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Application of the Laplace-Adomian Decomposition Method to Delay Fractional Partial Differential Equations

Mona Alsulami[1,*], Norah Sharif Al-Yazidi[1]

[1] *Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23218, Saudi Arabia*

Abstract. This paper applies the Laplace-Adomian decomposition method (LADM) to solve a class of delay fractional partial differential equations. The method combines the advantages of the Adomian decomposition method and Laplace transform to efficiently construct series solutions for complex fractional models involving time-delay effects. The proposed approach provides an effective analytical framework that simplifies the solution process and improves computational accuracy. Several test problems are presented to justify the method, including both linear and nonlinear cases. The results demonstrate that LADM can yield exact or highly accurate approximate solutions with only a few iterations. The efficiency and precision of the results obtained are compared with those of other analytical and numerical techniques, demonstrating the superiority of the proposed approach in terms of convergence and accuracy.

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Key Words and Phrases: Laplace-Adomian decomposition method, delay fractional differential equations, closed-form solutions, comparison analysis

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1. Introduction

Fractional calculus, which explores integrals and derivatives of arbitrary (non-integer) orders, has a deep and complex history spanning centuries. The concept was first proposed in the late 17th century by Gottfried Wilhelm Leibniz, and was later developed by prominent mathematicians such as Leonhard Euler[1–3]. Over time, fractional derivative theory has been used to describe various physical phenomena in scientific and technological fields, such as fluid mechanics, optical physics [4], control theory [5], biomedical [6], viscoelastic materials [7], and many other fields [8, 9]. Its expanding role in these areas highlights its increasing significance in modern scientific research. Thus, fractional partial differential equations (FPDEs) represent a natural advancement of conventional fractional differential

*Corresponding Author.
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Email addresses: mralsolami@uj.edu.sa (M. Alsulami), 2300414@uj.edu.sa (N. S. Al-Yazidi)

equations (FDEs). They have proven to be valuable tools for understanding and modeling intricate systems in domains such as physics, engineering, finance, and biology [10, 11]. The mathematical framework of FPDEs relies on various characterizations of non-integer derivatives, such as the Riemann-Liouville (RL) and the Caputo definitions, among others, allowing for flexibility in addressing diverse problems across scientific and engineering applications.

Furthermore, delay refers to the time lag between an action or event and its corresponding effect. In mathematical equations, delay is often represented in delay differential equations (DDEs), where the change rate of a system relies on its past states. Delay occurs in many real-world phenomena, where the present behavior or stage of the dynamical system relies on its preceding stages, influencing its performance, response, or development [12]. It is also evident in population dynamics, where growth depends on both the current and past population sizes [13]. Delay is a crucial factor in modeling systems where outcomes are influenced by previous states, allowing for the creation of more complex and accurate models. Delay ordinary differential equations (DODEs) are fundamental tools for describing dynamical systems, such that the change rate relies on the delayed and the present states. These equations are widely used in modeling processes with time delays, such as control systems and population dynamics. When this concept is extended to include fractional-order derivatives, delay fractional differential equations (DFDEs) arise. DFDEs provide greater flexibility by representing systems with memory and long-term behavior in addition to time delays. They are particularly useful for describing phenomena like viscoelasticity and vibration theory [14, 15]. Recently, delay fractional partial differential equations (DFPDEs) have engrossed momentous scientific thought in favour of their ability to describe complex systems with higher precision. These equations integrate partial derivatives with time delays, making them ideal for modeling systems with spatial and temporal interdependencies. DFPDEs possess vast relevance in contemporary science, when past states are pivotal, see [16–18]. Even though partial differential equations with delay terms are extensively studied by many scientists, however, to our knowledge, the open literature is sparse with regard to substantial research concerning DFPDEs. In particular, uniqueness and existence of results concerning FPDEs with nonlinearity and delay terms have been studied in [19]. Moreover, the Rihan [20] proposed an effective numerical approach for tackling DFPDEs, while Hosseinpour [21] introduced a promising method for solving the class of DFPDEs. In addition, in [22], a numerical method using the homotopy perturbation technique was introduced to treat a class of nonlinear DFPDEs, while some sets of FPDEs were analyzed in [23] through the application of the coupling between the natural integral transform and the classical decomposition method, in [24] a local radial basis function (RBF) technique coupled with the Laplace transform was developed to numerically solve time-fractional delay partial differential equations.

Moreover, among the various techniques available for addressing the chosen class of FPDEs, the present manuscript thus considers its foundational approach to be the Adomian decomposition method (ADM) [25]. ADM was introduced in 1984 by George Adomian [26]. The ADM has gained widespread recognition for its effectiveness in solving a wide range of functional equations. Over the years, ADM has passed through a series of modifications with the intention of improving its convergence, in addition to extending its applicability to complex models. The modifications have greatly improved the efficiency of obtaining solutions compared to the classical ADM, highlighting considerable progress in the technique's development. In [27], authors have employed the ADM with the variational iteration method (VIM) for tackling nonlinear FPDEs, while in [28], an effective approach was presented for coupled systems of nonlinear FPDEs with the application of the ADM and the Laplace transform (LT); see also [29], for a novel method for nonlinear FPDEs using the coupling of the ADM and the Sumudu transform (ST). In addition, [30] deployed the ADM for tackling FDEs with a delay term. Subsequently, the ADM was combined with the LT and the Aboodh integral transform, respectively, to address FDEs with delay terms in [31, 32], enhancing the solution process. However, in this study, we aim to integrate the Laplace transform (LT) and the ADM, termed as the Laplace-Adomian decomposition method (LADM), to solve the class of DFPDEs. The combination of these two methods offers several key benefits in this context. The ADM allows for the efficient handling of FDEs by breaking them into convergent series, thus providing accurate solutions for complex systems. When paired with the LT, which simplifies and transforms the equations, it facilitates the analysis of DDEs, a task that is typically challenging. This integration not only enhances the precision of the solutions but also improves computational efficiency, as it reduces the complexity of solving these equations. Additionally, the method is particularly useful for dealing with the delays inherent in FPDEs, which traditional methods struggle to address. The synergy between the ADM and the LT provides a powerful and reliable tool for solving DFPDEs, offering a more effective and efficient approach than conventional techniques. Lastly, the present manuscript is arranged as follows: Section 2 gives certain basics of non-integer order derivative and the LT. Section 3 gives the methodology of the devised approach and explains its details. In Section 4, the derived computational schemes are demonstrated on a range of diverse test problems. Section 5 offers a conclusion, summarizing the key findings and providing final remarks on the work.

2. Preliminaries

Some fractional calculus and LT definitions with some notations that are needed in this study are discussed in this section.

