimation. Theorem 1 yields a principled default prior for parameter estimation of visibility from a sequence of binary outcomes. In the Bures-angle coordinate $\theta = \arcsin(V)$, the invariant density is Uniform$(-\pi/2,\pi/2)$, which simplifies elicitation and posterior computation in quantum metrology [10, 14]. This is in contrast to the uniform density in V, which is not a half-density with respect to Fisher–Rao volume and is therefore not covariant under smooth reparameterizations [1, 9].

Table 1: Retention ratio $R(r) = \langle I_{\rm cl}\rangle_{S^2}/F_Q$ for representative Bloch radii. The function is monotone from $R(0) = 1/3$ to $R(1) = 1/2$.

r	$R(r)$
0.0	0.3333
0.1	0.3340
0.3	0.3396
0.5	0.3521
0.7	0.3756
0.9	0.4254
0.99	0.4830
1.0	0.5000

The Gudermannian as a canonical information-geometric bridge. Theorem 2 locates the Gudermannian function as the unique arclength-transfer map between the two natural metrics on the qubit equator: the Bures metric (intrinsic to the quantum state space) and the Beltrami–Klein metric (intrinsic to hyperbolic geometry) [2]. The Gudermannian has appeared in classical cartography and in relativistic rapidity calculations [12, 13]; here it appears as the unique map reconciling these two quantum-geometric structures. General background on Fisher–Rao geometry is given by [5, 6]; for invariant-prior theory see [7, 8].

Design of isotropic measurement protocols. Theorem 3 answers a practical question: when a measurement axis cannot be optimally chosen (e.g. because the parameter direction is unknown or because measurements are averaged over a random ensemble), how much Fisher information is retained? The retention ratio $R(r)$ varies only from 1/3 at the maximally mixed limit to 1/2 at the pure-state limit, a factor-of-3/2 range. This quantifies the robustness of randomized or averaged measurement strategies in quantum metrology and tomography. Extending the retention integral to higher-dimensional probes and to non-isotropic axis distributions is a natural direction for further work.

Related work by the author. Equation (1) also serves as the starting point for further identities the author has explored, developed in two Zenodo preprints [3, 4] that are not peer-reviewed. The three theorems established here are self-contained and independent of that material.

Declarations

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