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Class of Groups Close to Groups with the Basis Property

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Abstract. Let G be a finite group. In [1, 2], Jones P. introduced and initiated the study of semigroups and groups with the Basis Property, after that Alkhalaf A. investigated the properties of semigroups and groups exhibiting the Basis Property. In our work, we aim to investigate whether certain classes of groups lack this property. Specifically, we demonstrate that the symmetric groups S_n for $n > 3$, cyclic groups of non-prime-power order, non-abelian simple groups, and groups constructed as extensions of a p-group P by a generalised quaternion group Q, do not possess the Basis Property.

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1. Introduction

A group G is said to possess the Basis Property if, for every subgroup $H \subseteq G$, there exists a basis (also called a minimal generating set) for H, and all such bases are equivalent [1]. This concept is analogous to finite-dimensional vector spaces, where all minimal generating sets (bases) have the same size.

Research by Jones [1, 2] focused on inverse semigroups and groups with this property, establishing that a completely semisimple inverse semigroup has the Basis Property. He also examined various properties associated with groups possessing this property.

Alkhalaf [3–5] classified groups with the Basis Property, demonstrating that every such group must be a Frobenius group. This classification allows for the application of known estimates regarding the nilpotency class of p-subgroups within Fitting $\{p, q\}$-groups, depending on the parameter q.

Alkhalaf also explored the relationship between the Basis Property and the Exchange Property, which relates to how subgroups can be exchanged within generating sets [5].

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Furthermore, I. Al-Dayel and A. Alkhalaf [6, 7] extended this study to completely simple and 0-simple semigroups, establishing necessary and sufficient conditions under which these semigroups exhibit the Basis Property.

2. Characteristics of Groups with the Basis Property

First, this manuscript discusses important properties of groups that possess the Basis Property. These properties highlight structural aspects such as periodicity, solubility, and subgroup behaviors.

A group with the Basis Property is a group in which certain structural conditions hold, notably related to element orders and subgroup structures.

Periodic Group: A group where every element has finite order.

Soluble Group: A group whose derived series terminates with the trivial subgroup after finitely many steps.

A simple group is a group whose only normal subgroups are the trivial groups (Containing just the identity element and the group itself) [8]. We will now introduce the following properties associated with groups possessing the Basis Property (see [9, 10]).

- Any group with the Basis Property is a periodic group, that is, every element has finite order.
- Every finite group with the Basis Property is soluble[1].
- Every finite p-group P (where p is prime) has the Basis Property (via Burnside's theorem [8]).
- Homomorphic images of a finite group with the Basis Property also have this property.

3. Structural Lemmas and Theorems

Lemma 1. *[8, Theorem 1]*

Suppose G is a finite, soluble, non-primary group in which every element has prime power order. In that case, there exist two elements p and q, $p \neq q$ such that the order of a group G is $p^a q^b, a, b \in G$ can be viewed as an extension of a p-group P by a subgroup H, which is one of the following:

- *A cyclic group of order q^b.*
- *A generalized quaternion group with $q = 2$ and $p \neq q$.*
- *A metacyclic group where all Sylow p-subgroups are cyclic, and automorphisms induced by elements from H are regular.*

Understanding the characteristics and restrictions of groups with the Basis Property facilitates the classification of finite groups and the analysis of their subgroup structures. The properties outlined above are fundamental in the theory of group structures, especially in relation to periodicity and solubility. On the other hand, we can prove that cyclic groups whose orders are non-prime powers do not satisfy the Basis Property, which can be shown via a certain lemma and a theorem.

