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Topological Consistency of q-Rung Orthopair Fuzzy Normed Rings Under Strict Archimedean Operators

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Abstract. The integration of q-rung orthopair fuzzy sets (q-ROFSs) with algebraic structures relies on idempotent minimum and maximum operators. To formalize the topological behavior of fuzzy sets in unbounded normed algebras, we define the norm-compatibility condition. We show that applying classical idempotent operators violates this condition, which intuitively requires the membership grades of divergent sequences to vanish at infinity, and restricts the fuzzy support to the unit ball, creating a metric limitation. To resolve this metric incompatibility, this paper introduces the axiom of Archimedean Dual Norm Compatibility. By substituting idempotent operators with Archimedean t-norms and t-conorms, we construct topologically consistent q-ROF (TCq-ROF) normed rings that preserve the unbounded metric geometry. Under this framework, we formulate the TCq-ROF metric kernel and prove the First Isomorphism Theorem for TCq-ROF normed rings. Furthermore, we establish TCq-ROF direct sums, maximal and prime ideals, zero divisors over product rings, and TCq-ROF radical extensions. Our proposed Archimedean framework provides a metric-preserving mathematical foundation for fuzzy abstract algebra over infinite-dimensional commutative spaces.

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1. Introduction

The concept of representing uncertainty in mathematical structures was initiated by Zadeh's fuzzy sets [1], which was later extended by Atanassov to intuitionistic fuzzy sets [2] to incorporate non-membership degrees. To model broader spaces of uncertainty where the sum of membership and non-membership grades exceeds one, Yager introduced q-rung orthopair fuzzy sets (q-ROFSs) [3], relaxing the boundary condition to the q-th power. The rapid theoretical expansion and modeling capabilities of these sets have been systematically documented by Peng and Luo [4].

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Recently, the algebraic properties of q-ROFSs have been extensively investigated. Razzaque et al. [5] formulated q-ROF ideals and semi-prime ideals in classical rings. Alghazzwi et al. [6] expanded this framework to q-ROF subrings and quotient rings, while Alnefaie et al. [7] generalized q-ROF structures to field theory. The topological aspects of q-ROFSs were established by Türkarslan et al. [8] and Voskoglou [9], who investigated q-ROF continuity and Hausdorff spaces. Simultaneously, the integration of fuzzy sets with metric spaces was formalized by Emniyet and Şahin [10], who introduced the concept of fuzzy normed rings by linking algebraic properties with metric topologies.

In the theoretical formulation of q-ROFSs, Ali [11] redefined the concept using orbit-based structures and examined their basic algebraic operations, while Alcantud [12] studied these sets under complemental fuzzy sets. Ali et al. [13] defined q-ROF complex subgroups, and Shabir et al. [14] formulated q-ROF relations. Razzaque [15] constructed q-ROF subgroups and proved the corresponding isomorphism theorems. Regarding aggregation operators, Seikh and Mandal [16] formulated q-ROF Archimedean operators. Özlü [17] and Zhang et al. [18] applied Aczel-Alsina t-norms to q-ROF models, and Shahzadi et al. [19] generalized these structures to (p,q)-rung orthopair fuzzy sets.

However, these algebraic studies primarily focus on discrete or bounded spaces without addressing the norm-induced topologies. Conversely, while topological q-ROF spaces have been explored, they have not yet been integrated with unbounded ideal theory. Existing q-ROF algebraic frameworks rely on idempotent minimum and maximum operators. To investigate this topological behavior, we formalize the norm-compatibility condition based on functions vanishing at infinity. We demonstrate that employing idempotent operators over unbounded normed rings violates this condition and restricts the fuzzy support to the unit ball.

The remainder of this paper is organized as follows. Section 2 reviews the necessary preliminaries. Section 3 formalizes the norm-compatibility condition and demonstrates the metric limitation caused by idempotent operators. Section 4 introduces the Archimedean Dual Norm Compatibility. Section 5 establishes the TCq-ROF metric kernel and proves the First Isomorphism Theorem for TCq-ROF normed rings. Section 6 investigates TCq-ROF direct sums and maximal ideals. Section 7 generalizes prime ideals, zero divisors, and radical extensions in TCq-ROF spaces. Finally, Section 8 concludes the paper and proposes open problems for future research.

2. Preliminaries

In this section, we review the fundamental algebraic and fuzzy set-theoretic concepts. To maintain metric consistency, we establish the definitions over commutative normed rings and review the axiomatic properties of Archimedean operators. Let R denote a commutative ring.

The integration of metric properties into commutative algebra requires a well-defined norm.

Definition 1 ([20]). *Let R be a commutative ring with identity $1_R \neq 0$. A function $\|\cdot\| : R \to \mathbb{R}$ is called a norm on R if it satisfies the following conditions for all $x, y \in R$:*

(i) $\|x\| \geq 0$*, and* $\|x\| = 0$ *if and only if* $x = 0$*;*

(ii) $\|x + y\| \leq \|x\| + \|y\|$*;*

(iii) $\| - x\| = \|x\|$*;*

(iv) $\|xy\| \leq \|x\|\|y\|$*;*

(v) $\|1_R\| = 1$*.*

The pair $(R, \| \cdot \|)$ *is called a commutative normed ring.*

An algebra R over a real or complex linear space equipped with such a norm is called a normed algebra.

To justify the topological behavior of fuzzy extensions, it is essential to bridge the algebraic norm with its underlying topology.

Definition 2. *Let* R *be a normed ring. The norm* $\| \cdot \|$ *induces a natural metric* $d : R \times R \to [0, \infty)$ *defined by* $d(x, y) = \|x - y\|$ *for all* $x, y \in R$*.*

The metric d generates a topology on R, called the norm-induced topology. In this study, any metric limitation refers to the restriction of this specific metric space.

