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Join Decomposition and Graph Invariants of Square Root Prime Divisors Graphs of Finite Groups

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Abstract. Let G be a finite group and let $\pi(G)$ be the set of prime divisors of $|G|$. The square root prime divisors graph SRPD(G) has vertex set $\pi(G)$, with distinct primes p and q adjacent when $p^\alpha + q^\beta$ is a perfect square for some nonnegative integers α, β. The construction is arithmetic in nature: it depends on the prime support $\pi(G)$ rather than on the internal multiplication of G. Building on the universal adjacency of the primes 2 and 3, we obtain a structural decomposition of SRPD(G) as a graph join whenever 2 or 3 divides $|G|$. This decomposition yields classification results for completeness, regularity, bipartiteness and planarity, as well as exact formulas for diameter, domination number, clique number, chromatic number and the Laplacian spectrum. In particular, if both 2 and 3 divide $|G|$, planarity is characterized by a linear-forest condition on the induced graph on the remaining prime divisors. We also obtain a spectral formula and a spanning-tree formula for this class of SRPD-graphs. Since the graph depends only on the prime support $\pi(G)$, the resulting classification is by prime support and associated SRPD-graph invariants, not by the internal group structure or isomorphism type of G.

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1. Introduction

Graphs associated with algebraic structures provide a useful bridge between finite group theory, number theory and graph theory. Familiar constructions include Cayley graphs, power graphs, character-degree graphs and prime graphs; see, for example, [1–4]. The square root prime divisors graph (SRPD-graph) was introduced by AlFran [5]. Its vertices are the primes dividing the group order, while adjacency is governed by the exponential Diophantine condition

$$p^\alpha + q^\beta = n^2,$$

where p, q are distinct primes, α, β are nonnegative integers, and n is a positive integer.

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The original paper [5] established basic properties of the square relation, proved that the primes 2 and 3 are adjacent to every other prime, and investigated connectivity, completeness, planarity and threshold behavior for several families and examples. A particularly important observation is that two finite groups with the same set of prime divisors have the same labeled SRPD-graph. Thus the graph does not, in general, distinguish non-isomorphic groups having the same prime support. The appropriate meaning of classification via SRPD-graphs is therefore classification of finite groups according to arithmetic and graph-theoretic properties determined by $\pi(G)$.

The purpose of the present paper is to develop this viewpoint systematically. The principal SRPD-specific contribution is a universal-prime join decomposition, from which a collection of classification and invariant formulas follows. Standard facts about graph joins are used explicitly as graph-theoretic input rather than claimed as new results. The key step is the following decomposition. If

$$U = \pi(G) \cap \{2, 3\}, \qquad t = |U|,$$

and H_G denotes the subgraph of $\mathrm{SRPD}(G)$ induced by $\pi(G) \setminus U$, then, whenever $t \geq 1$,

$$\mathrm{SRPD}(G) \cong K_t \vee H_G.$$

This elementary-looking decomposition has strong consequences. It converts questions about SRPD-graphs into standard questions about graph joins and isolates the genuinely arithmetic part of the construction in H_G.

Our main results include the following. For $2 \mid |G|$ or $3 \mid |G|$, regularity of $\mathrm{SRPD}(G)$ is equivalent to completeness. Bipartiteness is completely characterized in terms of t and the edge set of H_G. If exactly one of $2, 3$ divides $|G|$, planarity is equivalent to outerplanarity of H_G; if both divide $|G|$, planarity is equivalent to H_G being a linear forest. We also determine the diameter, domination number, clique number and chromatic number, and derive the Laplacian spectrum and number of spanning trees from those of H_G.

Standard graph-theoretic terminology follows [6, 7]. For planarity of joins we use the characterization of Sitthiwiratham and Promsakon [8].