2.1. Preliminary concepts for fractional calculus[33]

Definition 1. *For the fractional-order $\alpha \in \mathbb{C}(\Re(\alpha) > 0)$, the right $I^{\alpha}_{a+}u(x)$ (for $x > a$) and left $I^{\alpha}_{b-}u(x)$ (for $x < b$) Riemann-Liouville fractional integrals are defined, respectively, as*

$$I^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(t)dt}{(x-t)^{1-\alpha}}, \quad \textit{and} \quad I^{\alpha}_{b-}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(t)dt}{(t-x)^{1-\alpha}}. \tag{1}$$

Definition 2. *For the fractional-order $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$, $n = [\Re(\alpha)] + 1$, the right $D^{\alpha}_{a+}u(x)$ (for $x > a$) and left $D^{\alpha}_{b-}u(x)$ (for $x < ba$) Riemann-Liouville fractional deriva-*

tives are defined, respectively, as

$$D^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{u(t)dt}{(x-t)^{\alpha-n+1}}, \text{ and } D^{\alpha}_{b-}u(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}\frac{u(t)dt}{(t-x)^{\alpha-n+1}}. \tag{2}$$

In addition, one can re-express the latter definitions from Eq. (1) as

$$D^{\alpha}_{a+}u(x) = \left(\frac{d}{dx}\right)^{n}(I^{n-\alpha}_{a+}u)(x), \quad \text{and} \quad D^{\alpha}_{b-}u(x) = \left(-\frac{d}{dx}\right)^{n}(I^{n-\alpha}_{b-}u)(x).$$

Definition 3. *For the fractional-order $\alpha \geq 0$, $\alpha \notin \mathbb{N}_0$, the right ${}^{C}D^{\alpha}_{a+}u(x)$ (for $x > a$), and left ${}^{C}D^{\alpha}_{b-}u(x)$ (for $x < b$), the Caputo fractional derivatives are defined, respectively as*

$${}^{C}D^{\alpha}_{a+}u(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{u^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, \quad \text{and} \quad {}^{C}D^{\alpha}_{b-}u(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}\frac{u^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}. \tag{3}$$

Moreover, with regard to the Caputo fractional derivative, one gets when $\alpha = n \in \mathbb{N}_0$ the following

$${}^{C}D^{\alpha}_{a+}u(x) = u^{(n)}(x), \qquad \text{and} \qquad {}^{C}D^{\alpha}_{b-}u(x) = (-1)^{n}u^{(n)}(x), \tag{4}$$

where $n \in \mathbb{N}$. Further, if $\alpha = 0$, one gets from the last equation ${}^{C}D^{0}_{a+}u(x) = u(x) = {}^{C}D^{0}_{b-}u(x)$.

Property 2.1. Among the noteworthy properties for the fractional derivatives and integrals are the following:

I). ${}^{C}D^{\alpha}_{*}\{k\} = 0$, where k is a real constant.

II). ${}^{C}D^{\alpha}_{*}\{x^n\} = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & n \geq m-1, \\ 0, & n < m-1. \end{cases}$

III). $I^{\alpha}\{x^n\} = \frac{\Gamma(1+n)}{\Gamma(\alpha+1+n)}x^{n+\alpha}, \quad n > -1, \quad m-1 < \alpha \leq m, \quad x > 0.$

IV). $I^{\alpha}\{I^{\gamma}u(x)\} = I^{\alpha+\gamma}u(x), \quad \alpha, \gamma \geq 0\,.$

V). $I^{\alpha}\left\{{}^{C}D^{\alpha}_{*}u(x)\right\} = u(x) - \sum_{l=0}^{m-1}u^{(l)}\left(0^{+}\right)\frac{x^{l}}{\Gamma(l+1)}, \quad m-1 < \alpha < m\,.$

VI). $I^{\alpha}\left\{{}^{C}D^{\gamma}_{*}u(x)\right\} = I^{\gamma-\alpha}u(x) - \sum_{l=0}^{m-1}u^{(l)}\left(0^{+}\right)\frac{x^{l+\gamma-\alpha}}{\Gamma(l+\gamma-\alpha+1)}, \quad m-1 < \gamma, \alpha \leq m, \ \alpha > \gamma.$

2.2. Preliminary concepts for Laplace transform

Pierre-Simon Laplace first introduced the LT, an integral transform that powerfully solves linear differential equations. Thus, this subsection reviews some of the basic definitions and properties associated with the said integral transform[34].

Definition 4. *The Laplace transform for the function $u(t)$ $(t > 0)$ is defined as follows*

$$\mathcal{L}[u(t)] = \int_{0}^{\infty}e^{-st}u(t)dt = U(s), \tag{5}$$

where $u(t), s \in \mathbb{R}$, or $\mathbb{C}$, with s denoting the LT parameter, while $\mathcal{L}$ is the LT operator. Notably, given the inverse LT operator $\mathcal{L}^{-1}$, it is then obvious for one to get

$$\mathcal{L}^{-1}[U(s)] = u(t).$$

Moreover, as an instance, given $u(t) = t^n$ $(t > 0)$, one gets from Definition 4 that

$$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0.$$

Definition 5. *Given the fractional-order $m-1 < \alpha < m$, the LT of Caputo fractional derivative for the function $u(t)$ is expressed as follows*

$$\mathcal{L}\left(D_t^\alpha u(t)\right) = s^\alpha U(s) - \sum_{l=0}^{m-1} s^{\alpha-l-1} u^{(k)}(0). \tag{6}$$

3. The LADM

This section makes a consideration of a generalized initial-value problem (IVP) for the nonlinear DFPDE to demonstrate the idea of the deployed LADM. The delay effect is introduced in two different forms, depending on the structure of the model.
Linear case: The delay appears explicitly as a constant time shift in the unknown function $u(x, t-\tau_1)$, where $\tau_1 > 0$ denotes a fixed delay.
Nonlinear case: The delay is incorporated through proportional or relative arguments inside a nonlinear function $Nu(x\tau_2, t\tau_3)$, where $\tau_2, \tau_3 > 0$. Here, the delay affects the solution implicitly through the nonlinear interaction. Accordingly, the general model can be expressed in a unified form as:

$$D_t^\alpha u(x,t) + Ru(x,t-\tau_1) + N\big(u(x\tau_2, t\tau_3)\big) = g(x,t); \quad m-1 < \alpha \le m, \quad a < x < b, \; 0 < t \le T \tag{7}$$
$$u(x,t) = \phi(x,t), \quad x \in [a,b], \quad t \in [-\tau, 0].$$

coupled with the imposed initial condition as follows:

$$u(x,0) = f(x),$$

where $D_t^\alpha u(x,t)$ is the time-Caputo type fractional differential operator earlier defined in the previous section; R and N are differential operators that are, respectively, linear and nonlinear; and the function g is the given source function. Where $\tau_i > 0$ for $i = 1, 2, 3$ and τ denotes the delay parameter. In addition, the function $f(x)$ in the last equation is a prescribed function.
Now, applying the LT, through the use of Eq. (6) to Eq. (7) reveals

$$\mathcal{L}\left[u(x,t)\right] = u(x,0) + \frac{1}{s^\alpha}\mathcal{L}\left[g(x,t)\right] - \frac{1}{s^\alpha}\mathcal{L}\left[R\left[u(x,t-\tau_1)\right] + N\left[u(x\tau_2, t\tau_3)\right]\right]. \tag{8}$$

Next, the application of the inverse LT, alongside utilizing the imposed initial condition on the latter equation yields:

$$u(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[g(x,t)\right]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[R\left[u(x,t-\tau_1)\right] + N\left[u(x\tau_2, t\tau_3)\right]\right]\right]. \quad (9)$$

Then, on using the ADM, where the resultant solution function $u(x,t)$ and the nonlinear term $N[u(x,t)]$ are represented using infinite series forms as below

$$u(x,t) = \sum_{n=0}^{\infty} u_n, \quad N[u(x\tau_2, t\tau_3)] = \sum_{n=0}^{\infty} A_n\left(u_0, u_1, \ldots, u_n\right), \quad (10)$$

then takes the following representation

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[g(x,t)]\right] - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[R\left[\sum_{n=0}^{\infty} u_n(x,t-\tau_1)\right] + \sum_{n=0}^{\infty} A_n\right]\right]. \quad (11)$$

Further, the ADM's procedure triggers the latter equation to admit the following recurrent scheme

$$\begin{cases} u_0(x,t) = f(x) + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}[g(x,t)]\right] = h(x,t), \\ u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[R\left[u_n(x,t-\tau_1)\right] + A_n\right]\right]; \quad n \geq 0\ . \end{cases}$$

In some cases, the Reliable modification can be used, resulting in the following recursive relations.

$$\begin{cases} u_0(x,t) = h_0(x,t), \\ u_1(x,t) = h_1(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[R\left[u_0(x,t-\tau_1)\right] + A_0\right]\right], \\ u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[R\left[u_n(x,t-\tau_1)\right] + A_n\right]\right]; \quad n \geq 1\ . \end{cases} \quad (12)$$

By computing the components u_n from (12), the exact or approximate solution can be obtained.