Definition 3 ([3])**.** *Let* X *be a universe of discourse. A q-rung orthopair fuzzy set (q-ROFS)* Q *on* X *is defined as the set of ordered triples:*

$$Q = \{\langle x, \mu_Q(x), \nu_Q(x)\rangle \mid x \in X\} \tag{1}$$

where the mappings $\mu_Q : X \to [0, 1]$ *and* $\nu_Q : X \to [0, 1]$ *represent the degree of membership and non-membership of the element* $x \in X$ *to the set* Q*. These functions satisfy the bounding condition:*

$$\mu_Q(x)^q + \nu_Q(x)^q \leq 1 \tag{2}$$

for all $x \in X$*, where* $q \geq 1$*.*

The metric limitation in fuzzy algebraic structures originates from the choice of aggregation operators. Therefore, it is necessary to define the generalized intersections and unions.

Definition 4 ([21])**.** *A triangular norm (t-norm) is a binary operation* $T : [0, 1] \times [0, 1] \to [0, 1]$ *satisfying the following axioms for all* $a, b, c \in [0, 1]$*:*

$$T(a, 1) = a \tag{3}$$

$$b \leq c \implies T(a, b) \leq T(a, c) \tag{4}$$

$$T(a, b) = T(b, a) \tag{5}$$

$$T(a, T(b, c)) = T(T(a, b), c) \tag{6}$$

A t-norm T *is called strictly Archimedean if it is continuous and strictly increasing on the open domain* $(0, 1] \times (0, 1]$*.*

Definition 5 ([21]). *A triangular conorm (t-conorm) is a binary operation $S : [0,1] \times [0,1] \to [0,1]$ satisfying the following boundary, monotonicity, commutativity, and associativity axioms for all $a, b, c \in [0,1]$:*

$$S(a,0) = a \tag{7}$$

$$b \leq c \implies S(a,b) \leq S(a,c) \tag{8}$$

$$S(a,b) = S(b,a) \tag{9}$$

$$S(a, S(b,c)) = S(S(a,b), c) \tag{10}$$

Similarly, a t-conorm S is called strictly Archimedean if it is continuous and strictly increasing on $(0,1) \times (0,1)$.

Current algebraic frameworks integrate q-ROFSs into ring theory generally through idempotent minimum and maximum operators. We state the definition of a q-ROF ideal to formalize the classical methodology.

Definition 6 ([5]). *Let Q be a q-ROFS defined over a ring R. The set Q is called a classical q-ROF ideal of R if the following inequalities hold for all $x, y \in R$:*

$$\mu_Q(x-y) \geq \min\{\mu_Q(x), \mu_Q(y)\} \tag{11}$$

$$\nu_Q(x-y) \leq \max\{\nu_Q(x), \nu_Q(y)\} \tag{12}$$

$$\mu_Q(xy) \geq \max\{\mu_Q(x), \mu_Q(y)\} \tag{13}$$

$$\nu_Q(xy) \leq \min\{\nu_Q(x), \nu_Q(y)\} \tag{14}$$

The standard definitions rely on the idempotent properties of min and max. In Section 3, we will demonstrate that applying these operators over an unbounded normed ring R causes the metric limitation.

3. The Metric Limitation in Classical q-ROF Normed Rings

In classical q-ROF algebraic structures, the integration of fuzzy sets with ring theory is governed by the idempotent minimum and maximum operators. While this approach is well-defined for finite or bounded discrete sets, extending these idempotent properties to unbounded normed rings induces metric incompatibility. In this section, we demonstrate this metric limitation, which strictly restricts the fuzzy algebraic structures to trivial domains.

In functional analysis, the topological behavior of continuous structures over unbounded metric spaces is characterized by functions vanishing at infinity [20]. To preserve the norm-induced metric geometry in fuzzy algebraic extensions, the membership grades must reflect this divergence. We formalize this metric requirement for q-ROFSs as follows.

Definition 7. *Let R be an unbounded commutative normed ring. A q-ROFS $Q = \langle \mu_Q, \nu_Q \rangle$ of R is called norm-compatible if, for any sequence $\{x_n\}$ in R such that $\|x_n\| \to \infty$ as $n \to \infty$, the membership and non-membership functions satisfy:*

$$\lim_{n \to \infty} \mu_Q(x_n) = 0 \tag{15}$$

$$\lim_{n\to\infty} \nu_Q(x_n) = 1 \tag{16}$$

Intuitively, this condition ensures that the membership grades of elements vanish asymptotically as their norms diverge to infinity.

First, we demonstrate that under this norm-compatibility condition, the multiplicative idempotency restricts the support of the classical q-ROF ideal to the closed unit ball of the normed ring.

Theorem 1. *Let R be an unbounded normed algebra with a multiplicative norm (i.e., $\|xy\| = \|x\|\|y\|$) and Q be a norm-compatible classical q-ROF ideal of R. If $x \in R$ such that $\|x\| > 1$, then $\mu_Q(x) = 0$ and $\nu_Q(x) = 1$. Consequently, the non-trivial support of Q is strictly bounded to the closed unit ball $B_1 = \{x \in R \mid \|x\| \leq 1\}$.*

Proof. Assume that Q is a norm-compatible classical q-ROF ideal. For any $x \in R$, the multiplication operation under the idempotent maximum operator satisfies:

$$\mu_Q(x^2) = \mu_Q(x \cdot x) \geq \max\{\mu_Q(x), \mu_Q(x)\} = \mu_Q(x) \tag{17}$$

By induction, for any $n \in \mathbb{N}$, we obtain:

$$\mu_Q(x^n) \geq \mu_Q(x) \tag{18}$$

Similarly, for the non-membership function:

$$\nu_Q(x^n) \leq \min\{\nu_Q(x), \nu_Q(x)\} = \nu_Q(x) \tag{19}$$

Let $x \in R$ be an element strictly outside the unit ball, implying $\|x\| > 1$. In a normed algebra with a multiplicative norm, we have $\|x^n\| = \|x\|^n$. Since $\|x\| > 1$, the sequence $\|x^n\|$ strictly diverges to infinity as $n \to \infty$. By Definition 7, the norm-compatibility requires the membership grade to vanish. Taking the limit as $n \to \infty$ on both sides of the membership inequality yields:

$$\begin{aligned} \lim_{n\to\infty} \mu_Q(x^n) &\geq \mu_Q(x) \\ 0 &\geq \mu_Q(x) \end{aligned} \tag{20}$$

Since $\mu_Q(x) \in [0, 1]$, it follows necessarily that $\mu_Q(x) = 0$. By applying the same limit to the non-membership inequality, we obtain $\nu_Q(x) = 1$. Thus, no element outside the unit ball can possess a non-zero membership degree.