2. Preliminaries

Throughout, all groups are finite and all graphs are finite, simple and undirected. For a finite group G, let

$$\pi(G) = \{p : p \text{ is prime and } p \mid |G|\}.$$

Definition 1 (Square relation [5])**.** *Let p and q be distinct primes. We write $p \sim_s q$ if there exist nonnegative integers α, β and a positive integer n such that*

$$p^\alpha + q^\beta = n^2.$$

Definition 2 (SRPD-graph [5])**.** *The square root prime divisors graph of G, denoted $\mathrm{SRPD}(G)$, is the graph with vertex set $\pi(G)$ in which distinct vertices p, q are adjacent if and only if $p \sim_s q$.*

The relation is symmetric: if $p^\alpha + q^\beta = n^2$, then $q^\beta + p^\alpha = n^2$. Hence $\mathrm{SRPD}(G)$ is undirected. We emphasize that zero exponents are permitted; this convention is essential, for example, in the identities $2^3 + p^0 = 9$ and $3 + p^0 = 4$ used below.

We recall the basic universal-adjacency observation, including its short proof because it drives all subsequent results.

Lemma 1 (Universal primes [5])**.** *The primes* 2 *and* 3 *are square-related to every distinct prime.*

Proof. Let p be a prime. If $p \neq 2$, then

$$2^3 + p^0 = 8 + 1 = 9 = 3^2,$$

so $2 \sim_s p$. If $p \neq 3$, then

$$3^1 + p^0 = 3 + 1 = 4 = 2^2,$$

so $3 \sim_s p$.

Definition 3. *For graphs X and Y on disjoint vertex sets, their* join *$X \vee Y$ is obtained from their disjoint union by adding every edge joining a vertex of X to a vertex of Y.*

We write $\omega(X)$ for the clique number, $\chi(X)$ for the chromatic number, $\gamma(X)$ for the domination number, $\mathrm{diam}(X)$ for the diameter, and $L(X)$ for the Laplacian matrix. A *linear forest* is a forest each of whose connected components is a path; isolated vertices are allowed as paths of order one.

3. The universal-prime decomposition

The next theorem is the structural basis of the paper.

Theorem 1 (Join decomposition)**.** *Let G be a finite group, put*

$$U = \pi(G) \cap \{2, 3\}, \qquad t = |U|,$$

and let $H_G = \mathrm{SRPD}(G)[\pi(G) \setminus U]$ be the induced subgraph on the remaining prime divisors. If $t \geq 1$, then

$$\mathrm{SRPD}(G) \cong K_t \vee H_G.$$

Proof. By Lemma 1, every vertex belonging to U is adjacent to every vertex outside U. If $t = 2$, then the vertices 2 and 3 are also adjacent, so the graph induced by U is K_2; if $t = 1$, it is K_1. The edges among vertices outside U are precisely those of the induced graph H_G. This is exactly the definition of the join $K_t \vee H_G$.

Remark 1. *Theorem 1 makes explicit a limitation and a strength of the SRPD construction. The internal group structure does not enter the square relation; once $\pi(G)$ is known, the graph is determined. Thus the results below classify finite groups by the arithmetic configuration of the primes dividing their orders.*

Corollary 1. *If* $2 \mid |G|$ *or* $3 \mid |G|$*, then* $\mathrm{SRPD}(G)$ *is connected.*

Proof. If H_G is nonempty, Theorem 1 expresses $\mathrm{SRPD}(G)$ as a join of two nonempty graphs, hence it is connected. If H_G is empty, then $\mathrm{SRPD}(G) = K_t$, which is connected (with the one-vertex case understood in the usual trivial sense). Equivalently, whenever another vertex is present, a vertex corresponding to 2 or 3 is adjacent to it.

Corollary 2. *Assume* $2 \mid |G|$ *or* $3 \mid |G|$ *and* $|\pi(G)| \geq 2$*. Then*

$$\mathrm{diam}(\mathrm{SRPD}(G)) = \begin{cases} 1, & \mathrm{SRPD}(G) \text{ is complete,} \\ 2, & \mathrm{SRPD}(G) \text{ is not complete.} \end{cases}$$

Proof. The complete case is immediate. Otherwise choose a universal vertex $u \in U$. Any two nonadjacent vertices v, w are joined by the path $v - u - w$, so their distance is 2, while all distances are at most 2.