4. Numerical examples

The current section considers several examples, in the form of IVPs, and initial-boundary value problems (IBVPs), to demonstrate the application of the devised LADM. Some existing methods have also been used to assess the effectiveness of the adopted technique.

Example 1. *Consider the IBVP for the nonhomogeneous linear time DFPDE when* $0 < \alpha \leq 1$, $x \in [0,2]$, *and* $t \in [0,1]$ *as follows*

$$D_t^{\alpha}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - u(x,t-1) + \frac{\Gamma(3)}{\Gamma(3-\alpha)}\left(2x - x^2\right)t^{2-\alpha} + 2t^2 + x(2-x)(t-1)^2,$$
$$u(x,0) = 0, \quad and \quad u(0,t) = u(1,t) = 0, \quad (13)$$

where the model satisfies the following exact analytical solution

$$u(x,t) = t^2\left(2x - x^2\right). \tag{14}$$

Now, applying the LT to the governing equation in Eq. (13), alongside the application of Eq. (6) yields

$$L[u(x,t)] = \frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u(x,t) - u(x,t-1) + \frac{\Gamma(3)}{\Gamma(3-\alpha)}\left(2x - x^2\right)t^{2-\alpha} + 2t^2 + x(2-x)(t-1)^2\right]. \tag{15}$$

In addition, the deployment of the inverse LT on the above equation gives

$$u(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u(x,t) - u(x,t-1) + \frac{\Gamma(3)}{\Gamma(3-\alpha)}(2x - x^2)t^{2-\alpha} + 2t^2 + x(2-x)(t-1)^2\right]\right]. \tag{16}$$

What is more, an equivalent expression can be obtained by applying Eq. (10) to the aforementioned equation as

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t) = \mathcal{L}^{-1}&\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}(2x - x^2)t^{2-\alpha} + 2t^2 + x(2-x)(t-1)^2\right]\right]\\ &+ \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\sum_{n=0}^{\infty}\frac{\partial^2}{\partial x^2}u_n(x,t) - \sum_{n=0}^{\infty}u_n(x,t-1)\right]\right].\end{aligned} \tag{17}$$

Accordingly, utilizing the adopted LADM through the reliable modification of ADM, one thus obtains the following recurrent relation

$$\begin{cases} u_0(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\Gamma(3)}{\Gamma(3-\alpha)}(2x - x^2)t^{2-\alpha}\right]\right], \\ u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[2t^2 - x(x-2)(t-1)^2\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t-1)\right]\right], \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,t) - u_n(x,t-1)\right]\right] \quad n \geq 1\,. \end{cases} \tag{18}$$

Then, the latter recurrent relation when plainly performed yields

$$\begin{cases} u_0(x,t) = t^2(2x - x^2), \\ u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[2t^2 + x(2-x)(t-1)^2\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t-1)\right]\right], \\ \qquad\quad = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-2t^2 - (t-1)^2(2x - x^2) + 2t^2 + x(2-x)(t-1)^2\right]\right] = 0, \\ u_n(x,t) = 0, \; n \geq 1. \end{cases} \tag{19}$$

Lastly, the overall exact solution of the governing IBVP is obtained by computing the total sum of the solution components in the recurrent relation in Eq. (19) as

$$u(x,t) = t^2(2x - x^2), \tag{20}$$

which coincides with the earlier reported exact analytical solution of the model in Eq. (14), and is consistent with the results in [35].

Example 2. *Consider the IBVP for the nonhomogeneous linear time DFPDE when* $0 < \alpha \leq 1$, $x \in [0,1]$, *and* $t \in [0,2]$ *as follows*

$$D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t-1) + x^2 \left(\frac{\Gamma\left(\frac{8}{3}\right) t^{\frac{5}{3}-\alpha}}{\Gamma\left(\frac{8}{3}-\alpha\right)} + \frac{\Gamma\left(\frac{7}{3}\right) t^{\frac{4}{3}-\alpha}}{\Gamma\left(\frac{7}{3}-\alpha\right)} \right) - g(x,t) \qquad (21)$$
$$u(x,0) = 0, \quad \text{and} \quad u(0,t) = 0, \; u(1,t) = t^{\frac{5}{3}} + t^{\frac{4}{3}},$$

where $g(x,t) = 2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right) - x^2\left((t-1)^{\frac{5}{3}} + (t-1)^{\frac{4}{3}}\right)$, *that satisfies the following analytical solution*

$$u(x,t) = x^2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right). \qquad (22)$$

Accordingly, with the deployment of the LADM on model expressed in Eq. (21), one gets the modified reliable recursive relation as follows

$$\begin{cases} u_0(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[x^2\left(\frac{\Gamma\left(\frac{8}{3}\right)t^{\frac{5}{3}-\alpha}}{\Gamma\left(\frac{8}{3}-\alpha\right)} + \frac{\Gamma\left(\frac{7}{3}\right)t^{\frac{4}{3}-\alpha}}{\Gamma\left(\frac{7}{3}-\alpha\right)}\right)\right]\right], \\ u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[-2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right) - x^2\left((t-1)^{\frac{5}{3}} + (t-1)^{\frac{4}{3}}\right)\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0(x,t-1)\right]\right], \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 u_n(x,t)}{\partial x^2} + u_n(x,t-1)\right]\right], \quad n \geq 1, \end{cases} \qquad (23)$$

which when iteratively performed yields the following components

$$\begin{cases} u_0(x,t) = x^2(t^{\frac{5}{3}} + t^{\frac{4}{3}}), \\ u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[-2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right) - x^2\left((t-1)^{\frac{5}{3}} + (t-1)^{\frac{4}{3}}\right)\right]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} + u_0(x,t-1)\right]\right], \\ \qquad = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right) + x^2\left((t-1)^{\frac{5}{3}} + (t-1)^{\frac{4}{3}}\right) - 2\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right) - x^2\left((t-1)^{\frac{5}{3}} + (t-1)^{\frac{4}{3}}\right)\right]\right], \\ \qquad = 0, \\ u_n(x,t) = 0, \quad n \geq 1, \end{cases} \qquad (24)$$

that ultimately matched with the reported analytical solution in Eq. (22) upon summing the latter iterative components as follows

$$u(x,t) = x^2(t^{\frac{5}{3}} + t^{\frac{4}{3}}). \qquad (25)$$

Remarkably, the authors in [21] employed the Muntz-Legendre polynomials method (MLPM) and the Legendre polynomials method (LPM) to tackle the present model, while the spectral Galerkin method (SGM) was used in [35]. Each of these methods revealed an approximate iterative solution by computing five terms, whereas the proposed LADM attains the exact solution by utilizing only the first two iterates. Thus, this approach offers significant advantages over the deployed method in [21, 35], fostering swift convergence and greatly lessening computational intricacy. Furthermore, the study includes a comparison of errors produced by these methods with those obtained using the proposed LADM in Table 1.

Table 1: Comparison of the absolute errors for $u(x,t)$ via LADM against MLPM, LPM and SGM for the Example 4.2.

