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A Nonlinear Dynamical Model of Mathematics Achievement, Engagement, and Anxiety: Stability and Numerical Analysis

Francis Ohene Boateng

[1] *Department of Mathematics Education, University of Skills Training and Entrepreneurial Development (USTED),Kumasi, Ghana*

Abstract. This study developed a nonlinear dynamical model to examine the interactions among mathematics achievement, student engagement, and mathematics anxiety. The three educational constructs were represented as continuous state variables governed by a system of nonlinear ordinary differential equations. The feasible region was shown to be positively invariant, and equilibrium existence was established. Local stability was examined using the Jacobian matrix and Routh–Hurwitz criteria, while numerical simulations were performed using an adaptive Runge–Kutta method implemented through the ode45 solver. Under the baseline parameter configuration, the positive equilibrium (0.73795,0.57783,0.18449) was locally asymptotically stable, with all eigenvalues having negative real parts. Simulations showed increasing achievement, a slight initial decline followed by increasing engagement, and declining mathematics anxiety, with trajectories from different tested initial conditions approaching the same equilibrium. Sensitivity analysis revealed that increasing the engagement–achievement interaction parameter substantially increased equilibrium achievement while producing smaller changes in engagement and anxiety. The Routh–Hurwitz stability function remained positive for $\beta \in [0, 2]$ under fixed baseline values of the remaining parameters, and no Hopf bifurcation was detected in this one-parameter analysis. The model provides a theoretical framework for investigating the dynamic interdependence of cognitive, behavioral, and affective states in mathematics learning.

2020 Mathematics Subject Classifications: 34D20, 65L06, 97C20

Key Words and Phrases: Mathematics achievement, student engagement, mathematics anxiety, nonlinear dynamical system, stability analysis, numerical simulation.

1. Introduction

Mathematics learning is shaped by interconnected cognitive, behavioral, and affective processes that may change as learners interact with mathematical tasks and learning environments. Among these processes, mathematics achievement, student engagement, and

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Email address: foboateng@usted.edu.gh (F. O. Boateng)

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mathematics anxiety have received substantial research attention because of their relationships with participation, persistence, and performance. Student engagement encompasses behavioral, emotional, and cognitive involvement in learning activities [1], whereas mathematics anxiety refers to feelings of tension, apprehension, or discomfort associated with mathematics and mathematical activity [2, 3]. These constructs may therefore interact in ways that shape mathematics learning over time.

Evidence indicates a positive relationship between student engagement and mathematics achievement. Engagement can support participation, effort, strategy use, and sustained involvement in learning [1], while Fung et al. [4] reported meaningful associations between dimensions of engagement and mathematics achievement. Research has also connected engagement with affective experiences in mathematics learning. Bergqvist [5], Binyamina and Dubovi [6], Shabab [7], and Guo and Liao [8] examined relationships between mathematics anxiety and engagement across different learning conditions. Mathematics anxiety, in turn, has generally been negatively associated with mathematics performance, with meta-analytic evidence demonstrating an inverse relationship between mathematics anxiety and achievement across student populations [9].

These relationships may also unfold over time. Gunderson et al. [10] and Sun et al. [11] examined reciprocal relationships involving mathematics anxiety, motivational factors, and achievement, while Tomasetto et al. [12] investigated longitudinal associations between mathematics anxiety and achievement. Importantly, the evidence does not imply uniform reciprocity: Sorvo et al. [13] found that earlier arithmetic achievement predicted subsequent anxiety about failure, whereas anxiety did not predict later achievement in their sample. Control-value theory similarly conceptualizes achievement emotions as dynamically related to learners' appraisals, learning processes, and achievement outcomes [14]. Zhang et al. [15] found that mathematics anxiety and learning engagement mediated the relationship between teacher support and mathematics achievement among high school students. Collectively, these findings support examining achievement, engagement, and anxiety as interconnected processes while recognizing that their relationships may vary across populations, contexts, and time scales.

Recent studies have demonstrated the applicability of mathematical modelling to dynamic processes in mathematics education. Kaushik et al. [16], for example, applied nonlinear ordinary differential equations to teacher-student interaction and learning dynamics, while Dzramado et al. [17] developed a mathematical model of the dynamics of teachers' adoption of problem-based learning in mathematics education. These studies demonstrate the potential of mathematical modelling for representing educational processes that evolve through interacting mechanisms over time. More broadly, differential equations provide useful mathematical tools for representing time-dependent processes and examining their qualitative behavior. Their application extends across nonlinear systems in which stability, asymptotic behavior, and oscillatory properties are of interest. For example, Moaaz et al. [18] developed oscillation criteria for fourth-order neutral differential equations, while Moaaz et al. [19] examined asymptotic and oscillatory properties of solutions to fourth-order neutral differential equations. Although these studies address different dynamical systems, they demonstrate the broader applicability of differential-equation

methods for investigating the qualitative behavior of interacting processes. In the present study, nonlinear ordinary differential equations are used to represent the coupled evolution of mathematics achievement, student engagement, and mathematics anxiety.

Against this background, the present study develops a nonlinear dynamical model representing mathematics achievement, student engagement, and mathematics anxiety as simultaneous normalized states connected through theoretically specified feedback mechanisms. The analysis examines mathematical feasibility, existence and local stability of a positive equilibrium, parameter-dependent stability and potential Hopf bifurcation behavior, and numerical trajectories and equilibrium sensitivity. The study therefore extends existing empirical descriptions by translating documented relationships among the three constructs into an explicit nonlinear feedback system. The model is theoretical rather than an empirical causal model and provides a mathematical foundation for subsequent calibration and validation using longitudinal educational data.

2. Nonlinear Dynamical Model Formulation

The model represents the temporal interaction among three educational states: mathematics achievement, student engagement, and mathematics anxiety. Unlike population-compartment models in which individuals are assigned to mutually exclusive classes, the present formulation treats the three constructs as continuous states that may coexist and change simultaneously. Let A(t), E(t), and X(t) denote the normalized levels of mathematics achievement, student engagement, and mathematics anxiety, respectively, at time t. Each state variable is dimensionless and bounded within the interval [0,1].

The state variables are defined as follows.

i. Mathematics achievement, A(t). This represents the normalized state of mathematical achievement at time t. Its evolution is assumed to depend on external achievement-supporting influences, engagement-related enhancement, natural attenuation, and inhibition associated with mathematics anxiety.

ii. Student engagement, E(t). This represents the normalized state of students' involvement in mathematics learning at time t. Its evolution is assumed to depend on external engagement-supporting influences, natural attenuation, and inhibition associated with mathematics anxiety.

iii. Mathematics anxiety, X(t). This represents the normalized state of mathematics-related apprehension or tension at time t. Its evolution is assumed to depend on external anxiety-generating influences, natural attenuation, and anxiety-reducing interactions with achievement and engagement.

The normalization gives

$$0 \leq A(t) \leq 1, \quad 0 \leq E(t) \leq 1, \quad 0 \leq X(t) \leq 1$$

The values 0 and 1 represent theoretical lower and upper bounds of the normalized states and should not be interpreted as raw scores or percentages from particular educational instruments. Furthermore, because achievement, engagement, and anxiety are

distinct educational states rather than mutually exclusive student groups, no conservation requirement such as A + E + X = 1 is imposed.