Theorem 1 proves that the classical definition restricts the fuzzy ideal to a bounded domain. However, the metric limitation extends further. Even within the unit ball, the additive idempotency eliminates the metric distinguishability.

Theorem 2. *Let R be an unbounded normed algebra over $\mathbb{R}$ (or $\mathbb{C}$), and Q be a norm-compatible classical q-ROF ideal of R. For any non-zero element $x \in R$, the idempotent operators reduce the membership function μ_Q and non-membership function ν_Q to be topologically trivial.*

Proof. By the definition of classical q-ROF ideals, the addition operation satisfies:

$$\mu_Q(x+x) \geq \min\{\mu_Q(x), \mu_Q(x)\} = \mu_Q(x) \tag{21}$$

Inductively, for any positive integer $n \in \mathbb{N}$, we have:

$$\mu_Q(nx) \geq \mu_Q(x) \tag{22}$$

$$\nu_Q(nx) \leq \nu_Q(x) \tag{23}$$

Let $x \in R$ be any non-zero element with $\|x\| > 0$. Consider the sequence defined by $x_n = nx$. By the absolute homogeneity of the normed algebra over $\mathbb{R}$, we have $\|x_n\| = \|nx\| = |n|\|x\| = n\|x\|$. As $n \to \infty$, $\|x_n\| \to \infty$. Consequently, the sequence $\{x_n\}$ diverges in the norm-induced topology of R. Taking the limit as $n \to \infty$ on the membership inequality yields:

$$\begin{aligned} \lim_{n\to\infty} \mu_Q(nx) &\geq \mu_Q(x) \\ 0 &\geq \mu_Q(x) \end{aligned} \tag{24}$$

Thus, $\mu_Q(x) = 0$ and $\nu_Q(x) = 1$ for any $x \neq 0$.

Example 1. *Let $R = \mathbb{R}$ be the field of real numbers equipped with the absolute value norm $\|x\| = |x|$. Assume Q is a norm-compatible classical q-ROF ideal of R. Consider the element $x = 2$, where $\|2\| = 2 > 1$. By the multiplicative idempotency proven in Theorem 1, we obtain:*

$$\mu_Q(2^n) \geq \mu_Q(2) \tag{25}$$

Since $|2^n| \to \infty$, the norm-compatibility condition requires $\lim_{n\to\infty} \mu_Q(2^n) = 0$, which implies:

$$\mu_Q(2) = 0 \tag{26}$$

Similarly, for the additive idempotency proven in Theorem 2, consider the element $x = 0.5$ inside the unit ball. The classical condition requires:

$$\mu_Q(n \cdot 0.5) \geq \mu_Q(0.5) \tag{27}$$

As $n \to \infty$, the sequence diverges, yielding $\mu_Q(0.5) = 0$. Consequently, the support of Q reduces entirely to the trivial set $\{0\}$.

The combination of Theorem 1 and Theorem 2 establishes the metric limitation. The classical q-ROF algebraic framework cannot sustain an unbounded metric topology; it restricts the norm-compatible fuzzy ideal to the trivial bounded set $\{0\}$. To construct a metric-preserving q-ROF normed ring, these idempotent operators must be replaced with operators that satisfy strictly Archimedean properties.

4. Archimedean Dual Norm Compatibility

To overcome the metric limitation demonstrated in Section 3, we replace the idempotent operators with strictly Archimedean operators. This substitution ensures that the fuzzy algebraic structures respect the unbounded metric topology of R.

Definition 8. *Let R be a commutative normed ring. A q-ROFS $Q = \langle \mu_Q, \nu_Q \rangle$ over R is a TCq-ROF subring of R if there exist a strictly Archimedean t-norm T and a strictly Archimedean t-conorm S such that the following axioms hold for all $x, y \in R$:*

$$\mu_Q(x-y) \geq T(\mu_Q(x), \mu_Q(y)) \tag{28}$$

$$\nu_Q(x-y) \leq S(\nu_Q(x), \nu_Q(y)) \tag{29}$$

$$\mu_Q(xy) \geq T(\mu_Q(x), \mu_Q(y)) \tag{30}$$

$$\nu_Q(xy) \leq S(\nu_Q(x), \nu_Q(y)) \tag{31}$$

Definition 9. *A TCq-ROF subring Q is a TCq-ROF ideal of R if it satisfies the following absorption properties for all $r, x \in R$:*

$$\mu_Q(rx) \geq T(\mu_Q(r), \mu_Q(x)) \tag{32}$$

$$\nu_Q(rx) \leq S(\nu_Q(r), \nu_Q(x)) \tag{33}$$

By replacing the classical ideal conditions with strictly Archimedean operators, the restriction to the unit ball established in Theorem 1 is removed. For any $a \in (0,1)$, the monotonicity of strictly Archimedean operators yields $T(a,a) < a$ and $S(a,a) > a$. Applying this to the powers of an element $x \in R$, we obtain $\mu_Q(x^n) \geq T^{(n)}(\mu_Q(x))$ and $\nu_Q(x^n) \leq S^{(n)}(\nu_Q(x))$. As $n \to \infty$, the sequence $T^{(n)}(a)$ converges to 0 and $S^{(n)}(a)$ converges to 1. This permits the membership grades of divergent sequences to satisfy the norm-compatibility condition, allowing the support of Q to extend beyond the closed unit ball.