(x,t)	Exact	MLPM [21]	LPM [21]	SGM [35]	LADM
$(0.1, 1.8)$	0.0485310176	1.61838×10^{-3}	3.14707×10^{-4}	1.27101×10^{-4}	0
$(0.2, 1.8)$	0.1941240703	4.93482×10^{-4}	6.33806×10^{-4}	2.36635×10^{-4}	0
$(0.3, 1.8)$	0.4367791582	4.40890×10^{-5}	9.49793×10^{-4}	3.21843×10^{-4}	0
$(0.4, 1.8)$	0.7764962813	2.98649×10^{-4}	1.24365×10^{-4}	3.77345×10^{-4}	0
$(0.5, 1.8)$	1.2132754400	5.87367×10^{-4}	1.48602×10^{-3}	3.98910×10^{-4}	0
$(0.6, 1.8)$	1.7471166330	9.53604×10^{-4}	1.63839×10^{-3}	3.82941×10^{-4}	0
$(0.7, 1.8)$	2.3780198610	1.23816×10^{-3}	1.65424×10^{-3}	3.27556×10^{-4}	0
$(0.8, 1.8)$	3.1059851250	1.22565×10^{-3}	1.48023×10^{-3}	2.35492×10^{-4}	0
$(0.9, 1.8)$	3.9310124240	8.03846×10^{-4}	1.05738×10^{-3}	1.18178×10^{-4}	0
$(1.0, 1.8)$	4.8531017580	1.19071×10^{-4}	3.22235×10^{-4}	0	0

Example 3. *Consider the IVP for the nonhomogeneous linear time two-dimensional DF-PDE when $0 < \alpha \leq 1$ as follows*

$$D_t^{\alpha} u(x,y,t) = \frac{\partial^2}{\partial x^2} u(x,y,t) + \frac{\partial^2}{\partial y^2} u(x,y,t) + u(x,y,t-1) + \left(x^2+y^2\right) g(t) - 4\left(t^{5/3} + t^{\frac{4}{3}}\right),$$
$$u(x,0) = 0, \tag{26}$$

where $g(t) = \frac{\Gamma\left(\frac{8}{3}\right) t^{5/3-\alpha}}{\Gamma\left(\frac{8}{3}-\alpha\right)} + \frac{\Gamma\left(\frac{7}{3}\right) t^{4/3-\alpha}}{\Gamma\left(\frac{7}{3}-\alpha\right)} - (t-1)^{5/3} - (t-1)^{4/3}$, *and further satisifies the following analytical solution*

$$u(x,y,t) = \left(x^2+y^2\right)\left(t^{\frac{5}{3}} + t^{\frac{4}{3}}\right). \tag{27}$$

Consequently, with the deployment of the LADM, the IVP expressed in Eq. (26) reveals the following reliable modified recursive scheme

$$\begin{cases} u_0(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\left(x^2+y^2\right)\left[\frac{\Gamma\left(\frac{8}{3}\right)t^{5/3-\alpha}}{\Gamma\left(\frac{8}{3}-\alpha\right)} + \frac{\Gamma\left(\frac{7}{3}\right)t^{4/3-\alpha}}{\Gamma\left(\frac{7}{3}-\alpha\right)}\right]\right]\right], \\ u_1(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\left(x^2+y^2\right)\left[-\left((t-1)^{5/3} + (t-1)^{4/3}\right)\right] - 4\left(t^{5/3} + t^{\frac{4}{3}}\right)\right]\right] \\ \qquad + \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,y,t) + \frac{\partial^2}{\partial y^2}u_0(x,y,t) + u_0(x,y,t-1)\right]\right], \\ u_{n+1}(x,y,t) = L^{-1}\left[\frac{1}{s^{\alpha}}L\left[\frac{1}{s^{\alpha}}L\left[\frac{\partial^2}{\partial x^2}u_n(x,y,t) + \frac{\partial^2}{\partial y^2}u_n(x,y,t) + u_n(x,y,t-1)\right]\right]\right], \quad n \geq 1. \end{cases} \tag{28}$$

Further, the latter recurrent relation can be plainly expressed as follows

$$\begin{cases} u_0(x,y,t) & = \left(x^2+y^2\right)\left(t^{\frac{5}{3}}+t^{\frac{4}{3}}\right), \\ u_1(x,y,t) & = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-\left(x^2+y^2\right)\left[(t-1)^{5/3}+(t-1)^{4/3}\right]-4\left(t^{5/3}+t^{\frac{4}{3}}\right)\right]\right] \\ & + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,y,t)+\frac{\partial^2}{\partial y^2}u_0(x,y,t)+u_0(x,t-1)\right]\right], \\ & = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[2\left(t^{5/3}+t^{\frac{4}{3}}\right)+2\left(t^{5/3}+t^{\frac{4}{3}}\right)+\left(x^2+y^2\right)\left[(t-1)^{5/3}+(t-1)^{4/3}\right]\right.\right. \\ & \left.\left. -\ \left(x^2+y^2\right)\left[(t-1)^{5/3}+(t-1)^{4/3}\right]-4\left(t^{5/3}+t^{\frac{4}{3}}\right)\right]\right]=0, \\ u_n(x,y,t) & =0, \quad n\geq 1, \end{cases} \tag{29}$$

which when summed up corresponds to the earlier reported analytical solution in Eq. (27) as

$$u(x,y,t)=\left(x^2+y^2\right)\left(t^{\frac{5}{3}}+t^{\frac{4}{3}}\right). \tag{30}$$

Example 4. *Consider the IVP for the nonhomogeneous linear time DFPDE when* $0<\alpha\leq 1$ *as follows*

$$\begin{gathered} D_t^\alpha u(x,y,t)=\frac{\partial^2}{\partial a x^2}u(x,y,t)+\frac{\partial^2}{\partial y^2}u(x,y,t)-u(x,y,t-1)+g(x,y,t), \\ u(x,0)=0, \end{gathered} \tag{31}$$

where $g(x,y,t)=\left(\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+4t^2+(t-1)^2\right)\left[2(x+y)-\left(x^2+y^2\right)\right]$, *and satisfies the following exact analytical solution*

$$u\left(x,y,t\right)=t^2\left(2\left(x+y\right)-\left(x^2+y^2\right)\right). \tag{32}$$

What is more, the reliable modified recurrent LADM scheme for the governing model is accordingly obtained as follows

$$\begin{cases} u_0(x,y,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\left[2(x+y)-(x^2+y^2\right]\right]\right], \\ u_1(x,y,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[4t^2+(t-1)^2\left[2(x+y)-\left(x^2+y^2\right)\right]\right]\right] \\ \qquad + \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,y,t)+\frac{\partial^2}{\partial y^2}u_0(x,y,t)+u_0(x,y,t-1)\right]\right], \\ u_{n+1}(x,y,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,y,t)+\frac{\partial^2}{\partial y^2}u_n(x,y,t)+u_n(x,y,t-1)\right]\right], \quad n\geq 1, \end{cases} \tag{33}$$

that further yields the following components

$$\begin{cases} u_0(x,y,t) = t^2\left(2\left(x+y\right)-\left(x^2+y^2\right)\right), \\ u_1(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-4t^2-(t-1)^2\left(2(x+y)-\left(x^2+y^2\right)\right)\right.\right. \\ \qquad\qquad \left.\left.+\ 4t^2+(t-1)^2\left(2(x+y)-\left(x^2+y^2\right)\right)\right]\right]=0, \\ u_n(x,y,t)=0, \quad n\geq 1. \end{cases} \tag{34}$$

which eventually, after summing the latter components converges to the earlier expressed exact analytical solution in Eq. (32) as follows

$$u\left(x,y,t\right) = t^2\left(2\left(x+y\right)-\left(x^2+y^2\right)\right). \tag{35}$$