2.1. Basic Assumptions of the Model

The nonlinear model is constructed under the following assumptions.

(i) Mathematics achievement receives an external positive contribution at rate ρ_A. Because achievement is bounded above, the contribution is represented by $\rho_A(1 - A)$, so that its magnitude diminishes as A approaches its theoretical upper bound.

(ii) Student engagement receives an external positive contribution at rate η_E. The bounded contribution is represented by $\eta_E(1-E)$, allowing the external contribution to engagement to diminish as E approaches 1.

(iii) Mathematics anxiety receives an external contribution at rate ρ_X. This is represented by $\rho_X(1 - X)$, which permits anxiety-generating influences outside the explicitly modelled achievement–engagement interactions while preserving saturation near the upper bound.

(iv) Engagement contributes positively to achievement. The contribution is represented by $\beta E(1-A)$, where β controls the strength of the engagement–achievement interaction. The factor $(1-A)$ incorporates diminishing capacity for additional achievement growth as achievement approaches its upper bound.

(v) Mathematics anxiety inhibits achievement through the interaction term $-\gamma AX$. The multiplicative form assumes that the magnitude of the inhibitory contribution depends jointly on the prevailing levels of achievement and anxiety. The inclusion of this pathway represents a theoretical modelling assumption rather than a claim that anxiety inhibits later achievement in every population. Although broader evidence has identified negative associations between mathematics anxiety and achievement [9, 20], Sorvo et al. [13] found that anxiety did not predict later achievement in their longitudinal sample. The parameter γ therefore represents a hypothesized pathway whose magnitude, including the possibility of $\gamma = 0$, should ultimately be evaluated through empirical calibration.

(vi) Mathematics anxiety inhibits student engagement through $-\alpha EX$. Thus, the magnitude of anxiety-related inhibition of engagement depends on the simultaneous presence of engagement and anxiety.

(vii) Achievement contributes to a reduction in mathematics anxiety through $-\delta AX$. This term represents a feedback mechanism in which the anxiety-reducing contribution associated with achievement depends on both the existing achievement and anxiety states.

(viii) Engagement contributes to a reduction in mathematics anxiety through $-\theta EX$. This provides a second feedback mechanism linking engagement to the evolution of anxiety.

(ix) In the absence of reinforcing influences, achievement, engagement, and anxiety undergo proportional attenuation at rates μ_A, μ_E, and μ_X, respectively. These processes are represented by $-\mu_A A$, $-\mu_E E$, and $-\mu_X X$.

(x) All model parameters are nonnegative and are assumed constant over the time interval considered. The resulting formulation is therefore an autonomous nonlinear dynamical system.

(xi) The directions of interaction are theoretically motivated by the educational literature reviewed in the Introduction. However, the specific interaction functions and parameter magnitudes are modelling assumptions and should not be interpreted as causal effects estimated from longitudinal student data.

2.2. Parameter Definitions

Table 1: The Parameters Governing the System

Parameter	**Mathematical Interpretation**
ρ_A	Rate of external input supporting mathematics achievement
η_E	Rate of external input supporting student engagement
ρ_X	Rate of external input contributing to mathematics anxiety
μ_A	Proportional attenuation rate of mathematics achievement
μ_E	Proportional attenuation rate of student engagement
μ_X	Proportional attenuation rate of mathematics anxiety
β	Strength of engagement-related enhancement of achievement
γ	Strength of anxiety-related inhibition of achievement
α	Strength of anxiety-related inhibition of engagement
δ	Strength of achievement-related reduction in anxiety
θ	Strength of engagement-related reduction in anxiety

All parameters satisfy

$$\rho_A, \eta_E, \rho_X, \mu_A, \mu_E, \mu_X, \beta, \gamma, \alpha, \delta, \theta \geq 0$$

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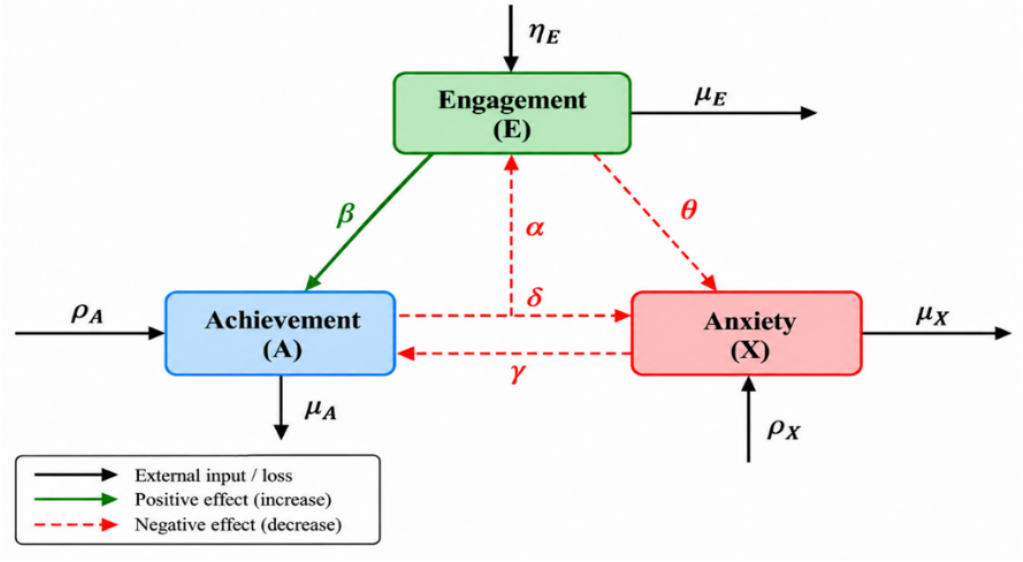

Figure 1: Schematic structure of the nonlinear achievement-engagement-anxiety model.

Note. A = normalized mathematics achievement; E = normalized student engagement; X = normalized mathematics anxiety. Positive pathways represent enhancement, whereas negative pathways represent inhibition or reduction. The pathways represent theoretical modelling assumptions rather than empirically estimated causal effects.

2.3. Derivation of the Governing Equations

Based on the assumptions stated above, the temporal evolution of each state variable is determined by the combined effects of external input, attenuation, and interactions with the other states. For mathematics achievement, the term $\rho_A(1-A)$ represents an external positive contribution, while $\beta E(1-A)$ represents the contribution associated with student engagement. The factor $(1-A)$ introduces saturation by reducing the magnitude of these positive contributions as achievement approaches its normalized upper bound. Achievement is reduced through proportional attenuation, $-\mu_A A$, and anxiety-related inhibition, $-\gamma AX$.

Student engagement receives the external positive contribution $\eta_E(1-E)$, where $(1-E)$ limits further growth as engagement approaches its upper bound. Engagement is reduced through proportional attenuation, $-\mu_E E$, and the interaction term $-\alpha EX$, which represents the inhibitory contribution associated with mathematics anxiety.

Mathematics anxiety receives the external contribution $\rho_X(1-X)$, with $(1-X)$ limiting further increase as anxiety approaches its normalized upper bound. Anxiety is reduced through proportional attenuation, $-\mu_X X$, and through the interaction terms $-\delta AX$ and $-\theta EX$, representing achievement-related and engagement-related reductions in anxiety, respectively.