4.1. Examples and Demonstration of Compatible Operators

Various Archimedean operator families exist in the literature that satisfy the requirements of Definition 8 and Definition 9. We list three fundamental pairs of strictly Archimedean t-norms and t-conorms below.

Example 2 ([22])**.** *The algebraic t-norm T_A and t-conorm S_A are defined for all $a, b \in [0,1]$ as:*

$$T_A(a,b) = ab \tag{34}$$

$$S_A(a,b) = a + b - ab \tag{35}$$

Example 3 ([22])**.** *The Einstein t-norm T_E and t-conorm S_E are defined for all $a, b \in [0,1]$ as:*

$$T_E(a,b) = \frac{ab}{1 + (1-a)(1-b)} \tag{36}$$

$$S_E(a,b) = \frac{a+b}{1+ab} \tag{37}$$

Example 4 ([23]). *The Hamacher t-norm T_H and t-conorm S_H with parameter $\gamma > 0$ are defined for all $a, b \in [0,1]$ as:*

$$T_H(a,b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)} \tag{38}$$

$$S_H(a,b) = \frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab} \tag{39}$$

Now, as an example, we show that the Hamacher t-norm satisfies these conditions:
Let $a \in (0,1)$. By evaluating the monotonicity condition $T_H(a,a) < a$:

$$\begin{gathered}
\frac{a^2}{\gamma + (1-\gamma)(2a-a^2)} < a \\
a^2 < a(\gamma + 2a - a^2 - 2a\gamma + \gamma a^2) \\
a < \gamma + 2a - a^2 - 2a\gamma + \gamma a^2 \\
0 < \gamma(1-2a+a^2) + a - a^2 \\
0 < \gamma(1-a)^2 + a(1-a)
\end{gathered} \tag{40}$$

Since $a \in (0,1)$ and $\gamma > 0$, both terms in inequality (40) are strictly positive. Thus, the inequality $T_H(a,a) < a$ holds.

This condition guarantees that $\lim_{n\to\infty} T_H^{(n)}(a) = 0$. The membership grade of x^n approaches 0 as $\|x^n\| \to \infty$, verifying that the fuzzy ideal is not bounded by the unit ball. Substituting the classical idempotent operators with any of these Archimedean pairs generates a TCq-ROF ideal.

5. TCq-ROF Metric Kernel and First Isomorphism Theorem

To establish an algebraic structure over the Archimedean TCq-ROF framework, we introduce the concept of the metric kernel. The preservation of the metric geometry allows the formulation of the isomorphism theorems without topological distortion. Let R be a commutative normed ring.

Definition 10. *Let $Q = \langle \mu_Q, \nu_Q \rangle$ be a TCq-ROF ideal of R. The metric kernel of Q, denoted by* $\ker(Q)$*, is defined as the set of elements in R achieving the maximal membership and minimal non-membership grades:*

$$\ker(Q) = \{x \in R \mid \mu_Q(x) = \mu_Q(0), \nu_Q(x) = \nu_Q(0)\} \tag{41}$$

Lemma 1. *If Q is a TCq-ROF ideal of R, then its metric kernel* $\ker(Q)$ *is a classical ideal of R.*

Proof. Let $x, y \in \ker(Q)$. By the definition of the metric kernel, we have:

$$\mu_Q(x) = \mu_Q(0), \quad \mu_Q(y) = \mu_Q(0) \tag{42}$$

$$\nu_Q(x) = \nu_Q(0), \quad \nu_Q(y) = \nu_Q(0) \tag{43}$$

Applying Definition 8, we obtain:

$$\mu_Q(x - y) \geq T(\mu_Q(x), \mu_Q(y)) = T(\mu_Q(0), \mu_Q(0)) \tag{44}$$

Since $\mu_Q(0)$ is the upper bound for the membership function in Q, we have:

$$\mu_Q(x - y) \leq \mu_Q(0) \tag{45}$$

To satisfy both inequalities under the continuous t-norm, the grade must be exact:

$$\mu_Q(x - y) = \mu_Q(0) \tag{46}$$

For the non-membership grade, we evaluate:

$$\nu_Q(x - y) \leq S(\nu_Q(x), \nu_Q(y)) = S(\nu_Q(0), \nu_Q(0)) = \nu_Q(0) \tag{47}$$

Thus, $x - y \in \ker(Q)$. For the absorption property, applying Definition 9 for any $r \in R$:

$$\mu_Q(rx) \geq T(\mu_Q(r), \mu_Q(x)) = T(\mu_Q(r), \mu_Q(0)) \tag{48}$$

By the boundary properties of fuzzy ideals, this forces $\mu_Q(rx) = \mu_Q(0)$ and analogously $\nu_Q(rx) = \nu_Q(0)$. Therefore, $rx \in \ker(Q)$, establishing that $\ker(Q)$ is a classical ideal of R.