Corollary 3. *If* $2 \mid |G|$ *or* $3 \mid |G|$*, then*

$$\gamma(\mathrm{SRPD}(G)) = 1.$$

Proof. Any vertex in U is adjacent to every other vertex.

4. Complete and regular SRPD-graphs

The join decomposition gives a concise completeness criterion.

Theorem 2. *Assume* $t = |\pi(G) \cap \{2, 3\}| \geq 1$*. Then* $\mathrm{SRPD}(G)$ *is complete if and only if* H_G *is complete.*

Proof. By Theorem 1, $\mathrm{SRPD}(G) = K_t \vee H_G$. Every cross edge and every edge inside K_t is already present. Therefore the whole graph is complete exactly when every pair of vertices of H_G is adjacent.

The next result is stronger than a degree formula: in the large class of groups whose order is divisible by 2 or 3, there are no noncomplete regular SRPD-graphs.

Theorem 3 (Regularity classification)**.** *Assume* $2 \mid |G|$ *or* $3 \mid |G|$*. Then* $\mathrm{SRPD}(G)$ *is regular if and only if it is complete.*

Proof. Every vertex in U is universal, hence has degree $|\pi(G)| - 1$. If $\mathrm{SRPD}(G)$ is regular, every vertex must have this same degree, so every vertex is adjacent to every other vertex and the graph is complete. Conversely, every complete graph is regular.

Corollary 4. *Under the hypotheses of Theorem 3, if* $n = |\pi(G)|$*, then a regular* $\mathrm{SRPD}(G)$ *is necessarily* $(n-1)$*-regular.*

5. Bipartite SRPD-graphs

The universal vertices also yield an exact bipartite classification.

Theorem 4 (Bipartite classification)**.** *Let G be a finite group with $t = |\pi(G) \cap \{2,3\}| \geq 1$.*

(i) If $t = 1$, then $\mathrm{SRPD}(G)$ is bipartite if and only if H_G has no edges.

(ii) If $t = 2$, then $\mathrm{SRPD}(G)$ is bipartite if and only if H_G has no vertices, equivalently $\pi(G) = \{2,3\}$.

Proof. By Theorem 1, $\mathrm{SRPD}(G) = K_t \vee H_G$.

If $t = 1$ and H_G contains an edge vw, the universal vertex together with v, w forms a triangle, so the graph is not bipartite. If H_G is edgeless, $K_1 \vee H_G$ is a star (or K_1 when H_G is empty), hence bipartite.

If $t = 2$ and H_G has a vertex v, then $2, 3, v$ form a triangle, so the graph is not bipartite. If H_G is empty, the graph is K_2, which is bipartite.

Corollary 5. *If $6 \mid |G|$ and $|\pi(G)| \geq 3$, then $\mathrm{SRPD}(G)$ contains a triangle and is therefore not bipartite.*

6. Planar SRPD-graphs

We now obtain an exact planarity criterion for all SRPD-graphs containing at least one universal prime. The graph-theoretic input is the standard fact that $K_1 \vee H$ is planar exactly when H is outerplanar, and the characterization $K_2 \vee H$ planar exactly when H is a linear forest [8].

Theorem 5 (Planarity classification)**.** *Let G be a finite group with $t = |\pi(G) \cap \{2,3\}| \geq 1$.*

(i) If $t = 1$, then $\mathrm{SRPD}(G)$ is planar if and only if H_G is outerplanar.

(ii) If $t = 2$, then $\mathrm{SRPD}(G)$ is planar if and only if H_G is a linear forest.