The resulting nonlinear dynamical system is therefore

$$\frac{dA}{dt} = \rho_A(1 - A) + \beta E(1 - A) - \mu_A A - \gamma AX. \tag{1}$$

$$\frac{dE}{dt} = \eta_E(1 - E) - \mu_E E - \alpha EX. \tag{2}$$

$$\frac{dX}{dt} = \rho_X(1 - X) - \mu_X X - \delta AX - \theta EX. \tag{3}$$

The system is supplied with the initial conditions

$$A(0) = A_0, \qquad E(0) = E_0, \qquad X(0) = X_0, \tag{4}$$

where

$$(A_0, E_0, X_0) \in [0, 1]^3.$$

All model parameters are assumed to be nonnegative and constant over the modelled time interval. The resulting formulation constitutes a system of coupled nonlinear ordinary differential equations. Such systems permit the application of qualitative mathematical tools for examining how interacting state variables evolve over time. In this study, positive invariance and equilibrium analysis are used to establish the mathematical feasibility of the model, while Jacobian linearization and the Routh–Hurwitz criterion are employed to investigate local stability. Bifurcation analysis is further used to examine parameter-dependent changes in stability, and numerical integration is employed to investigate the temporal behavior of the system. These approaches are consistent with the broader application of differential-equation methods to the analysis of stability, asymptotic behavior, and oscillatory dynamics [18, 19]. Consequently, the system constitutes an autonomous nonlinear dynamical system of ordinary differential equations [21–23]. The nonlinear interaction terms represent theoretically specified feedback mechanisms among achievement, engagement, and mathematics anxiety rather than empirically estimated causal effects.

Figure 1 summarizes the directional structure of the proposed nonlinear model. The arrows represent the enhancement, inhibition, and reduction mechanisms linking mathematics achievement, student engagement, and mathematics anxiety. Each pathway corresponds to a term incorporated into Equations (1)-(3). Positive pathways represent enhancement, whereas negative pathways represent inhibition or reduction. The external mechanisms represent inputs to and attenuation of the corresponding states. The directions shown in the figure constitute theoretical modelling assumptions informed by the educational literature and should not be interpreted as empirically estimated causal pathways.

3. Mathematical Properties of the Model

3.1. Feasible Region and Positive Invariance

Establishing a positively invariant region is important in nonlinear dynamical-system analysis because it ensures that trajectories initiated within the admissible state space

remain mathematically meaningful under the model dynamics [24, 25]. For the present system, the feasible state space is defined by the normalized bounds imposed on achievement, engagement, and mathematics anxiety.

The mathematically feasible state space of the model is

$$\Omega = \left\{(A, E, X) \in \mathbb{R}^3 : 0 \leq A \leq 1,\ 0 \leq E \leq 1,\ 0 \leq X \leq 1\right\} = [0, 1]^3. \tag{5}$$

Lemma 1. *For nonnegative model parameters, the region $\Omega = [0, 1]^3$ is positively invariant under the flow generated by the nonlinear system (1)–(3).*

Proof. Let

$$Y(t) = \begin{pmatrix} A(t) \\ E(t) \\ X(t) \end{pmatrix}.$$

System (1)–(3) can be expressed compactly as

$$\frac{dY}{dt} = F(Y), \tag{6}$$

where

$$F(Y) = \begin{pmatrix} \rho_A(1 - A) + \beta E(1 - A) - \mu_A A - \gamma AX \\ \eta_E(1 - E) - \mu_E E - \alpha EX \\ \rho_X(1 - X) - \mu_X X - \delta AX - \theta EX \end{pmatrix}. \tag{7}$$

The boundary $\partial\Omega$ of the closed cube Ω consists of six faces:

$$\begin{aligned} P_1 &= \{(0, E, X) : 0 \leq E \leq 1,\ 0 \leq X \leq 1\}, \\ P_2 &= \{(1, E, X) : 0 \leq E \leq 1,\ 0 \leq X \leq 1\}, \\ P_3 &= \{(A, 0, X) : 0 \leq A \leq 1,\ 0 \leq X \leq 1\}, \\ P_4 &= \{(A, 1, X) : 0 \leq A \leq 1,\ 0 \leq X \leq 1\}, \\ P_5 &= \{(A, E, 0) : 0 \leq A \leq 1,\ 0 \leq E \leq 1\}, \\ P_6 &= \{(A, E, 1) : 0 \leq A \leq 1,\ 0 \leq E \leq 1\}. \end{aligned}$$

Corresponding outward normal vectors are

$$\begin{aligned} n_1 &= (-1, 0, 0)^T, \quad n_2 = (1, 0, 0)^T, \\ n_3 &= (0, -1, 0)^T, \quad n_4 = (0, 1, 0)^T, \\ n_5 &= (0, 0, -1)^T, \quad n_6 = (0, 0, 1)^T. \end{aligned}$$

It is sufficient to show that the vector field has no outward-pointing component on any face of $\partial\Omega$.

On P_1, $A = 0$. Therefore,

$$F(Y) \cdot n_1 = -\left[\rho_A + \beta E\right] \leq 0. \tag{8}$$

On P_2, $A = 1$, giving

$$F(Y) \cdot n_2 = -\mu_A - \gamma X \leq 0. \tag{9}$$

On P_3, $E = 0$, and hence

$$F(Y) \cdot n_3 = -\eta_E \leq 0. \tag{10}$$

On P_4, $E = 1$, so that

$$F(Y) \cdot n_4 = -\mu_E - \alpha X \leq 0. \tag{11}$$

On P_5, $X = 0$, yielding

$$F(Y) \cdot n_5 = -\rho_X \leq 0. \tag{12}$$

Finally, on P_6, $X = 1$, and therefore

$$F(Y) \cdot n_6 = -\mu_X - \delta A - \theta E \leq 0. \tag{13}$$

Consequently,

$$F(Y) \cdot n_i \leq 0, \qquad i = 1, \ldots, 6,$$

for every Y belonging to the corresponding boundary face P_i. Thus, the vector field is directed inward or tangent to $\partial\Omega$ at every boundary point. A solution originating in Ω therefore cannot leave the feasible region through any of its boundary faces. Hence,

$$Y(0) \in \Omega \quad \Longrightarrow \quad Y(t) \in \Omega \qquad \text{for all } t \geq 0.$$

Therefore, Ω is positively invariant under system (1)–(3).

Corollary 1. *If $(A_0, E_0, X_0) \in \Omega$, then*

$$0 \leq A(t) \leq 1, \qquad 0 \leq E(t) \leq 1, \qquad 0 \leq X(t) \leq 1$$

for all $t \geq 0$ for which the solution exists.

3.2. Existence and Uniqueness of Solutions

Theorem 1. *For every initial condition $Y(0) \in \Omega$, system (1)–(3) admits a unique solution defined for all $t \geq 0$, and this solution remains in Ω.*

Proof. The vector field $F(Y)$ defined in Equation (7) consists of polynomial combinations of A, E, and X. Consequently, F is continuously differentiable and therefore locally Lipschitz continuous with respect to Y. By the standard existence and uniqueness theory for ordinary differential equations, local Lipschitz continuity guarantees a unique local solution through every initial state $Y(0) \in \Omega$ [22, 26].