Given that $\ker(Q)$ is a classical ideal of R, we construct the quotient ring $R/\ker(Q)$. To establish the TCq-ROF quotient ring, we define a mapping from $R/\ker(Q)$ to $[0, 1]$. Let $\tilde{Q} = \langle \mu_{\tilde{Q}}, \nu_{\tilde{Q}} \rangle$ be a TCq-ROFS on $R/\ker(Q)$ defined by:

$$\mu_{\tilde{Q}}(x + \ker(Q)) = \mu_Q(x) \tag{49}$$

$$\nu_{\tilde{Q}}(x + \ker(Q)) = \nu_Q(x) \tag{50}$$

Definition 11. *Let $f : R \to R'$ be a surjective ring homomorphism and $Q = \langle \mu_Q, \nu_Q \rangle$ be a TCq-ROF ideal of R. The image $f(Q) = \langle \mu_{f(Q)}, \nu_{f(Q)} \rangle$ is defined for all $y \in R'$ by:*

$$\mu_{f(Q)}(y) = \sup\{\mu_Q(x) \mid f(x) = y\} \tag{51}$$

$$\nu_{f(Q)}(y) = \inf\{\nu_Q(x) \mid f(x) = y\} \tag{52}$$

Theorem 3. *Let $f : R \to R'$ be a surjective ring homomorphism. If Q is a TCq-ROF ideal of R such that $\ker(f) \subseteq \ker(Q)$, then $f(Q)$ is a TCq-ROF ideal of R'.*

Proof. Since $\ker(f) \subseteq \ker(Q)$, for any $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$, we have:

$$x_1 - x_2 \in \ker(f) \subseteq \ker(Q) \tag{53}$$

Thus, the membership and non-membership grades are preserved:

$$\mu_Q(x_1) = \mu_Q(x_2) \tag{54}$$
$$\nu_Q(x_1) = \nu_Q(x_2) \tag{55}$$

This establishes that the following mappings are well-defined:

$$\mu_{f(Q)}(f(x)) = \mu_Q(x) \tag{56}$$
$$\nu_{f(Q)}(f(x)) = \nu_Q(x) \tag{57}$$

To show $f(Q)$ is a TCq-ROF ideal of R', let $y_1, y_2 \in R'$. Since f is surjective, there exist $x_1, x_2 \in R$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By Definition 8, for the strictly Archimedean t-norm T:

$$\mu_{f(Q)}(y_1 - y_2) = \mu_Q(x_1 - x_2) \geq T(\mu_Q(x_1), \mu_Q(x_2)) = T(\mu_{f(Q)}(y_1), \mu_{f(Q)}(y_2)) \tag{58}$$

The proofs for the multiplication and non-membership grades under the strictly Archimedean t-conorm S follow analogously. Thus, $f(Q)$ is a TCq-ROF ideal of R'.

Theorem 4 (First Isomorphism Theorem for TCq-ROF Normed Rings)**.** *Let $f : R \to R'$ be a surjective ring homomorphism and Q be a TCq-ROF ideal of R with $\ker(f) \subseteq \ker(Q)$. Then:*

$$R/\ker(Q) \cong R'/\ker(f(Q)) \tag{59}$$

Proof. By the classical First Isomorphism Theorem, the mapping $\psi : R/\ker(f) \to R'$ given by $\psi(x + \ker(f)) = f(x)$ is an isomorphism. Extending this to the fuzzy structures, the corresponding metric kernels satisfy:

$$\ker(f(Q)) = f(\ker(Q)) \tag{60}$$

The bijection preserves the algebraic operations and the TCq-ROF bounds under the Archimedean norms, establishing $R/\ker(Q) \cong R'/\ker(f(Q))$.

Example 5. *Consider the commutative normed ring of dual numbers $R = \mathbb{R}[\epsilon] = \{a + b\epsilon \mid a, b \in \mathbb{R}, \epsilon^2 = 0\}$ equipped with the norm $\|a + b\epsilon\| = |a| + |b|$. Let $f : R \to \mathbb{R}$ be the projection homomorphism defined by $f(a + b\epsilon) = a$. The mapping f is a surjective ring homomorphism with $\ker(f) = \langle \epsilon \rangle = \{b\epsilon \mid b \in \mathbb{R}\}$.*

To observe the First Isomorphism Theorem in the fuzzy domain, we construct a TCq-ROF ideal $Q = \langle \mu_Q, \nu_Q \rangle$ over R using the algebraic t-norm $T_A(x, y) = xy$ and t-conorm $S_A(x, y) = x + y - xy$. We define the membership and non-membership grades as:

$$\mu_Q(a + b\epsilon) = \frac{1}{1 + |a|} \tag{61}$$

$$\nu_Q(a+b\epsilon) = \frac{|a|}{1+|a|} \tag{62}$$

Evaluating the metric kernel of Q yields:

$$\ker(Q) = \{a+b\epsilon \in R \mid \mu_Q(a+b\epsilon) = 1, \nu_Q(a+b\epsilon) = 0\} = \{a+b\epsilon \in R \mid a = 0\} = \langle \epsilon \rangle \tag{63}$$

Since $\ker(f) = \ker(Q) = \langle \epsilon \rangle$, *Theorem 4 applies. We determine the image $f(Q)$ on $\mathbb{R}$:*

$$\mu_{f(Q)}(a) = \sup\{\mu_Q(a+b\epsilon) \mid f(a+b\epsilon) = a\} = \frac{1}{1+|a|} \tag{64}$$

$$\nu_{f(Q)}(a) = \inf\{\nu_Q(a+b\epsilon) \mid f(a+b\epsilon) = a\} = \frac{|a|}{1+|a|} \tag{65}$$

The metric kernel of the image evaluates to $\ker(f(Q)) = \{0\}$. *By the First Isomorphism Theorem for TCq-ROF normed rings:*

$$R/\ker(Q) \cong R/\ker(f(Q)) \implies \mathbb{R}[\epsilon]/\langle \epsilon \rangle \cong \mathbb{R}/\{0\} \cong \mathbb{R} \tag{66}$$

This demonstrates that the Archimedean framework successfully preserves the ring isomorphism while accommodating the norm-induced topological divergence.

6. TCq-ROF Direct Sums and Maximal Ideals

We now define the direct sum of fuzzy ideals over the norm-induced topology. The topological consistency requires the aggregation of operators to be governed by the Archimedean operators. Let R be a commutative normed ring.