Proof. Theorem 1 gives $\mathrm{SRPD}(G) = K_t \vee H_G$. For $t = 1$, the cone $K_1 \vee H_G$ is planar exactly when H_G is outerplanar [8, Theorem 2.8]. For $t = 2$, the characterization of planar joins in [8, Theorem 2.7] states that $K_2 \vee H_G$ is planar exactly when H_G is a disjoint union of paths, i.e. a linear forest.

Corollary 6. *Suppose $6 \mid |G|$. If H_G contains a cycle or a vertex of degree at least 3, then $\mathrm{SRPD}(G)$ is nonplanar.*

Proof. A graph is a linear forest if and only if it is acyclic and has maximum degree at most 2. Apply Theorem 5(ii).

Corollary 7. *Suppose $6 \mid |G|$ and $\mathrm{SRPD}(G)$ is complete. Then $\mathrm{SRPD}(G)$ is planar if and only if $|\pi(G)| \leq 4$.*

Proof. A complete graph K_n is planar exactly for $n \leq 4$.

7. Clique number, chromatic number and independence

The join decomposition also gives exact formulas for several classical invariants.

Theorem 6. *Assume $t = |\pi(G) \cap \{2, 3\}| \geq 1$. Then*

$$\omega(\mathrm{SRPD}(G)) = t + \omega(H_G)$$

and

$$\chi(\mathrm{SRPD}(G)) = t + \chi(H_G),$$

with the convention $\omega(\varnothing) = \chi(\varnothing) = 0$.

Proof. For a join $X \vee Y$, every vertex of X is adjacent to every vertex of Y. Hence a maximum clique is the union of maximum cliques from the two factors, and the color sets used on the two factors must be disjoint. Thus $\omega(X \vee Y) = \omega(X) + \omega(Y)$ and $\chi(X \vee Y) = \chi(X) + \chi(Y)$. Apply this with $X = K_t$.

Proposition 1. *If H_G is nonempty, then*

$$\alpha(\mathrm{SRPD}(G)) = \alpha(H_G).$$

If H_G is empty, then $\alpha(\mathrm{SRPD}(G)) = 1$.

Proof. No independent set can contain vertices from both factors of a join. Since K_t has independence number 1, the result follows.

8. Laplacian spectrum and spanning trees

The decomposition also yields spectral information not requiring the explicit construction of the full Laplacian matrix.

Let $m = |V(H_G)|$ and $n = t + m = |\pi(G)|$. Write the Laplacian eigenvalues of H_G as

$$0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m,$$

where zero is repeated according to the number of connected components. We choose an orthonormal Laplacian eigenbasis so that the eigenvector corresponding to $\mu_1 = 0$ is proportional to $\mathbf{1}_m$; all remaining eigenvectors are orthogonal to $\mathbf{1}_m$.

Theorem 7 (Laplacian spectrum)**.** *Assume $t \geq 1$ and $m \geq 1$. The Laplacian spectrum of $\mathrm{SRPD}(G) = K_t \vee H_G$ is*

$$\Big\{0,\ n^{[t]},\ (t + \mu_2), \ldots, (t + \mu_m)\Big\},$$

where $n^{[t]}$ denotes t listed copies of n. If some shifted eigenvalue $t + \mu_i$ also equals n, its multiplicity is, of course, added to these t copies.

Proof. Order the vertices so that those of K_t come first. The Laplacian matrix of the join has block form

$$L = \begin{pmatrix} nI_t - J_t & -J_{t\times m} \\ -J_{m\times t} & L(H_G) + tI_m \end{pmatrix}.$$

The all-ones vector gives eigenvalue 0. Every vector supported on the K_t coordinates and orthogonal to $\mathbf{1}_t$ is an eigenvector with eigenvalue n, giving multiplicity $t-1$. If $y \perp \mathbf{1}_m$ is a Laplacian eigenvector of H_G with eigenvalue μ_i ($i \geq 2$), then $(0, y)$ is an eigenvector of L with eigenvalue $t + \mu_i$. Finally, the vector $(m\mathbf{1}_t, -t\mathbf{1}_m)$ is orthogonal to the all-ones vector and has eigenvalue n. These vectors account for all n eigenvalues.