By Lemma 1, Ω is positively invariant. Hence, a solution beginning in Ω remains in the compact set Ω and cannot become unbounded in finite time. The local solution can therefore be continued for every $t \geq 0$. Thus, system (1)–(3) possesses a unique global forward solution satisfying

$$Y(t) \in \Omega \qquad \text{for all } t \geq 0.$$

3.3. Existence of a Positive Equilibrium

An equilibrium of system (1)–(3) is a state at which the rates of change of mathematics achievement, student engagement, and mathematics anxiety are simultaneously zero. Let

$$P^* = (A^*, E^*, X^*)$$

denote a positive equilibrium. Setting the right-hand sides of system (1)–(3) equal to zero gives

$$\rho_A(1 - A^*) + \beta E^*(1 - A^*) - \mu_A A^* - \gamma A^* X^* = 0, \tag{14}$$

$$\eta_E(1 - E^*) - \mu_E E^* - \alpha E^* X^* = 0, \tag{15}$$

and

$$\rho_X(1 - X^*) - \mu_X X^* - \delta A^* X^* - \theta E^* X^* = 0. \tag{16}$$

Rearranging Equation (14) gives

$$A^* = \frac{\rho_A + \beta E^*}{\rho_A + \beta E^* + \mu_A + \gamma X^*}. \tag{17}$$

Similarly, Equations (15) and (16) yield

$$E^* = \frac{\eta_E}{\eta_E + \mu_E + \alpha X^*}, \tag{18}$$

and

$$X^* = \frac{\rho_X}{\rho_X + \mu_X + \delta A^* + \theta E^*}. \tag{19}$$

Equations (17)–(19) characterize any positive equilibrium of the nonlinear achievement–engagement–anxiety system. The following theorem establishes the existence of at least one such equilibrium.

Theorem 2. *Suppose that*

$$\rho_A > 0, \qquad \eta_E > 0, \qquad \rho_X > 0,$$

and

$$\mu_A > 0, \qquad \mu_E > 0, \qquad \mu_X > 0,$$

with

$$\beta, \gamma, \alpha, \delta, \theta \geq 0.$$

Then system (1)–(3) possesses at least one equilibrium

$$P^* = (A^*, E^*, X^*) \in (0, 1)^3.$$

Proof. Define the mapping

$$T : \Omega \to \Omega$$

by

$$T(A, E, X) = \left(\frac{\rho_A + \beta E}{\rho_A + \beta E + \mu_A + \gamma X}, \frac{\eta_E}{\eta_E + \mu_E + \alpha X}, \frac{\rho_X}{\rho_X + \mu_X + \delta A + \theta E} \right). \tag{20}$$

For every $(A, E, X) \in \Omega = [0, 1]^3$, all denominators in Equation (20) are strictly positive under the stated parameter conditions. Hence, T is well defined and continuous on Ω.

Consider the first component of T,

$$T_1(A, E, X) = \frac{\rho_A + \beta E}{\rho_A + \beta E + \mu_A + \gamma X}.$$

Since $\rho_A > 0$, $\mu_A > 0$, and all other parameters and state variables are nonnegative,

$$0 < T_1(A, E, X) < 1. \tag{21}$$

Similarly,

$$T_2(A, E, X) = \frac{\eta_E}{\eta_E + \mu_E + \alpha X}.$$

Because $\eta_E > 0$ and $\mu_E > 0$,

$$0 < T_2(A, E, X) < 1. \tag{22}$$

For the third component,

$$T_3(A, E, X) = \frac{\rho_X}{\rho_X + \mu_X + \delta A + \theta E}.$$

Since $\rho_X > 0$ and $\mu_X > 0$,

$$0 < T_3(A, E, X) < 1. \tag{23}$$

Therefore,

$$T(\Omega) \subset (0, 1)^3 \subset \Omega. \tag{24}$$

The set $\Omega = [0, 1]^3$ is nonempty, compact, and convex. Since T is continuous and maps Ω into itself, Brouwer's fixed-point theorem guarantees the existence of at least one fixed point of T in Ω [27]. Hence, there exists at least one point

$$P^* = (A^*, E^*, X^*) \in \Omega$$

such that

$$T(P^*) = P^*. \tag{25}$$

Moreover, because each component of T lies strictly between 0 and 1, the fixed point satisfies

$$0 < A^* < 1, \qquad 0 < E^* < 1, \qquad 0 < X^* < 1. \tag{26}$$

Hence,

$$P^* \in (0,1)^3.$$

Therefore, system (1)–(3) possesses at least one positive equilibrium.

Theorem 2 establishes the existence of at least one positive equilibrium but does not establish uniqueness. From Equation (18), E^* is a strictly decreasing function of X^* when $\alpha > 0$, while Equations (17) and (19) introduce additional coupled dependence among A^*, E^*, and X^*. Although this monotone structure suggests that uniqueness may be established under suitable parameter conditions, a complete uniqueness proof for the full admissible parameter space is not obtained here. Accordingly, the subsequent local stability analysis applies to a specified positive equilibrium rather than asserting uniqueness or global attraction of the equilibrium.

3.4. Local Asymptotic Stability of the Positive Equilibrium

Local stability criterion. Let $P^* = (A^*, E^*, X^*)$ be a positive equilibrium of system (1)–(3). For notational convenience, define

$$p = \rho_A + \beta E^* + \mu_A + \gamma X^*,$$

$$q = \eta_E + \mu_E + \alpha X^*,$$

$$r = \rho_X + \mu_X + \delta A^* + \theta E^*,$$

together with

$$b = \beta(1 - A^*), \qquad c = \gamma A^*, \qquad d = \alpha E^*, \qquad e = \delta X^*, \qquad s = \theta X^*.$$

Applying the Routh–Hurwitz criterion to the Jacobian evaluated at P^*, the positive equilibrium is locally asymptotically stable if

$$pq + pr + qr > ce + ds, \tag{27}$$

$$pqr > pds + ceq + bde, \tag{28}$$

and

$$(p + q + r)(pq + pr + qr - ce - ds) > pqr - pds - ceq - bde. \tag{29}$$

Conditions (27)–(29) are the Routh–Hurwitz conditions expressed in terms of the equilibrium-dependent quantities defined above. Thus, they provide an operational criterion for evaluating local stability at a specified positive equilibrium rather than independent sufficient conditions stated solely in terms of the original model parameters. Deriving parameter-only conditions that guarantee stability throughout the admissible parameter space would require additional analytical arguments.

To obtain this criterion, the Jacobian matrix of system (1)–(3), evaluated at P^*, can be written as

$$J(P^*) = \begin{pmatrix} -p & b & -c \\ 0 & -q & -d \\ -e & -s & -r \end{pmatrix}. \tag{30}$$

The local behavior of the nonlinear system near the equilibrium P^* can be characterized through linearization and the eigenvalues of the Jacobian matrix evaluated at the equilibrium, provided the equilibrium is hyperbolic [28].

These eigenvalues are the roots of

$$\det\left(\lambda I - J(P^*)\right) = 0.$$

Expanding the determinant gives the cubic characteristic polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{31}$$

where

$$a_1 = p + q + r, \tag{32}$$

$$a_2 = pq + pr + qr - ce - ds, \tag{33}$$

and

$$a_3 = pqr - pds - ceq - bde. \tag{34}$$

The Routh–Hurwitz criterion provides algebraic conditions under which all roots of a characteristic polynomial have negative real parts, thereby permitting local stability to be assessed without explicitly solving for all eigenvalues [28].