Definition 12. *Let $I = \langle \mu_I, \nu_I \rangle$ and $J = \langle \mu_J, \nu_J \rangle$ be TCq-ROF ideals of R. The TCq-ROF direct sum $I \oplus J$ is defined by the following membership and non-membership functions for all $x \in R$:*

$$\mu_{I\oplus J}(x) = \sup_{x=y+z} T(\mu_I(y), \mu_J(z)) \tag{67}$$

$$\nu_{I\oplus J}(x) = \inf_{x=y+z} S(\nu_I(y), \nu_J(z)) \tag{68}$$

Theorem 5. *Let T distribute over the supremum and S distribute over the infimum. If I and J are TCq-ROF ideals of R, then their direct sum $I \oplus J$ is a TCq-ROF ideal of R.*

Proof. To establish the additive conditions, let $a, b \in R$. For the membership function $\mu_{I\oplus J}$, we evaluate $T(\mu_{I\oplus J}(a), \mu_{I\oplus J}(b))$. By Definition 12, the distribution of the continuous Archimedean t-norm T over the supremum, its commutativity and monotonicity, and the ideal properties of I and J, we obtain:

$$T(\mu_{I\oplus J}(a), \mu_{I\oplus J}(b)) = T\left(\sup_{a=y_1+z_1} T(\mu_I(y_1), \mu_J(z_1)), \sup_{b=y_2+z_2} T(\mu_I(y_2), \mu_J(z_2))\right)$$

$$
\begin{aligned}
&= \sup_{\substack{a=y_1+z_1\\ b=y_2+z_2}} T\left(T(\mu_I(y_1),\mu_J(z_1)),T(\mu_I(y_2),\mu_J(z_2))\right)\\
&= \sup_{\substack{a=y_1+z_1\\ b=y_2+z_2}} T\left(T(\mu_I(y_1),\mu_I(y_2)),T(\mu_J(z_1),\mu_J(z_2))\right)\\
&\leq \sup_{\substack{a=y_1+z_1\\ b=y_2+z_2}} T(\mu_I(y_1-y_2),\mu_J(z_1-z_2))\\
&\leq \sup_{a-b=y+z} T(\mu_I(y),\mu_J(z))\\
&= \mu_{I\oplus J}(a-b) \qquad (69)
\end{aligned}
$$

where the final inequality follows by setting $y = y_1 - y_2$ and $z = z_1 - z_2$, noting that $y+z = (y_1+z_1)-(y_2+z_2) = a-b$, and observing that the supremum over this restricted subset of components is bounded by the supremum over all pairs summing to $a-b$.

Similarly, for the non-membership function $\nu_{I\oplus J}$, using the distribution of the continuous t-conorm S over the infimum, along with its commutativity, monotonicity, and the properties of TCq-ROF ideals, we have:

$$
\begin{aligned}
S(\nu_{I\oplus J}(a),\nu_{I\oplus J}(b)) &= S\left(\inf_{a=y_1+z_1} S(\nu_I(y_1),\nu_J(z_1)), \inf_{b=y_2+z_2} S(\nu_I(y_2),\nu_J(z_2))\right)\\
&= \inf_{\substack{a=y_1+z_1\\ b=y_2+z_2}} S\left(S(\nu_I(y_1),\nu_J(z_1)),S(\nu_I(y_2),\nu_J(z_2))\right)\\
&= \inf_{\substack{a=y_1+z_1\\ b=y_2+z_2}} S\left(S(\nu_I(y_1),\nu_I(y_2)),S(\nu_J(z_1),\nu_J(z_2))\right)\\
&\geq \inf_{\substack{a=y_1+z_1\\ b=y_2+z_2}} S(\nu_I(y_1-y_2),\nu_J(z_1-z_2))\\
&\geq \inf_{a-b=y+z} S(\nu_I(y),\nu_J(z))\\
&= \nu_{I\oplus J}(a-b) \qquad (70)
\end{aligned}
$$

where the final inequality is derived using the same substitution $y = y_1-y_2$ and $z = z_1-z_2$.

The multiplicative absorption properties for $\mu_{I\oplus J}(rx)$ and $\nu_{I\oplus J}(rx)$ follow analogously from the continuous distribution of Archimedean operators over suprema and infima. Consequently, $I\oplus J$ is a TCq-ROF ideal of R.

Definition 13. *A TCq-ROF ideal $M = \langle \mu_M, \nu_M\rangle$ is a TCq-ROF maximal ideal of R if its metric kernel* $\ker(M)$ *forms a maximal ideal of the ring R in the classical sense.*

Theorem 6. *Let R be a commutative normed ring with identity 1_R. If M is a TCq-ROF maximal ideal of R, then the TCq-ROF quotient space $R/\ker(M)$ forms a TCq-ROF normed field.*

Proof. By Definition 13, $\ker(M)$ is a maximal ideal of R. From commutative algebra, the quotient ring $R/\ker(M)$ is a field. By Theorem 4, the TCq-ROF quotient ring $\tilde{M}$ is well-defined over $R/\ker(M)$ such that the Archimedean norms are preserved. Since the underlying algebraic structure is a field, $\tilde{M}$ constitutes a TCq-ROF normed field.

Example 6. *To demonstrate the construction of TCq-ROF normed fields while resolving the metric limitation, consider the polynomial ring $R[x]$ equipped with the norm:*

$$\left\|\sum a_i x^i\right\| = \sum |a_i| \tag{71}$$

Let $K = \langle x \rangle$ be the classical maximal ideal generated by x, consisting of all polynomials with a zero constant term. We construct a TCq-ROF subset $M = \langle \mu_M, \nu_M \rangle$ governed by strictly Archimedean operators. Let $p(x) \in R[x]$. We define the grades as:

$$\mu_M(p(x)) = \begin{cases} 1, & p(x) \in K \\ \frac{1}{1+\|p(x)\|}, & p(x) \notin K \end{cases} \tag{72}$$

$$\nu_M(p(x)) = \begin{cases} 0, & p(x) \in K \\ 1 - \frac{1}{1+\|p(x)\|}, & p(x) \notin K \end{cases} \tag{73}$$

The metric kernel evaluates to:

$$\ker(M) = \{p(x) \in R[x] \mid \mu_M(p(x)) = 1, \nu_M(p(x)) = 0\} = K \tag{74}$$

Since K is a maximal ideal in $R[x]$, M is a TCq-ROF maximal ideal. The structure ensures that polynomials diverging in norm satisfy the norm-compatibility condition in their membership grades, preserving the metric consistency.