Example 5. *Consider the IVP for the nonhomogeneous linear time DFPDE when* $0<\alpha\leq 1$ *as follows*

$$\begin{gathered} D_t^\alpha u\left(x,t\right)+D_t^\alpha u\left(x,t-0.1\right)=\frac{1}{2}\frac{\partial^2 u\left(x,t\right)}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u\left(x,t-0.1\right)}{\partial x^2}+\cos(\pi x)g(t), \\ u(x,0)=0, \end{gathered} \tag{36}$$

where $g(t)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{2}{\Gamma(3-\alpha)}(t-0.1)^{2-\alpha}+\frac{\pi^2}{2}\left(t^2+(t-0.1)^2\right)$, *and satisfies the following exact solution*

$$u\left(x,t\right)=t^2\cos(\pi x). \tag{37}$$

Accordingly, the model admits the reliable modified LADM scheme as follows

$$\begin{cases} u_0(x,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\cos(\pi x)\frac{2}{\Gamma(3-\alpha)}(t)^{2-\alpha}\right]\right], \\ u_1(x,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[\cos(\pi x)\left[\frac{2}{\Gamma(3-\alpha)}(t-0.1)^{2-\alpha}+\frac{\pi^2}{2}(t^2+(t-0.1)^2)\right]\right]\right] \\ \qquad +\ \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-D_t^\alpha u_0\left(x,t-0.1\right)\frac{1}{2}\frac{\partial^2 u_0(x,t)}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_0(x,t-0.1)}{\partial x^2}\right]\right], \\ u_{n+1}(x,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-D_t^\alpha u_n\left(x,t-0.1\right)+\frac{1}{2}\frac{\partial^2 u_n(x,t)}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_n(x,t-0.1)}{\partial x^2}\right]\right], \quad n\geq 0, \end{cases} \tag{38}$$

which further expressed more plainly as

$$\begin{cases} u_0(x,t)=t^2\cos(\pi x), \\ u_1(x,t)=\mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[-\frac{2(t-0.1)^{2-\alpha}}{\Gamma(3-\alpha)}\cos(\pi x)-\frac{\pi^2}{2}t^2\cos(\pi x)-\frac{\pi^2}{2}(t-0.1)^2\cos(\pi x)\right.\right. \\ \qquad\qquad \left.\left.+\ \cos(\pi x)\left[\frac{2}{\Gamma(3-\alpha)}(t-0.1)^{2-\alpha}+\frac{\pi^2}{2}(t^2+(t-0.1)^2)\right]\right]\right]=0, \\ u_n(x,t)=0, \quad n\geq 1, \end{cases} \tag{39}$$

that sums up to the earlier stated exact solution as follows

$$u(x,t) = t^2 \cos(\pi x). \tag{40}$$

Remarkably, it is important to emphasize that in Examples 4.3, 4.4, and 4.5, the application of the LADM resulted in exact analytical solutions, while the same examples solved in [24] provided only approximate results. This remarkable improvement clearly demonstrates the superiority and precision of the proposed approach. The ability of LADM to produce exact results with minimal computational effort highlights its potential as a more efficient and powerful analytical tool for solving DFPDEs.

Example 6. *Consider the IVP for the nonlinear proportional DFPDE featuring the generalized Burger's equation when* $0 < \alpha \leq 1$, $x \in [0,1]$, *and* $t \in [0,1]$ *as follows*

$$\begin{aligned} D_t^{\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + u\left(\frac{x}{2}, \frac{t}{2}\right) \frac{\partial}{\partial x} u\left(x, \frac{t}{2}\right) + \frac{1}{2} u(x,t), \\ u(x,0) = x, \end{aligned} \tag{41}$$

where for a special value of the fractional-order, that is, at $\alpha = 1$, *the model satisfies the following exact analytical solution*

$$u(x,t) = xe^t. \tag{42}$$

As proceed, the application of the adopted LADM yields the following recursive relation

$$\begin{cases} u_0(x,t) = x, \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}} \mathcal{L}\left[\frac{\partial^2}{\partial x^2} u_n(x,t) + A_n + \frac{1}{2} u_n(x,t)\right]\right], \ n \geq 0, \end{cases} \tag{43}$$

where $\{A_n\}$ are the Adomian polynomials for the delay nonlinear expression $u\left(\frac{x}{2}, \frac{t}{2}\right) \frac{\partial}{\partial x} u\left(x, \frac{t}{2}\right)$, which are then expressed from [25, 26] as follows

$$\begin{aligned} A_0 &= u_0(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_0(x, \frac{t}{2})}{\partial x}, \\ A_1 &= u_0(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_1(x, \frac{t}{2})}{\partial x} + u_1(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_0(x, \frac{t}{2})}{\partial x}, \\ A_2 &= u_0(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_2(x, \frac{t}{2})}{\partial x} + u_1(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_1(x, \frac{t}{2})}{\partial x} + u_2(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_0(x, \frac{t}{2})}{\partial x}, \\ A_3 &= u_0(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_3(x, \frac{t}{2})}{\partial x} + u_1(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_2(x, \frac{t}{2})}{\partial x} + u_2(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_1(x, \frac{t}{2})}{\partial x} + u_3(\frac{x}{2}, \frac{t}{2}) \frac{\partial u_0(x, \frac{t}{2})}{\partial x}, \\ &\vdots \end{aligned} \tag{44}$$

Furthermore, without much delay, some of the components are plainly expressed from the latter scheme as follows

$$\begin{aligned}
u_0(x,t) &= x,\\
u_1(x,t) &= \frac{t^{\alpha}x}{\Gamma(\alpha+1)},\\
u_2(x,t) &= \frac{t^{2\alpha}x(2+2^{\alpha})}{2^{\alpha+1}\Gamma(2\alpha+1)},\\
u_3(x,t) &= \frac{xt^{3\alpha}\left(2+2^{\alpha}\right)}{2^{3\alpha+1}\Gamma(3\alpha+1)} + \frac{xt^{3\alpha}\left(2+2^{\alpha}\right)}{2^{\alpha+2}\Gamma(3\alpha+1)} + \frac{xt^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha+1}\Gamma(3\alpha+1)\Gamma(\alpha+1)^2},\\
&\vdots
\end{aligned} \tag{45}$$

More so, upon summing the above iterates, one thus gets the following series solution

$$\begin{aligned}
u(x,t) = x &+ \frac{t^{\alpha}x}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}x(2+2^{\alpha})}{2^{\alpha+1}\Gamma(2\alpha+1)} + \frac{xt^{3\alpha}\left(2+2^{\alpha}\right)}{2^{3\alpha+1}\Gamma(3\alpha+1)} + \frac{xt^{3\alpha}\left(2+2^{\alpha}\right)}{2^{\alpha+2}\Gamma(3\alpha+1)}\\
&+ \frac{xt^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha+1}\Gamma(3\alpha+1)\Gamma(\alpha+1)^2} + \ldots
\end{aligned} \tag{46}$$

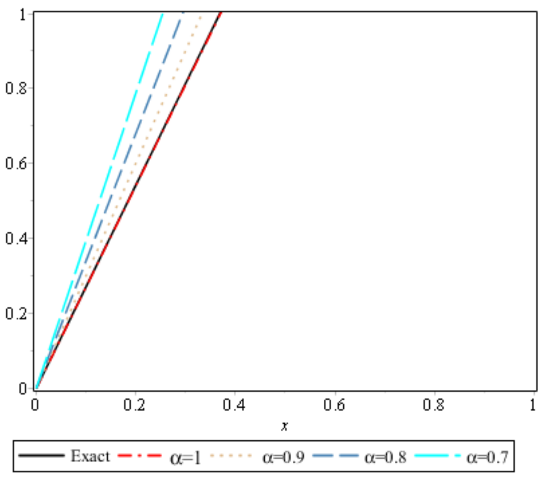
or equally when fixing $\alpha = 1$, the following convergent solution

$$u(x,t) = x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n\right) = xe^t, \tag{47}$$