For a cubic polynomial of the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

the Routh–Hurwitz conditions for all roots to have strictly negative real parts are

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_1a_2 > a_3. \tag{35}$$

Because all model parameters are nonnegative and the positive equilibrium satisfies

$$A^* > 0, \qquad E^* > 0, \qquad X^* > 0,$$

we have

$$p > 0, \qquad q > 0, \qquad r > 0.$$

Consequently,

$$a_1 = p + q + r > 0. \tag{36}$$

Condition (27) implies

$$a_2 = pq + pr + qr - ce - ds > 0.$$

Condition (28) implies

$$a_3 = pqr - pds - ceq - bde > 0.$$

Finally, Condition (29) is equivalent to

$$a_1 a_2 > a_3.$$

Thus, all four Routh–Hurwitz conditions are satisfied. Therefore, every eigenvalue of $J(P^*)$ has a strictly negative real part. It follows that the positive equilibrium P^* is locally asymptotically stable.

The local stability result should not be interpreted as a global stability result. In particular, the present analysis does not establish that every admissible initial condition converges to a unique positive equilibrium. Establishing uniqueness and global asymptotic stability through an appropriate monotonicity, Lyapunov, or related analytical argument remains an important extension of the model.

In the context of the model, local asymptotic stability means that sufficiently small perturbations of achievement, engagement, or mathematics anxiety around P^* decay with time, causing the theoretical system to return toward the equilibrium state. This interpretation concerns the behavior of the mathematical model under its specified parameter configuration and should not be interpreted as evidence that individual students necessarily follow the same temporal trajectory.

4. Local Stability Boundary Analysis with Respect to β

The local stability analysis in this section is restricted to variation in the engagement–achievement interaction parameter β, with the remaining model parameters held at their baseline values. Accordingly, the analysis is intended to determine whether variation in β alone causes the positive equilibrium to approach or cross a local stability boundary within the investigated interval. It should not be interpreted as a general bifurcation analysis over the complete parameter space. In particular, stability could in principle be lost through violation of the Routh–Hurwitz conditions $a_2 > 0$, $a_3 > 0$, or $a_1 a_2 > a_3$ [25, 29]. These conditions therefore provide the relevant stability boundaries for interpreting the β-based analysis. In the present model, the engagement-to-achievement interaction parameter

β is selected as the bifurcation parameter because it controls the strength with which engagement contributes to mathematics achievement. The equilibrium and the coefficients of the characteristic polynomial are therefore regarded as functions of β.

At the positive equilibrium $P^* = (A^*, E^*, X^*)$, the characteristic equation obtained in Section 3 is

$$\lambda^3 + a_1(\beta)\lambda^2 + a_2(\beta)\lambda + a_3(\beta) = 0. \tag{37}$$

According to the Routh–Hurwitz criterion, the equilibrium is locally asymptotically stable when

$$a_1(\beta) > 0, \qquad a_2(\beta) > 0, \qquad a_3(\beta) > 0,$$

and

$$a_1(\beta)a_2(\beta) > a_3(\beta). \tag{38}$$

Define

$$H(\beta) = a_1(\beta)a_2(\beta) - a_3(\beta). \tag{39}$$

Because loss of local stability is not restricted to the Hopf boundary $H(\beta) = 0$, the quantities $a_2(\beta)$ and $a_3(\beta)$ are also relevant. In particular, $a_3(\beta) = 0$ represents another possible stability boundary and must be distinguished from a Hopf-type crossing. Consequently, the absence of a zero of $H(\beta)$ alone is insufficient to establish stability unless the remaining Routh–Hurwitz conditions also remain positive throughout the investigated interval.

A potential Hopf bifurcation occurs at a critical value $\beta = \beta_c$ when

$$H(\beta_c) = 0, \tag{40}$$

while

$$a_1(\beta_c) > 0, \qquad a_2(\beta_c) > 0, \qquad a_3(\beta_c) > 0. \tag{41}$$

At the critical point,

$$a_3(\beta_c) = a_1(\beta_c)a_2(\beta_c),$$

and the characteristic polynomial can be written as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_1a_2 = (\lambda + a_1)(\lambda^2 + a_2) = 0. \tag{42}$$

Hence, the eigenvalues at the critical value are

$$\lambda_1 = -a_1(\beta_c), \qquad \lambda_{2,3} = \pm i\sqrt{a_2(\beta_c)}. \tag{43}$$

Thus, the critical angular frequency associated with the imaginary eigenvalue pair is

$$\omega_c = \sqrt{a_2(\beta_c)}. \tag{44}$$

The presence of purely imaginary eigenvalues alone is not sufficient to establish a Hopf bifurcation. A transversality condition must also be satisfied so that the conjugate eigenvalues cross the imaginary axis with nonzero speed as β passes through β_c [22, 29, 30]. Accordingly, the required condition is

$$\left.\frac{d\ \mathrm{Re}\,\lambda(\beta)}{d\beta}\right|_{\beta=\beta_c} \neq 0. \tag{45}$$

For the Routh–Hurwitz boundary considered here, the condition

$$H'(\beta_c) \neq 0 \tag{46}$$

provides a convenient numerical check that the stability boundary is crossed nondegenerately, provided the remaining Routh–Hurwitz conditions remain positive near β_c. A Hopf bifurcation additionally requires verification of the eigenvalue transversality condition in Equation (45).

Therefore, a candidate Hopf point is identified by Equations (40), (41), and (46), while confirmation of a Hopf bifurcation requires the transversality condition in Equation (45). If these conditions are verified, changes in the strength of the engagement–achievement interaction can cause the positive equilibrium to lose or gain local stability and may generate oscillatory dynamics in achievement, engagement, and mathematics anxiety.

Because A^*, E^*, and X^* themselves depend on β, the critical value β_c is determined from the coupled equilibrium relations

$$\begin{aligned} A^* &= \frac{\rho_A + \beta E^*}{\rho_A + \beta E^* + \mu_A + \gamma X^*}, \\ E^* &= \frac{\eta_E}{\eta_E + \mu_E + \alpha X^*}, \\ X^* &= \frac{\rho_X}{\rho_X + \mu_X + \delta A^* + \theta E^*}, \end{aligned} \tag{47}$$

together with the critical condition

$$a_1(\beta_c)a_2(\beta_c) - a_3(\beta_c) = 0. \tag{48}$$

These equations can be solved numerically for specified parameter values. The resulting equilibrium branches and eigenvalues can then be tracked as β varies to determine whether the theoretical conditions for a Hopf bifurcation are actually attained.

Over the investigated interval $0 \leq \beta \leq 2$, the computed positive equilibrium remained locally asymptotically stable under the fixed baseline values of the remaining parameters. No loss of stability through the investigated Routh–Hurwitz boundaries was observed, and no candidate Hopf point was detected. This result is restricted to the one-parameter continuation in β and does not establish unconditional stability of the positive equilibrium over the full parameter space.