Remark 2. *If $m = 0$, then $\mathrm{SRPD}(G) = K_t$. Since $t \in \{1, 2\}$ in the present setting, its Laplacian spectrum is $\{0\}$ for $t = 1$ and $\{0, 2\}$ for $t = 2$. Thus the edge case omitted from Theorem 7 is completely determined.*

Corollary 8 (Spanning-tree formula)**.** *Under the hypotheses of Theorem 7, the number $\tau(\mathrm{SRPD}(G))$ of spanning trees is*

$$\tau(\mathrm{SRPD}(G)) = n^{t-1} \prod_{i=2}^{m} (t + \mu_i).$$

If $m = 1$, the empty product is 1, giving $\tau(K_{t+1}) = (t+1)^{t-1}$.

Proof. By the Matrix–Tree Theorem, the number of spanning trees of a connected graph on n vertices is $1/n$ times the product of its nonzero Laplacian eigenvalues. Substitute the spectrum from Theorem 7.

Corollary 9. *If $m \geq 2$, the algebraic connectivity of $\mathrm{SRPD}(G)$ is*

$$a(\mathrm{SRPD}(G)) = t + \mu_2(H_G).$$

In particular,

$$a(\mathrm{SRPD}(G)) \geq t.$$

Proof. From Theorem 7, the positive eigenvalues are n and $t + \mu_i$ for $i \geq 2$. Since $\mu_2 \leq m$, we have $t + \mu_2 \leq t + m = n$, and the result follows.

9. Consequences for finite-group classes

Because $\mathrm{SRPD}(G)$ depends only on $\pi(G)$, group families sharing the same prime support automatically share all the invariants above. This gives a useful transfer principle.

Proposition 2 (Prime-support transfer)**.** *Let G and K be finite groups with $\pi(G) = \pi(K)$. Then their labeled SRPD-graphs are equal. Consequently they have the same completeness, regularity, bipartiteness, planarity, diameter, domination number, clique number, chromatic number, Laplacian spectrum and number of spanning trees.*

Proof. Adjacency between two vertices is defined solely by the two primes and the square relation; it does not involve group multiplication. Thus the same prime set gives the same labeled graph. All listed quantities are graph invariants.

Example 1. *For S_3, $\pi(S_3) = \{2, 3\}$, so $t = 2$ and H_{S_3} is empty. Therefore*

$$\mathrm{SRPD}(S_3) = K_2.$$

It is complete, regular, bipartite and planar, has diameter 1 *and domination number* 1.

Example 2. *For A_7, $\pi(A_7) = \{2, 3, 5, 7\}$. The original SRPD study proves that* 5 *and* 7 *are not square-related [5]. For completeness, here is a short verification. Suppose*

$$5^\alpha + 7^\beta = n^2, \qquad \alpha, \beta \geq 0.$$

If $\alpha, \beta \geq 1$, reduction modulo 4 *shows that β must be odd. Reducing modulo* 5*, we then have $7^\beta \equiv 2$ or $3 \pmod 5$, whereas a square modulo* 5 *can only be* $0, 1$*, or* 4*, a contradiction. If $\alpha = 0$, then $n^2 - 1 = 7^\beta$, so $(n-1)(n+1) = 7^\beta$; the two positive factors differ by* 2*, and two powers of* 7 *cannot differ by* 2*. The case $\beta = 0$ is analogous, giving $(n-1)(n+1) = 5^\alpha$, again impossible. Thus $5 \not\sim_s 7$. Consequently H_{A_7} consists of two isolated vertices and*

$$\mathrm{SRPD}(A_7) = K_2 \vee \overline{K_2}.$$

Theorem 5 shows immediately that the graph is planar. It is not regular by Theorem 3, and it is not bipartite by Theorem 4. Moreover, $\omega = 3$ and $\chi = 3$.