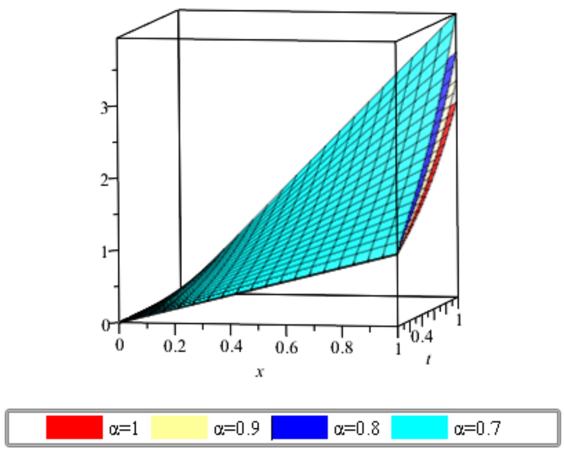
which greatly coincides with the earlier stated exact analytical solution of the corresponding integer-order problem in Eq. (42). However, scientists in [22] and [23] deployed the natural transform decomposition method (NTDM), and the homotopy perturbation method (HPM) to acquire accurate approximate solutions of the governing model by utilizing up to the fourth term. Nevertheless, the proposed LADM algorithm provided results that are very close to those obtained by the aforementioned methods upon utilizing the same number of terms. This demonstrates that the proposed approach yields highly consistent outcomes. The absolute error of these solutions is then analyzed and compared in Table 2; see also Figures 1 for the two-dimensional (2D) and three-dimensional (3D) views put forward by the method with the variation of the fractional-order. Certainly, the results indicate that the LADM method achieves accuracy comparable to the studied methods, demonstrating its effectiveness.

Table 2: Comparison of the absolute errors for $u(x,t)$ via LADM against NTDM, and HPM for the Example 4.6 at $\alpha = 1$.

x	t	Exact	NTDM [23]	HPM [22]	LADM
0.25	0.25	0.3210063542	2.12240×10^{-6}	2.12300×10^{-6}	2.12240×10^{-6}
	0.50	0.4121803178	7.09428×10^{-5}	7.09428×10^{-5}	7.09428×10^{-5}
	0.75	0.5292500042	5.63480×10^{-4}	5.63483×10^{-4}	5.63481×10^{-4}
	1.00	0.6795704570	2.48712×10^{-3}	2.48712×10^{-3}	2.48712×10^{-3}
0.50	0.25	0.6420127085	4.24500×10^{-6}	4.24500×10^{-6}	4.24500×10^{-6}
	0.50	0.8243606355	1.41885×10^{-4}	1.41885×10^{-4}	1.41886×10^{-4}
	0.75	1.0585000080	1.12696×10^{-3}	1.12697×10^{-3}	1.12696×10^{-3}
	1.00	1.3591409140	4.97424×10^{-3}	4.97425×10^{-3}	4.97424×10^{-3}
0.75	0.25	0.9630190628	6.36750×10^{-6}	6.36750×10^{-6}	6.36750×10^{-6}
	0.50	1.2365409530	2.12828×10^{-4}	2.12830×10^{-4}	2.12828×10^{-4}
	0.75	1.5877500130	1.69044×10^{-3}	1.69045×10^{-3}	1.69044×10^{-3}
	1.00	2.0387113710	7.46137×10^{-3}	7.46137×10^{-3}	7.46137×10^{-3}
1	0.25	1.2840254170	8.49000×10^{-6}	8.49100×10^{-6}	8.49000×10^{-6}
	0.50	1.6487212710	2.83771×10^{-4}	2.83770×10^{-4}	2.83772×10^{-4}
	0.75	2.1170000170	2.25392×10^{-3}	2.25392×10^{-3}	2.25392×10^{-3}
	1.00	2.7182818280	9.94849×10^{-3}	9.94849×10^{-3}	9.94849×10^{-3}



(a) 2D plot for $u(x,t)$ at different values of α.



(b) 3D plot for $u(x,t)$ at different values of α.

Figure 1: Plots of the fifth-order LADM solution of Eq. (41) when $\alpha \in \{0.7, 0.8, 0.9, 1\}$, and $t = 1$.

Example 7. *Consider the IVP for the nonlinear proportional DFPDE featuring the generalized Burger's equation when* $0 < \alpha \le 1$, $x \in [0,1]$ *and* $t \in [0,1]$ *as follows*

$$D_t^{\alpha} u(x,t) = u\left(x, \frac{t}{2}\right) \frac{\partial^2}{\partial x^2} u(x, \frac{t}{2}) - u(x,t), \qquad u(x,0) = x^2, \tag{48}$$

where for a special fractional-order, that is, at $\alpha = 1$, the model satisfies the following analytical solution

$$u(x,t) = x^2 e^t. \tag{49}$$

Consequently, LADM yields the resulting recurrent scheme as follows

$$\begin{cases} u_0(x,t) = x^2, \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left[B_n - u_n(x,t)\right]\right], \ n \geq 0, \end{cases} \tag{50}$$

where $\{B_n\}$ are the polynomials for the nonlinear term $u\left(x, \frac{t}{2}\right) \frac{\partial^2}{\partial x^2} u\left(x, \frac{t}{2}\right)$, which are accordingly determined as follows [25, 26]

$$\begin{aligned}
B_0 &= u_0(x, \frac{t}{2}) \frac{\partial^2 u_0(x, \frac{t}{2})}{\partial x^2}, \\
B_1 &= u_0(x, \frac{t}{2}) \frac{\partial^2 u_1(x, \frac{t}{2})}{\partial x^2} + u_1(x, \frac{t}{2}) \frac{\partial^2 u_0(x, \frac{t}{2})}{\partial x^2}, \\
B_2 &= u_0(x, \frac{t}{2}) \frac{\partial^2 u_2(x, \frac{t}{2})}{\partial x^2} + u_1(x, \frac{t}{2}) \frac{\partial^2 u_1(x, \frac{t}{2})}{\partial x^2} + u_2(x, \frac{t}{2}) \frac{\partial^2 u_0(x, \frac{t}{2})}{\partial x^2}, \\
B_3 &= u_0(x, \frac{t}{2}) \frac{\partial^2 u_3(x, \frac{t}{2})}{\partial x^2} + u_1(x, \frac{t}{2}) \frac{\partial^2 u_2(x, \frac{t}{2})}{\partial x^2} + u_2(x, \frac{t}{2}) \frac{\partial^2 u_1(x, \frac{t}{2})}{\partial x^2} + u_3(x, \frac{t}{2}) \frac{\partial^2 u_0(x, \frac{t}{2})}{\partial x^2}, \\
&\vdots
\end{aligned} \tag{51}$$