The present one-parameter analysis leaves open the stronger mathematical question of whether instability can occur for other admissible parameter combinations. Establishing unconditional local or global stability would require an analytical argument over the complete parameter space, while a broader numerical assessment would require simultaneous variation of the remaining parameters. These questions are beyond the scope of the present β-focused analysis and constitute important directions for further investigation.

5. Parameter Selection and Preliminary Stability Analysis

5.1. Selection and Justification of Baseline Parameter Values

The numerical values used in the model were specified as a nominal reference configuration rather than estimated from an empirical dataset. This approach was necessary because the educational literature provides evidence regarding the direction and theoretical plausibility of the relationships among mathematics achievement, student engagement, and mathematics anxiety, but does not provide continuous-time rate parameters that correspond directly to the terms in the proposed differential-equation system. Consequently, correlation coefficients, regression coefficients, and structural-equation path coefficients reported in previous educational studies were not treated as direct estimates of the model parameters.

Parameter selection was guided by mathematical admissibility, theoretical consistency, and dynamical nondegeneracy. First, all parameter values were required to satisfy the nonnegativity assumptions of the model. Second, the selected configuration was required to generate an equilibrium strictly within the normalized feasible region $\Omega = [0, 1]^3$. Third, the selected parameter configuration was examined to ensure that it produced a mathematically admissible interior equilibrium and nondegenerate system dynamics. Stability was not treated as an independently discovered property of the baseline configuration; rather, the subsequent eigenvalue and Routh–Hurwitz calculations provide a consistency check showing that the selected reference configuration lies within a locally stable regime. Fourth, the parameter magnitudes were selected so that the external-input, attenuation, enhancement, and inhibition mechanisms all contributed meaningfully to the system dynamics, thereby avoiding a reference configuration dominated by a single mechanism. Fifth, the resulting numerical trajectories were required to remain within the feasible region and exhibit nontrivial transient dynamics rather than being forced immediately toward the boundaries.

These criteria were used to construct a mathematically admissible benchmark from which the qualitative behavior of the proposed system could be investigated. The baseline values should therefore not be interpreted as empirically established rates or effect sizes for a particular student population. Rather, they provide a transparent reference configuration for examining equilibrium behavior, local stability, numerical trajectories, and parameter-dependent changes. Because multiple parameter combinations may satisfy the stated admissibility conditions, the baseline parameter vector is not assumed to be

unique. The selected parameter vector was therefore

$$\Theta_0 = (\rho_A, \eta_E, \rho_X, \mu_A, \mu_E, \mu_X, \beta, \gamma, \alpha, \delta, \theta)$$

with

$$\Theta_0 = (0.15, 0.20, 0.10, 0.08, 0.10, 0.15, 0.40, 0.30, 0.25, 0.20, 0.25). \tag{49}$$

The choice $\rho_A = 0.15$ should therefore not be interpreted as implying that 0.15 is an empirically established rate of external contribution to mathematics achievement. Rather, 0.15 is the nominal value assigned to this parameter in the reference configuration. Its suitability is assessed by the mathematical behavior it generates jointly with the remaining parameters. At the selected parameter vector, the model produces the interior equilibrium

$$P^* = (A^*, E^*, X^*) = (0.73795, 0.57783, 0.18449), \tag{50}$$

which satisfies

$$0 < A^* < 1, \qquad 0 < E^* < 1, \qquad 0 < X^* < 1. \tag{51}$$

Thus, the reference configuration does not force any equilibrium state outside its theoretically admissible normalized range.

The same principle applies to $\eta_E = 0.20$ and $\rho_X = 0.10$. These values define the baseline external contributions to engagement and anxiety, respectively. They were selected as components of the reference parameter vector rather than as independently estimated educational effect sizes. Their mathematical adequacy is evaluated jointly because equilibrium values in a coupled nonlinear system depend on combinations of parameters rather than on a single parameter in isolation.

The attenuation parameters

$$\mu_A = 0.08, \qquad \mu_E = 0.10, \qquad \mu_X = 0.15 \tag{52}$$

were similarly specified as nominal rates controlling the proportional reduction of the respective states. Their values were kept below the strongest interaction coefficients so that the baseline system retained meaningful contributions from both the direct attenuation mechanisms and the nonlinear interaction mechanisms. These values therefore define the time scale and relative strength of the attenuation processes within the reference system rather than empirically measured rates of decline.

For the interaction structure, the baseline values were specified as

$$\beta = 0.40, \qquad \gamma = 0.30, \qquad \alpha = 0.25, \qquad \delta = 0.20, \qquad \theta = 0.25. \tag{53}$$

The value $\beta = 0.40$ was selected to make the engagement-related enhancement of achievement a prominent interaction in the reference configuration because this pathway is the principal continuation mechanism examined in the subsequent bifurcation analysis. The value $\gamma = 0.30$ represents a substantial but smaller anxiety-related inhibitory

contribution to achievement, while $\alpha = 0.25$ specifies the corresponding anxiety-related inhibition of engagement. The feedback parameters $\delta = 0.20$ and $\theta = 0.25$ allow both achievement and engagement to contribute to reductions in anxiety without making either feedback pathway dominant over the engagement–achievement interaction. Hence, the interaction coefficients satisfy the reference ordering

$$\beta > \gamma > \alpha = \theta > \delta. \tag{54}$$

This ordering is a modelling specification used to construct the baseline dynamical regime; it is not presented as an empirically established ordering of educational effect sizes.

The mathematical admissibility of the complete baseline vector was additionally examined through the stability conditions. At Θ_0, the Jacobian evaluated at the equilibrium is

$$J(P^*) = \begin{pmatrix} -0.51648 & 0.10482 & -0.22138 \\ 0 & -0.34612 & -0.14446 \\ -0.03690 & -0.04612 & -0.54205 \end{pmatrix}, \tag{55}$$

with eigenvalues

$$\lambda_1 = -0.62355, \qquad \lambda_2 = -0.48043, \qquad \lambda_3 = -0.30067. \tag{56}$$

All three eigenvalues have negative real parts. Furthermore, the characteristic-polynomial coefficients are

$$a_1 = 1.40465, \qquad a_2 = 0.63150, \qquad a_3 = 0.09007, \tag{57}$$

and

$$a_1 a_2 - a_3 = 0.79697 > 0. \tag{58}$$

Therefore,

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0, \qquad a_1 a_2 > a_3. \tag{59}$$

The baseline vector satisfies the Routh–Hurwitz conditions for local asymptotic stability. Because stability formed part of the construction and assessment of the nominal reference configuration, this result should be interpreted as a consistency check rather than as an independent finding of the numerical analysis. The calculation confirms that the selected baseline provides an appropriate stable reference regime for examining the subsequent dynamics and parameter variation.

The baseline parameter vector is not claimed to be unique. Other parameter combinations may also generate admissible and stable equilibria. In the absence of longitudinal data for empirical calibration, the selected values should therefore be interpreted as a nominal reference configuration for investigating the qualitative dynamics of the model. The robustness of the principal dynamical conclusions is examined subsequently through

parameter variation, with β selected as the continuation parameter because it directly controls the engagement-related contribution to achievement.