7. Prime Ideals, Zero Divisors, and Radical Extensions in TCq-ROF Spaces

To establish an algebraic hierarchy within the TCq-ROF framework, we generalize the concepts of prime ideals, zero divisors, and radicals. The classical characterizations of fuzzy prime ideals were studied by Dixit et al. [24] and Mukherjee and Sen [25]. Malik and Mordeson [26] formalized the fuzzy radical of an ideal. Extending these notions to unbounded normed spaces requires the application of the Archimedean dual norm compatibility.

First, we define the product of TCq-ROF ideals to establish prime structures.

Definition 14. *Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be TCq-ROF ideals of R. The TCq-ROF product $A \circ B$ is defined by the following membership and non-membership functions for all $x \in R$:*

$$\mu_{A\circ B}(x) = \sup_{x=yz} T(\mu_A(y), \mu_B(z)) \tag{75}$$

$$\nu_{A\circ B}(x) = \inf_{x=yz} S(\nu_A(y), \nu_B(z)) \tag{76}$$

where T and S are strictly Archimedean t-norm and t-conorm, respectively.

Definition 15. *Let R be a commutative normed ring. A TCq-ROF ideal $P = \langle \mu_P, \nu_P \rangle$ of R is called a TCq-ROF prime ideal if it is not constant, and for any two TCq-ROF ideals A and B of R, the condition $A \circ B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.*

Every maximal ideal in a commutative ring with identity is a prime ideal. We prove that this structural property is preserved in the Archimedean TCq-ROF formulation.

Theorem 7. *Let R be a commutative normed ring with identity 1_R. If M is a TCq-ROF maximal ideal of R, then M is a TCq-ROF prime ideal of R.*

Proof. Let $M = \langle \mu_M, \nu_M \rangle$ be a TCq-ROF maximal ideal. By Definition 13, its metric kernel $\ker(M)$ is a maximal ideal in R.

Assume there exist two TCq-ROF ideals A and B such that $A \circ B \subseteq M$. We first establish the inclusion at the kernel level. By the definition of the TCq-ROF product, their respective metric kernels satisfy:

$$\ker(A)\ker(B) \subseteq \ker(A \circ B) \subseteq \ker(M) \tag{77}$$

Since $\ker(M)$ is a maximal ideal in a commutative ring with identity, it is necessarily a prime ideal. Therefore, the relation $\ker(A)\ker(B) \subseteq \ker(M)$ classically implies that either $\ker(A) \subseteq \ker(M)$ or $\ker(B) \subseteq \ker(M)$.

We now demonstrate that this kernel inclusion enforces the fuzzy inclusion. Suppose $\ker(A) \subseteq \ker(M)$. Because M is a TCq-ROF maximal ideal, it constitutes the maximal fuzzy extension over the proper ideal $\ker(M)$. Any TCq-ROF ideal A satisfying $\ker(A) \subseteq \ker(M)$ must satisfy the bounding grades $\mu_A(x) \leq \mu_M(x)$ and $\nu_A(x) \geq \nu_M(x)$ for all $x \in R$. Otherwise, the union $M \cup A$ would form a proper TCq-ROF ideal strictly containing M, which directly contradicts the fuzzy maximality of M.

Consequently, $\ker(A) \subseteq \ker(M)$ requires $A \subseteq M$. Similarly, $\ker(B) \subseteq \ker(M)$ requires $B \subseteq M$. Thus, the condition $A \circ B \subseteq M$ yields either $A \subseteq M$ or $B \subseteq M$. Hence, M is a TCq-ROF prime ideal.

Kute et al. [27] investigated fuzzy zero divisors in polynomial subrings. We formulate the concept of TCq-ROF zero divisors under the norm-induced topology.

Definition 16. *A non-zero element $x \in R$ is a TCq-ROF zero divisor if there exists a non-zero $y \in R$ such that for a given TCq-ROF ideal I, the membership and non-membership grades satisfy:*

$$\mu_I(xy) = \mu_I(0) \tag{78}$$

$$\nu_I(xy) = \nu_I(0) \tag{79}$$

while $\|x\| > 0$ and $\|y\| > 0$.

Example 7. *Consider the commutative product ring $R \times R$ equipped with the norm:*

$$\|(a, b)\| = |a| + |b| \tag{80}$$

Let I be a TCq-ROF ideal formulated using strictly Archimedean operators with the metric kernel $\ker(I) = \{(0,0)\}$. *Choose $x = (1,0)$ and $y = (0,1)$. We evaluate the norms:*

$$\|x\| = 1$$
$$\|y\| = 1$$

Since the elements are non-zero, the strictly Archimedean operators position the grades below the bounds of the kernel:

$$\mu_I(x) < \mu_I(0)$$
$$\nu_I(x) > \nu_I(0)$$

However, their product is $xy = (1,0) \cdot (0,1) = (0,0)$. Consequently, the membership grade evaluates to:

$$\mu_I(xy) = \mu_I(0,0) = \mu_I(0) \tag{81}$$

This satisfies Definition 16. The Archimedean topological framework identifies zero divisors in q-ROF product rings without bounding the support to the unit ball.

To conclude the structural hierarchy, we define the fuzzy radical, based on the intersection property established by Malik and Mordeson [26].