Furthermore, when the latter recursive scheme is fully expressed, one thus gets as follows

$$\begin{aligned}
u_0(x,t) &= x^2, \\
u_1(x,t) &= \frac{t^\alpha x^2}{\Gamma(\alpha+1)}, \\
u_2(x,t) &= \frac{4t^{2\alpha} x^2}{2^\alpha \Gamma(2\alpha+1)} - \frac{t^{2\alpha} x^2}{\Gamma(2\alpha+1)}, \\
u_3(x,t) &= \frac{16x^2 t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} - \frac{4x^2 t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} + \frac{2x^2 t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4x^2 t^{3\alpha}}{2^\alpha \Gamma(3\alpha+1)} + \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
&\vdots
\end{aligned} \tag{52}$$

which eventually sums up to yield

$$\begin{aligned}
u(x,t) = x^2 &+ \frac{x^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{4t^{2\alpha} x^2}{2^\alpha \Gamma(2\alpha+1)} - \frac{t^{2\alpha} x^2}{\Gamma(2\alpha+1)} + \frac{16x^2 t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} - \frac{4x^2 t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)} \\
&+ \frac{2x^2 t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{4x^2 t^{3\alpha}}{2^\alpha \Gamma(3\alpha+1)} + \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)} + ...
\end{aligned} \tag{53}$$

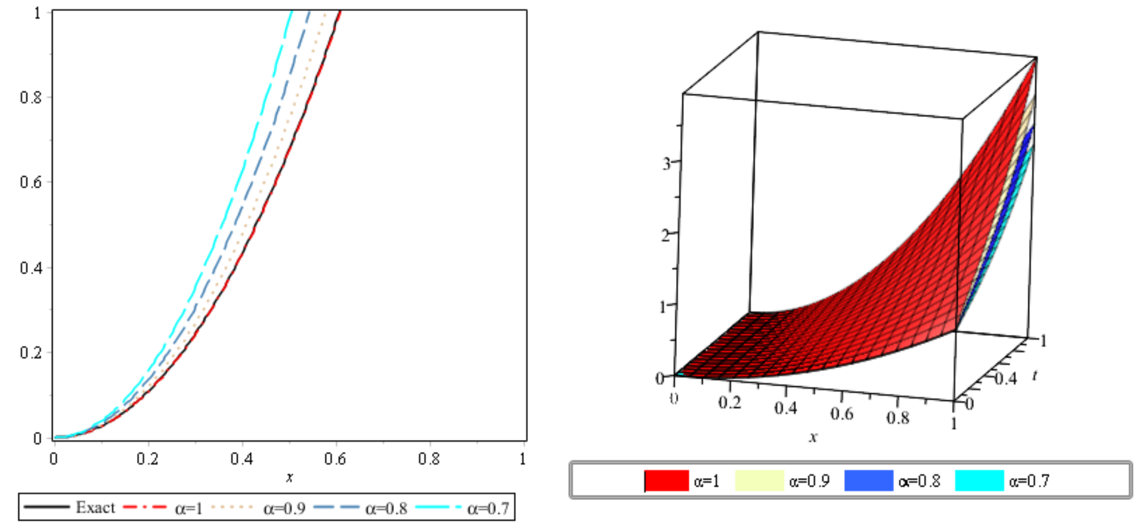
Moreover, when fixing $\alpha = 1$, the above LADM solution becomes

$$u(x,t) = x^2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots + \frac{1}{n!}t^n\right) = x^2e^t, \tag{54}$$

which coincides with Eq. (49), the earlier given solution for the corresponding integer-order model.
Table 3 gives the comparison between the results put forward by the adopted LADM, and the contending approaches [22, 23]; while Figures 2, give the graphical views of the derived approximate solution using the first four terms.

Table 3: Comparison of the absolute errors for $u(x,t)$ via LADM against NTDM, and HPM for the Example 4.7 at $\alpha = 1$.

x	t	Exact	NTDM [23]	HPM[22]	LADM
0.25	0.25	0.08025158856	3.00000×10^{-11}	5.30000×10^{-7}	5.30610×10^{-7}
	0.50	0.10304507940	6.50000×10^{-9}	1.77350×10^{-5}	1.77357×10^{-5}
	0.75	0.13231250110	1.69200×10^{-7}	1.40870×10^{-4}	1.40870×10^{-4}
	1.00	0.16989261420	1.74110×10^{-6}	6.21780×10^{-4}	6.21781×10^{-4}
0.50	0.25	0.32100635420	1.0000×10^{-10}	2.12300×10^{-6}	2.12240×10^{-6}
	0.50	0.41218031780	2.58000×10^{-8}	7.09430×10^{-5}	7.09428×10^{-5}
	0.75	0.52925000420	6.76700×10^{-7}	5.63483×10^{-4}	5.63481×10^{-4}
	1.00	0.67957045700	6.96490×10^{-6}	2.48712×10^{-3}	2.48712×10^{-3}
0.75	0.25	0.72226429710	4.77560×10^{-6}	4.77600×10^{-6}	4.77560×10^{-6}
	0.50	0.92740571490	1.59621×10^{-4}	1.59620×10^{-4}	1.59621×10^{-4}
	0.75	1.19081251000	1.26783×10^{-3}	1.26783×10^{-3}	1.26783×10^{-3}
	1.00	1.52903352800	5.59602×10^{-3}	5.59603×10^{-3}	5.59603×10^{-3}
1	0.25	1.28402541700	1.00000×10^{-9}	8.49000×10^{-6}	8.49000×10^{-6}
	0.50	1.64872127100	1.03000×10^{-7}	2.83772×10^{-4}	2.83771×10^{-4}
	0.75	2.11700001700	2.70700×10^{-6}	2.25392×10^{-3}	2.25392×10^{-3}
	1.00	2.71828182800	2.78590×10^{-5}	9.94845×10^{-3}	9.94849×10^{-3}



(a) 2D plot for $u(x,t)$ at different values of α.

(b) 3D plot for $u(x,t)$ at different values of α.

Figure 2: Plots of the fifth-order LADM solution of Eq. (48) when $\alpha \in \{0.7, 0.8, 0.9, 1\}$, and $t = 1$.

Example 8. *Consider the IVP for the nonlinear proportional DFPDE when* $0 < \alpha \leq 1$, $x \in [0,1]$ *and* $t \geq 0$ *as follows*

$$D_t^{\alpha} u(x,t) = \frac{\partial^2}{\partial x^2} u\left(\frac{x}{2}, \frac{t}{2}\right) \frac{\partial}{\partial x} u\left(\frac{x}{2}, \frac{t}{2}\right) - \frac{1}{8}\frac{\partial}{\partial x} u(x,t) - u(x,t), \quad u(x,0) = x^2, \tag{55}$$

where for a special fractional-order, that is, at $\alpha = 1$, *the problem satisfies the analytical solution as follows*

$$u(x,t) = x^2 e^{-t}. \tag{56}$$

As proceed, the application of the LADM yields the resultant scheme as follows

$$\begin{cases} u_0(x,t) = x^2, \\ u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}\left[C_n - \frac{1}{8}\frac{\partial}{\partial x} u_n(x,t) - u_n(x,t)\right]\right], \ n \geq 0, \end{cases} \tag{57}$$

where $\{C_n\}$ are the Adomian polynomials associated with the nonlinear term $\frac{\partial^2}{\partial x^2} u\left(\frac{x}{2}, \frac{t}{2}\right) \frac{\partial}{\partial x} u\left(\frac{x}{2}, \frac{t}{2}\right)$, which are accordingly expressed as follows

$$\begin{aligned} C_0 &= \frac{\partial^2 u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x}, \\ C_1 &= \frac{\partial^2 u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_1\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x} + \frac{\partial^2 u_1\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x}, \\ C_2 &= \frac{\partial^2 u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_2\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x} + \frac{\partial^2 u_1\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_1\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x} + \frac{\partial^2 u_2\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x^2} \frac{\partial u_0\left(\frac{x}{2}, \frac{t}{2}\right)}{\partial x}, \\ &\vdots \end{aligned} \tag{58}$$