Table 2: Baseline Parameter Values and Their Basis in the Reference Configuration

Parameter	**Description**	**Baseline value**	**Basis for baseline specification**
ρ_A	External input supporting mathematics achievement	0.15	Nominal achievement-input rate used in the admissible reference configuration
η_E	External input supporting student engagement	0.20	Nominal engagement-input rate used in the reference configuration
ρ_X	External input contributing to mathematics anxiety	0.10	Nominal anxiety-input rate used in the reference configuration
μ_A	Attenuation rate of mathematics achievement	0.08	Nominal achievement-attenuation rate
μ_E	Attenuation rate of student engagement	0.10	Nominal engagement-attenuation rate
μ_X	Attenuation rate of mathematics anxiety	0.15	Nominal anxiety-attenuation rate
β	Engagement-related enhancement of achievement	0.40	Principal positive interaction and continuation parameter
γ	Anxiety-related inhibition of achievement	0.30	Substantial inhibitory interaction below the baseline β
α	Anxiety-related inhibition of engagement	0.25	Intermediate inhibitory interaction
δ	Achievement-related reduction in anxiety	0.20	Moderate achievement–anxiety feedback
θ	Engagement-related reduction in anxiety	0.25	Intermediate engagement–anxiety feedback

Note. The parameter values are dimensionless nominal values defining the baseline reference configuration. They were not estimated from longitudinal student data and should not be interpreted as empirical effect sizes or uniquely determined parameter values.

6. Numerical Method

System (1)–(3) was solved numerically in GNU Octave using the `ode45` solver, which implements an adaptive explicit Runge–Kutta approach for the numerical solution of ordinary differential equations. Adaptive Runge–Kutta methods control numerical error by adjusting the integration step in response to prescribed error tolerances, making them suitable for the numerical investigation of nonstiff initial-value problems [31, 32]. Accordingly, unlike a fixed-step classical fourth-order Runge–Kutta scheme, `ode45` automatically adjusts its internal integration step according to the specified error tolerances. The relative and absolute error tolerances were set to 10^{-8} and 10^{-10}, respectively, to obtain high numerical accuracy.

The numerical simulations were conducted over the dimensionless time interval

$$0 \leq t \leq 100$$

using the baseline initial conditions

$$A(0) = 0.30, \qquad E(0) = 0.40, \qquad X(0) = 0.60.$$

Solutions were evaluated at 1,001 equally spaced output points over the simulation interval. These output points correspond to a spacing of 0.1 dimensionless time units; however, this spacing should not be interpreted as the internal integration step because `ode45` determines its integration steps adaptively. The baseline parameter values reported in Table 2 were used unless otherwise specified. Because the model parameters were not calibrated using longitudinal data, the dimensionless time variable cannot presently be mapped directly onto days, weeks, school terms, or other educational time scales. The simulation interval is therefore used to examine the qualitative behavior of the theoretical system rather than to represent a specific duration of student learning.

The numerical analysis examined baseline temporal dynamics, convergence from different admissible initial conditions, three-dimensional phase-space behavior, equilibrium sensitivity to the engagement–achievement interaction parameter β, variation in the Routh–Hurwitz stability function, and parameter–time response surfaces. Agreement between the simulated long-run states and the numerically computed equilibrium was also examined as a numerical consistency check on the analytical stability results. The parameter analysis is restricted to β and should therefore be regarded as a one-parameter sensitivity analysis rather than a comprehensive assessment of parameter uncertainty. A full parameter sweep or global sensitivity analysis would be required to evaluate the robustness of the model across the complete parameter space.

7. Numerical Results and Discussion

7.1. Baseline Dynamics

The numerical solution of the system was first obtained using the baseline parameter values in Table 2 and the initial condition

$$(A_0, E_0, X_0) = (0.40, 0.50, 0.45).$$

The trajectories showed that mathematics achievement increased with time, mathematics anxiety decreased, and student engagement exhibited a slight initial decline before increasing toward its equilibrium level. Selected numerical values are presented in Table 3.

As t increased, the numerical solution approached

$$(A(t), E(t), X(t)) \longrightarrow (0.73795, 0.57783, 0.18449) \qquad \text{as } t \to \infty. \tag{60}$$

This limiting state agrees with the positive equilibrium P^* computed numerically from the equilibrium equations. The numerical convergence is consistent with the locally stable baseline regime established during parameter selection and provides a numerical consistency check rather than an independent stability finding. Under the baseline configuration,

Table 3: Selected State Values Along the Baseline Numerical Trajectory

t	$A(t)$	$E(t)$	$X(t)$
0	0.4000	0.5000	0.4500
2	0.5673	0.5074	0.3028
5	0.6735	0.5379	0.2194
10	0.7255	0.5669	0.1898
20	0.7374	0.5772	0.1847
40	0.7379	0.5778	0.1845
100	0.7379	0.5778	0.1845

perturbations from the equilibrium diminish with time rather than generating sustained oscillations or divergent trajectories.

Figure 2 illustrates the temporal evolution of mathematics achievement, student engagement, and mathematics anxiety under the baseline parameter configuration. Achievement increased rapidly from its initial value of 0.40 and approached an equilibrium value of approximately 0.738, while engagement exhibited a small initial decline from 0.50 before increasing and converging to approximately 0.578. In contrast, mathematics anxiety declined from 0.45 to approximately 0.184. The three trajectories became effectively stationary as time increased, indicating convergence toward the positive equilibrium $P^* = (0.73795, 0.57783, 0.18449)$. This numerical behavior is consistent with the negative eigenvalues and the Routh–Hurwitz stability conditions obtained analytically.

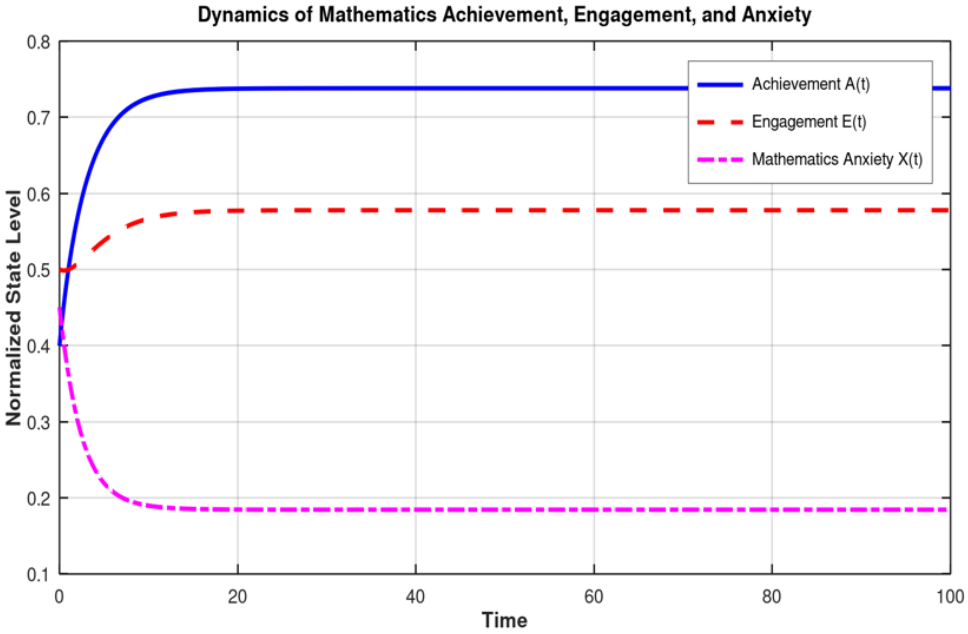

Figure 2: Time evolution of mathematics achievement A(t), student engagement E(t), and mathematics anxiety X(t) under the baseline parameter configuration.