Lemma 2. *If $G = C_{pq}$ where, p and q are co-prime, then $C_{pq} = C_p \times C_q$.*

proof.
Suppose that $G = C_{pq} = \langle a \rangle$, then $|a| = pq$. Define subgroups H, K of G such that H is a group generated by a^p and K is a group generated by a^q , then $|G : H| = q$ and $|G : K| = p$, hence the quotient group G/H isomorphic to $\langle Ha \rangle$ and G/K isomorphic to $\langle Ka \rangle$. Now since $|H| = q$ and $|K| = p$ and $(p, q) = 1$, then $H \cap K = \{e\}$. In addition, $G/(H \cap K) \cong G/H \times G/K$ (by the mapping $g \to (Hg, Kg)$), but since $|G/(H \cap K)| = pq = |G/H \times G/K|$ they are isomorphic. Thus,

$$C_{pq} \cong G/(H \cap K) \cong G/H \times G/K \cong C_p \times C_q.$$

Let G be a group, we define $d(G)$ to be the least number of generators required to generate G (where every minimal generating set has size $d(G)$). For example, given a cyclic group C_n, then $d(C_n) = 1$, and if G is any non-trivial dihedral group, then $d(G) = 2$. Now, based on the preceding introduction, we can present the following theorem.

Theorem 1. *Let G be a finite cyclic group of non-prime-power order n, then G does not have the Basis Property.*

proof.
Assume that $G = \langle a \rangle$ is a finite cyclic group. Since G is cyclic $d(G) = 1$. Now, as $|G| = n$, which can be written as a product of prime powers,

$$n = {n_1}^{i_1} {n_2}^{i_2} \dots {n_s}^{i_s},$$

where each n_j is a proper divisor of n. From the well-established result $C_{pq} \cong C_p \times C_q$. Then G can be expressed as a direct product of cyclic subgroups (see Lemma 3.2), hence we can establish that

$$G = G_{{n_1}^{i_1}} \times G_{{n_2}^{i_2}} \times \dots \times G_{{n_s}^{i_s}},$$

where $G_{{n_j}^{i_j}}$ generated by a_j say. Hence, the set $A = \{a_1, a_2, \dots, a_s\}$. is minimal generating set for G of size $s > 1$. Since omitting any of a_j would mean that the factor $\langle a_j \rangle$ would not generate the whole G. Then the result holds.

However, because G requires more than one generator, its minimal generating set necessarily contains more than one element. This contradicts the initial assumption that the group is cyclic (which would require only one generator). It means that G is not a group with the Bases Property.

Now we would like to discuss the permutation groups S_n.

Theorem 2. *Let S_n be a symmetric group on the set of n elements, where $n > 3$. Then S_n does not have the Basis Property.*

proof.
The symmetric group S_n can be generated by the set of transpositions

$$\{(1), (12), (23), \ldots, ((n-1)n)\}.$$

We can see that this is minimal since obviously none of the individual transpositions can be generated by a combination of any of the other transpositions. In fact, if we omit one of these transpositions, $\{(i(i+1))\}$ say, then the set of remaining transpositions

$$\{(1), (12), \ldots, ((i-1)i), ((i+1)(i+2)), \ldots, ((n-1)n)\}.$$

Generates $S_i \times S_{n-i}$ with S_i benig the symmetric group on points $\{1, 2, ..., i\}$ and S_{n-i} is the symmetric group on points $\{i+1, ..., n\}$. Then the set of transpositions is of size greater than $d(S_n) = 2$ and so the symmetric group does not have the Basis Property.

We notice that the symmetric group S_n does not satisfy the Basis Property for $n > 3$, but we will show that the smallest non-p-group with the Basis Property is the symmetric group on three points.

Example 1. *The symmetric group S_3 on three points $\{1, 2, 3\}$ has the Basis Property with $d(G) = 2$.*

Now as S_3 is not cyclic $d(S_3) \neq 1$, and since $\{(12), (23)\}$ generates S_3 minimally $d(S_3) = 2$. It remains to show that any generating set of S_3 contains a subset of size two that generates S_3.