Definition 17. *Let I be a TCq-ROF ideal of R. The q-ROF radical of I, denoted as $\sqrt{I}$, is defined as the intersection of all TCq-ROF prime ideals of R that contain I:*

$$\sqrt{I} = \bigcap \{P \mid P \textit{ is a TCq-ROF prime ideal and } I \subseteq P\} \tag{82}$$

Example 8. *To demonstrate the construction of the TCq-ROF radical, we utilize the algebraic t-norm $T_A(a,b) = ab$ and t-conorm $S_A(a,b) = a + b - ab$ from Section 4. Let $R = \mathbb{Z}$ be the commutative normed ring of integers equipped with the absolute value norm $\|x\| = |x|$.*

Consider the classical ideal $K = \langle 4 \rangle$. We construct a TCq-ROF ideal $Q = \langle \mu_Q, \nu_Q \rangle$ over $\mathbb{Z}$ by defining the grades for any $x \in \mathbb{Z}$ as:

$$\mu_Q(x) = \begin{cases} 1, & x \in \langle 4 \rangle \\ \frac{1}{1+|x|}, & x \notin \langle 4 \rangle \end{cases} \tag{83}$$

$$\nu_Q(x) = \begin{cases} 0, & x \in \langle 4 \rangle \\ \frac{|x|}{1+|x|}, & x \notin \langle 4 \rangle \end{cases} \tag{84}$$

For any $x, y \notin \langle 4 \rangle$, the algebraic t-norm yields:

$$T_A(\mu_Q(x), \mu_Q(y)) = \left(\frac{1}{1+|x|}\right)\left(\frac{1}{1+|y|}\right) = \frac{1}{1+|x|+|y|+|xy|} \tag{85}$$

Since the absolute value satisfies $|x-y| \leq |x|+|y| \leq |x|+|y|+|xy|$, *it follows that:*

$$\mu_Q(x-y) \geq \frac{1}{1+|x-y|} \geq T_A(\mu_Q(x), \mu_Q(y)) \tag{86}$$

For the non-membership grade under the algebraic t-conorm $S_A(a,b) = a+b-ab$, *we evaluate:*

$$\begin{aligned} S_A(\nu_Q(x), \nu_Q(y)) &= \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} - \frac{|x||y|}{(1+|x|)(1+|y|)} \\ &= \frac{|x|+|y|+|xy|}{1+|x|+|y|+|xy|} \end{aligned} \tag{87}$$

The function $g(t) = t/(1+t)$ *is strictly increasing for* $t \geq 0$. *Since the absolute value satisfies* $|x-y| \leq |x|+|y| \leq |x|+|y|+|xy|$, *it follows that:*

$$\nu_Q(x-y) \leq \frac{|x-y|}{1+|x-y|} \leq S_A(\nu_Q(x), \nu_Q(y)) \tag{88}$$

Thus, Q *is a TCq-ROF ideal with* $\ker(Q) = \langle 4 \rangle$.

By Definition 17, the TCq-ROF radical $\sqrt{Q}$ *is the intersection of all TCq-ROF prime ideals containing* Q. *In the classical sense, the radical of* $\langle 4 \rangle$ *is the prime ideal* $\langle 2 \rangle$. *Consequently, the metric kernel of the fuzzy intersection strictly expands to* $\langle 2 \rangle$:

$$\ker(\sqrt{Q}) = \sqrt{\ker(Q)} = \langle 2 \rangle \tag{89}$$

We evaluate the membership grade of the element $x = 2$. *In the original ideal* Q, *since* $2 \notin \langle 4 \rangle$, *its grade satisfies the norm-compatibility condition:*

$$\mu_Q(2) = \frac{1}{1+|2|} = \frac{1}{3} \tag{90}$$

However, in the TCq-ROF radical $\sqrt{Q}$, *because* $2 \in \ker(\sqrt{Q})$, *its membership grade is elevated to the supremum:*

$$\mu_{\sqrt{Q}}(2) = 1 \tag{91}$$

This demonstrates that the TCq-ROF radical operator extracts the nilpotent-like elements in the normed space, elevating their membership degrees to the absolute bound without causing metric limitation in the divergent sequences.

8. Conclusions and Future Directions

In this paper, we investigated a structural limitation in the algebraic frameworks of q-ROFSs when applied to unbounded topological spaces. We proved that defining q-ROF subrings and ideals using classical idempotent operators restricts the fuzzy support to the unit ball, violating the norm-compatibility condition on infinite-dimensional commutative normed algebras.

To resolve this metric limitation, we introduced the Archimedean Dual Norm Compatibility. By replacing the idempotent operators with strictly Archimedean t-norms and t-conorms, we constructed topologically consistent (TCq-ROF) normed rings. Under this framework, the metric kernel and the TCq-ROF quotient topologies were formulated, enabling the proof of the First Isomorphism Theorem for TCq-ROF normed rings while preserving the metric consistency. Furthermore, the structural hierarchy was extended by defining TCq-ROF maximal and prime ideals, fuzzy direct sums, TCq-ROF zero divisors over product rings, and TCq-ROF radical extensions.

To further advance the intersection of fuzzy abstract algebra and functional analysis, we propose the following open problems for future research:

(i) **Topological Localization and Rings of Fractions:** While this study establishes quotient structures, the construction of q-ROF rings of fractions using multiplicatively closed TCq-ROF subsets remains unresolved. Determining whether the strictly Archimedean operators preserve the norm-compatibility condition during localization is an algebraic challenge.

(ii) **Banach Completeness in TCq-ROF Fields:** Assuming R is a Banach ring (a complete normed ring), determining the analytical conditions on the generating strictly Archimedean t-norms under which the resulting TCq-ROF normed field $R/\ker(M)$ preserves topological completeness remains an open question.

(iii) **Generalization to TCq-ROF Modules and Algebras:** The metric limitation demonstrated in this study addresses commutative normed rings. Extending this investigation to infinite-dimensional normed modules and algebras remains an open problem. Determining how the Archimedean dual norm compatibility maintains metric consistency during scalar multiplication and module operations presents a fundamental theoretical direction for fuzzy abstract algebra.

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