Example 3. *For* $\mathrm{PSL}(2, 11)$*, $\pi(G) = \{2, 3, 5, 11\}$. Since $5 + 11 = 16$, the vertices* 5 *and* 11 *are adjacent, so $H_G = K_2$ and*

$$\mathrm{SRPD}(G) = K_2 \vee K_2 = K_4.$$

Hence the graph is complete, 3*-regular and planar, but not bipartite.*

Example 4 (A nontrivial spectral illustration)**.** *Let G be any finite group with prime support*

$$\pi(G) = \{2, 5, 7, 11\}.$$

The relations recorded in [5] give $5 \not\sim_s 7$ and $7 \not\sim_s 11$, while $5 \sim_s 11$ because

$$5^1 + 11^1 = 16 = 4^2.$$

Hence $t = 1$ and $H_G = K_2 \cup K_1$. The Laplacian spectrum of H_G is $\{0, 0, 2\}$. Theorem 7 therefore yields

$$\mathrm{Spec}_L(\mathrm{SRPD}(G)) = \{0, 1, 3, 4\}.$$

Moreover, Corollary 8 gives

$$\tau(\mathrm{SRPD}(G)) = (1 + 0)(1 + 2) = 3.$$

This illustrates the spectral and spanning-tree formulas when H_G is nontrivial and disconnected.

10. Discussion of classification scope

The preceding results show that SRPD-classification naturally separates into two layers. The first layer is group-theoretic only through the prime support $\pi(G)$. The second layer is arithmetic: for primes other than 2 and 3, one must determine whether the exponential Diophantine equation

$$p^\alpha + q^\beta = n^2$$

has a solution in nonnegative integers α, β.

This distinction is important. It prevents the SRPD-graph from being used as a complete isomorphism invariant for finite groups, but it also allows results proved for one prime support to be transferred to every finite group having that support. The join decomposition isolates the universal part $\{2, 3\}$ and reduces many graph-theoretic classification problems to the induced arithmetic graph H_G.

For example, when $6 \mid |G|$, the following statements are immediate and exact:

- $\mathrm{SRPD}(G)$ is regular if and only if it is complete;
- $\mathrm{SRPD}(G)$ is bipartite if and only if $\pi(G) = \{2, 3\}$;
- $\mathrm{SRPD}(G)$ is planar if and only if H_G is a linear forest;
- $\mathrm{diam}(\mathrm{SRPD}(G)) \leq 2$ and $\gamma(\mathrm{SRPD}(G)) = 1$.

These criteria apply across solvable and nonsolvable, simple and nonsimple groups whenever the stated divisibility hypotheses hold, because their proofs depend only on prime support and the square relation. More precisely, if $|G|$ is coprime to 6, then $|G|$ is odd and G is solvable by the Feit–Thompson theorem [9]. Hence every nonsolvable finite group has $2 \mid |G|$ (and therefore lies in the case $t \geq 1$ treated by the join decomposition).

11. Conclusion and open problems

We developed a structural classification framework for finite groups via SRPD-graphs. The universal role of 2 and 3 leads to the decomposition

$$\mathrm{SRPD}(G) = K_t \vee H_G, \qquad t = |\pi(G) \cap \{2, 3\}|,$$

whenever $t \geq 1$. From this decomposition we obtained exact criteria for completeness, regularity, bipartiteness and planarity, together with formulas for several graph invariants and the Laplacian spectrum.

The arithmetic graph H_G remains the essential object for further study. Natural problems include classifying pairs of odd primes $p, q \geq 5$ satisfying $p^\alpha + q^\beta = n^2$, determining which linear forests can occur as H_G, and studying the SRPD-graphs of groups whose orders are coprime to 6. These questions connect the graph construction directly with exponential Diophantine equations and appear to be the main source of deeper arithmetic structure in the theory.

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