Notice that, if a set X generates S_3, it cannot consist only of even permutations, otherwise it would generate A_3, not S_3. Thus, it must contain at least one odd permutation. Choose β to be that odd permutation and γ to be a permutation not found in β, then the two-element subset $\{\beta, \gamma\}$ of X contains either two odd permutations or an even permutation and an odd permutation. If γ is an even permutation and β is odd, then we have $S_3 = \langle \gamma, \beta \rangle$ as $\langle \gamma \rangle = A_3$. If γ and β are both odd, then

$$\langle \gamma, \beta \rangle \geq \langle \gamma\beta^{-1}, \beta \rangle = S_3,$$

as $\gamma\beta^{-1}$ is even and so $\langle \gamma, \beta \rangle$ is as in the first case. Hence, S_3 is a group with the Basis Property.

Theorem 3. *Let G be a finite simple and non-abelian group, then G does not have the Basis Property.*

proof.
It is well known that if G is a non-abelian simple group, then $d(G) = 2$. In fact, much more is true, for example, Guralnick-Kantor [11] showed that given any non-trivial element of a non-abelian simple group, one can find another element of G such that the two generate G. Now let T be the set of all elements of order two. Now $\langle T\rangle = G$ since $\langle T\rangle$ is normal in G. Choose a minimal subset T_0 of T such that $\langle T_0\rangle = G$. We will show that the size of T_0 is greater than two. If $x, y \in T_0$ have order 2 and $a = xy$, then

$$x^{-1}ax = yx = (xy)^{-1} \in \langle a\rangle,$$

and

$$y^{-1}ay = y^{-1}ay^2 = yx = (xy)^{-1} \in \langle a\rangle.$$

Thus, $\langle a\rangle$ is normal in $\langle x, y\rangle$ and so $\langle x, y\rangle = \langle a, x\rangle$. This is a dihedral group, as from above, $x^{-1}ax = a^{-1}$ gives us the required form, hence, T_0 must be of size three or greater. Thus we have two minimal generating sets of different size, and so G does not have the Basis Property.

4. An extension of the group P by the Generalised Quaternion group

A generalised quaternion group is defined by the following presentation

$$\langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, xyx = y\rangle, n > 1.$$

Now we will state another class of groups, which does not satisfy the Basis Property

Theorem 4. *Suppose that G is a finite group (where $p > 2$) as an extension of p-group P by the Generalised Quaternion group H. Then G does not satisfy the Basis Property.*

proof.
Suppose that G is a group with the Basis Property and $G = P \rtimes H$, where P is a p–group, $p > 2$ and H is a group given by

$$\langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, xyx = y\rangle, n > 1. \tag{1}$$

Let $\delta, \gamma \in P$, and consider the subgroup

$$H_1 = \langle x\gamma, y\delta\rangle.$$

First, we will prove

$$\langle x\gamma, y\delta\rangle \cap P = \{e\}, \tag{2}$$

for that, assume

$$K = \langle x\gamma, y\delta\rangle \cap P \neq \{e\}.$$

Since $P \lhd G$, then $K \lhd H_1$, and P is a normal sylow $p-$ subgroups of G, this gives that K must contain all p-element of H_1. Hence K is a sylow $p-$ subgroups of H_1.

Take a homomorphism $\varphi : G \to G/P$, since

$$\varphi(H_1) = \varphi(\langle x\gamma, y\delta\rangle) = \langle x, y\rangle,$$

and $\varphi^{-1}(e) \cap H_1 = K$, then we conclude that H_1 is an extension of normal subgroup K by quaternion group $H = \langle x, y\rangle$. By Theorem Schur [[12] Lemma 11] and since $p > 2$, the group H_1 contains a subgroup H_2 as a complement to K and isomorphic to H. Therefore, H_1 is a semidirect product of a group K by H_2. Assume

$$H_2 = \langle u, v : u^{2^n} = 1, v^2 = u^{2^{n-1}}, uvu = v\rangle, u, v \in H_1,$$

then, $\langle u, v\rangle \cap K = \{e\}$.