Furthermore, on expressing the above scheme explicitly, one gets some of the resulting components as follows

$$\begin{aligned}
u_0(x,t) &= x^2,\\
u_1(x,t) &= -\frac{x^2 t^\alpha}{\Gamma(\alpha+1)},\\
u_2(x,t) &= -\frac{x t^{2\alpha}}{2^{\alpha+1}\Gamma(2\alpha+1)} + \frac{x t^{2\alpha}}{4\Gamma(2\alpha+1)} + \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_3(x,t) &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left[-\frac{1}{2^{3\alpha+3}} + \frac{1}{2^{2\alpha+4}} + \frac{x}{2^{2\alpha+1}} + \frac{1}{2^{\alpha+4}} - \frac{1}{2^5} - \frac{x}{2} + \frac{x}{2^{\alpha+1}} - x^2 + \frac{x\Gamma(2\alpha+1)}{2^{2\alpha+2}\Gamma(\alpha+1)^2}\right],\\
&\vdots
\end{aligned} \tag{59}$$

where in the same fashion, the latter component sum to the following

$$u(x,t) = x^2 - \frac{x^2 t^\alpha}{\Gamma(\alpha+1)} - \frac{x t^{2\alpha}}{2^{\alpha+1}\Gamma(2\alpha+1)} + \frac{x t^{2\alpha}}{4\Gamma(2\alpha+1)} + \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)} +$$
$$\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\left[-\frac{1}{2^{3\alpha+3}} + \frac{1}{2^{2\alpha+4}} + \frac{x}{2^{2\alpha+1}} + \frac{1}{2^{\alpha+4}} - \frac{1}{2^5} - \frac{x}{2} + \frac{x}{2^{\alpha+1}} - x^2 + \frac{x\Gamma(2\alpha+1)}{2^{2\alpha+2}\Gamma(\alpha+1)^2}\right] + \ldots \tag{60}$$

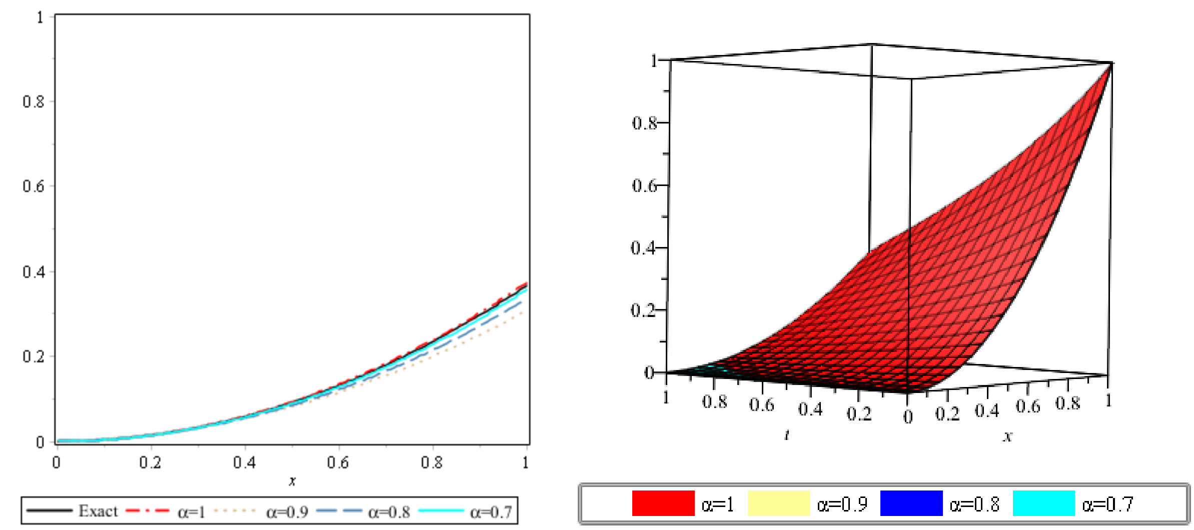
Accordingly, at $\alpha = 1$, the latter series solution reduces to the following

$$u(x,t) = x^2\left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 + \cdots + (-1)^n \frac{1}{n!}t^n\right) = x^2 e^{-t}, \tag{61}$$

eventually converging to the already mentioned exact solution expression in Eq. (56). In the same way, Table 4 presents a comparison of the absolute errors obtained by LADM and the other methods. The results demonstrate that the proposed method provides accurate and reliable approximations for the considered problem. Figures 3 further illustrate the behavior of the derived approximate solution.

Table 4: Comparison of the absolute errors for $u(x,t)$ via LADM against NTDM, and HPM for the Example 4.8 at $\alpha = 1$.

x	t	Exact	NTDM [23]	HPM [22]	LADM
0.25	0.25	0.04867504894	3.00000×10^{-11}	4.88000×10^{-7}	4.88180×10^{-7}
	0.50	0.03790816623	5.72000×10^{-9}	1.50109×10^{-5}	1.50109×10^{-5}
	0.75	0.02952290954	1.43160×10^{-7}	1.09659×10^{-4}	1.09659×10^{-4}
	1.00	0.02299246508	1.39364×10^{-6}	4.45035×10^{-4}	4.45035×10^{-4}
0.50	0.25	0.19470019580	1.00000×10^{-10}	1.95200×10^{-6}	1.95250×10^{-6}
	0.50	0.15163266490	2.29000×10^{-8}	6.00430×10^{-5}	6.00435×10^{-5}
	0.75	0.11809163820	5.72700×10^{-7}	4.38630×10^{-4}	4.38635×10^{-4}
	1.00	0.09196986030	5.57458×10^{-6}	1.78014×10^{-3}	1.78014×10^{-3}
0.75	0.25	0.43807544050	4.33934×10^{-6}	4.39340×10^{-6}	4.39340×10^{-6}
	0.50	0.34117349610	1.35098×10^{-4}	1.35080×10^{-4}	1.35098×10^{-4}
	0.75	0.26570618590	9.86929×10^{-4}	9.86929×10^{-4}	9.86929×10^{-4}
	1.00	0.20693218570	4.00531×10^{-3}	4.00531×10^{-3}	4.00531×10^{-3}
1	0.25	0.77880078310	4.00000×10^{-10}	7.81060×10^{-6}	7.81060×10^{-6}
	0.50	0.60653065970	9.17000×10^{-8}	2.40174×10^{-4}	2.40174×10^{-4}
	0.75	0.47236655270	2.29050×10^{-6}	1.75454×10^{-3}	1.75454×10^{-3}
	1.00	0.36787944120	2.22983×10^{-5}	7.12056×10^{-3}	7.12056×10^{-3}



(a) 2D plot for $u(x,t)$ at different values of α.

(b) 3D plot for $u(x,t)$ at different values of α.

Figure 3: Plots of the fifth-order LADM solution of Eq. (55) when $\alpha \in \{0.7, 0.8, 0.9, 1\}$, and $t = 1$.

5. Conclusion

In conclusion, the LADM was applied to a variety of DFPDEs. The results confirm that the adopted technique is accurate, simple, and efficient for handling the intricacy of fractional derivatives and time-delay terms. Several numerical examples were solved using Maple software, through which tables and graphical representations were supplied to showcase the applicability of the sought approach. The obtained solutions include both exact and approximate results, and a detailed comparison with other existing methods reveals that while some competing techniques provided only approximate solutions, the proposed method was able to produce exact or more accurate outcomes. Consequently, the study demonstrates the effectiveness and reliability of LADM as a valuable analytical tool for solving delay fractional models in various applied sciences. Furthermore, the presented framework may be extended in future research to tackle coupled systems of DFPDEs and other complex nonlinear problems involving memory and delay effects.

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