Let $\{\beta_1, \beta_2, ..., \beta_m\}$ be a fixed basis of a group K, then

$$X = \{u, v, \beta_1, \beta_2, ..., \beta_m\}$$

is a generated set of H_1. Now let us choose a minimal generating set of a group H_1, which contains the elements u, v and X. Assume that

$$Y = \{u, v, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k}\},$$

and we will prove that the set Y is a basis of H_1. Moreover, if we take

$$\beta_{i_j} \in \{u, v, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_{j-1}}, \beta_{i_{j+1}}, ..., \beta_{i_k}\},$$

where $1 \leq j \leq k$, then the element β_{i_j} can be omitted from the basis Y and we have a new basis of H_1. Thus, we get a contradiction with the minimal generating set. Hence,

$$\beta_{i_j} \notin \langle Y - \beta_{i_j}\rangle.$$

If

$$u \in \{v, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k}\},$$

then

$$H_1 = \{v, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k}\} = \langle v\rangle K,$$

and H_1 is a semidirect product p-subgroup K by the cyclic group $\langle v\rangle$. Hence,

$$|H_1| = |\langle v\rangle K| = |\langle v\rangle||K| = 2^n|K|.$$

On other side

$$|H_1| = |\langle u, v\rangle K| = |\langle u, v\rangle||K| = 2^{n+1}|K|.$$

Thus, we get a contradiction. Hence,

$$u \notin \{v, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k}\},$$

moreover, using the same previous consideration, we find that

$$v \notin \{u, \beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k}\}.$$

Thus, Y is an independent generating set, hence it is a basis of H_1. From

$$|H_1| = |\langle u, v\rangle K| > 2^{n+1},$$

hence, we have $k > 0$ and $|Y| > 2$. On other side, the set $\{x\gamma, y\delta\}$ is a basis of H_1.

Now if $x\gamma \in \langle y\delta\rangle$ or $y\delta \in \langle x\gamma\rangle$, then H_1 is a cyclic group, which is impossible. Thus, H_1 must be a group with the Basis Property as a subgroup of a group with the Basis Property of G, and we get a contradiction, and equation (2) has been proven.

Take now the element

$$\lambda = (y\delta)^{-1}xy\delta x,$$

where $\delta \in P$ and $\delta \neq e$. Since

$$\varphi(\lambda) = \varphi((y\delta)^{-1}xy\delta x)$$

we have

$$\varphi((y\delta)^{-1}xy\delta x) = (\varphi(y\delta)^{-1}\varphi(x)\varphi(y\delta)\varphi(x)).$$

Thus

$$\varphi((y\delta)^{-1}xy\delta x)y^{-1}xyx = y^{-1}xyx.$$

Now from (1) we have $xyx = y$, so $y^{-1}xyx = y^{-1}y = e$.

This means $\lambda \in P$, which is why $\lambda \in P \cap \langle x, y\delta\rangle$. However, from (1) we have

$$P \cap \langle x, y\delta\rangle = \{e\}$$

It follows that $\lambda = e$. So we have the equation

$$(y\delta)^{-1}xy\delta x = e \tag{3}$$

Using equations (1) and (3), we get

$$e = (y\delta)^{-1}xy\delta x = \delta^{-1}y^{-1}xy\delta x,$$

since from (1) $xy = yx^{-1}$ we have

$$\delta^{-1}x^{-1}y^{-1}y\delta x = \delta^{-1}x^{-1}\delta x.$$

Thus,

$$\delta^{-1}x^{-1}\delta x = e.$$

Hence $\delta x = x\delta$ and 2-element x commute with p-element δ. Thus, we get a contradiction with a group G which has a prime power order. Then, by [Lemma 4.7 [10]] G does not have the Basis Property.

5. Conclusion

This paper discussed a class of finite groups, which are non-simple groups, non-abelian simple groups, and a group that is an extension of a p-group P by a generalised quaternion group Q that do not have the Basis Property.

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