The simulated pattern is consistent with the broader educational literature on the relationships among mathematics achievement, student engagement, and mathematics anxiety. Student engagement has been associated positively with mathematics achievement because greater behavioral, emotional, and cognitive involvement can support participation, persistence, and productive involvement in learning activities [1, 4]. In contrast, mathematics anxiety has consistently shown a negative association with mathematics performance [2, 9, 20]. The simultaneous increase in achievement and engagement and decline in anxiety observed in Figure 2 therefore represents an educationally plausible pattern within the proposed model.

The result also reflects the feedback structure of the system: engagement contributes positively to achievement through $\beta E(1-A)$, while anxiety inhibits achievement and engagement through $-\gamma AX$ and $-\alpha EX$. Achievement and engagement, in turn, contribute to anxiety reduction through $-\delta AX$ and $-\theta EX$. As anxiety declines, its inhibitory influence becomes weaker, allowing achievement and engagement to approach their stable equilibrium levels. These numerical trajectories should, however, be interpreted as theoretical consequences of the specified model and parameter configuration rather than as empirically estimated changes in students over time.

7.2. Convergence from Different Initial Conditions

To further examine the convergence behavior of the system, numerical simulations were conducted using different initial conditions within the feasible region $\Omega = [0,1]^3$. The initial states considered were

$$(A_0, E_0, X_0) = (0.10, 0.20, 0.80), \quad (0.80, 0.20, 0.20), \quad (0.20, 0.80, 0.60), \quad (0.70, 0.70, 0.70). \tag{61}$$

Despite the differences in starting values, all simulated trajectories converged numerically to

$$P^* = (0.73795, 0.57783, 0.18449). \tag{62}$$

This numerical convergence indicates that the equilibrium attracts trajectories from the tested initial conditions. These simulations provide evidence of attraction over the investigated initial states, although they should not be interpreted as an analytical proof of global asymptotic stability.

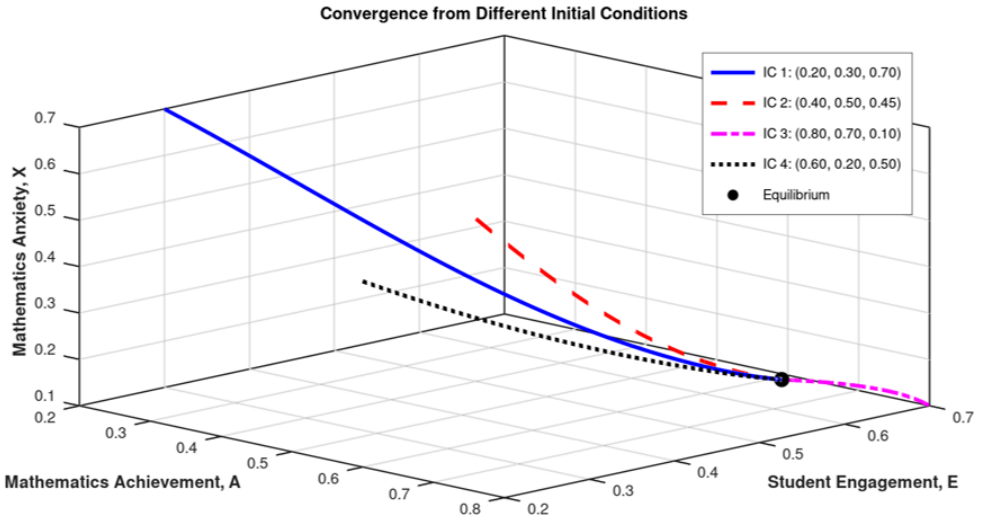

Figure 3: Convergence of the achievement-engagement-anxiety system from different initial conditions.

The convergence observed from the tested initial conditions is primarily a consequence of the specified model structure and common parameter configuration. It should therefore not be interpreted as evidence that students with different initial levels of achievement, engagement, and mathematics anxiety will necessarily approach the same long-term state. Longitudinal research indicates that individual differences and earlier achievement-related experiences may persist and influence subsequent outcomes [3, 9, 13]. The present model does not incorporate student-specific parameters, persistent heterogeneity, or time-varying external influences that could maintain such differences. Accordingly, Figure 3 demonstrates convergence for the selected initial conditions under a shared parameter configuration rather than an empirical claim about long-term student development.

7.3. Phase-Space Behavior

The interaction among the three state variables was further examined in the three-dimensional phase space (A, E, X). Beginning from $(A_0, E_0, X_0) = (0.40, 0.50, 0.45)$, the trajectory moved progressively toward the positive equilibrium

$$P^* = (0.73795, 0.57783, 0.18449).$$

The trajectory approached the equilibrium without sustained oscillatory behavior. This result is consistent with the three negative real eigenvalues obtained at the baseline equilibrium,

$$\lambda_1 = -0.62355, \qquad \lambda_2 = -0.48043, \qquad \lambda_3 = -0.30067.$$

Phase-space representations are particularly useful in nonlinear dynamical-system analysis because they display the joint evolution of state variables and permit trajectories, equilibria, and qualitative convergence behavior to be examined geometrically [21, 22].

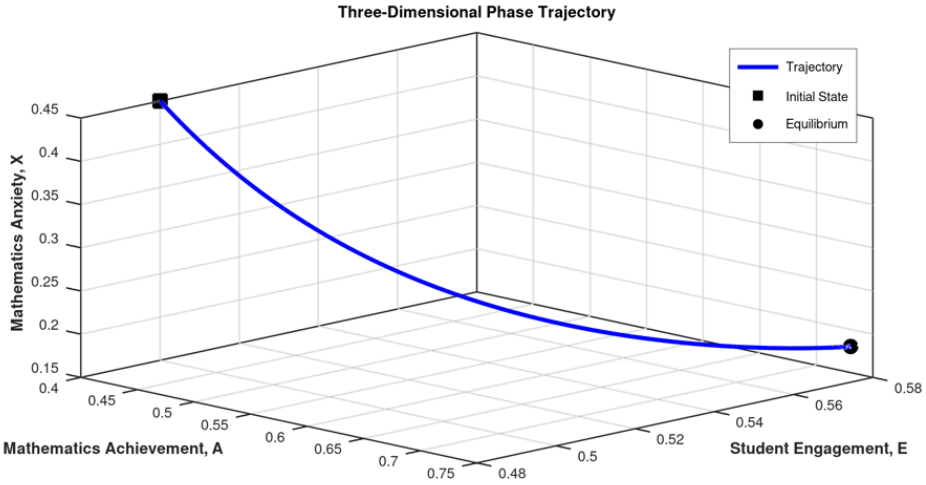

Figure 4: Three-dimensional phase trajectory of A(t), E(t), and X(t)approaching the positive equilibrium.

Educationally, the trajectory illustrates that achievement, engagement, and mathematics anxiety evolve as interconnected rather than independent states. Fredricks et al. [1] emphasized the role of engagement in students' learning experiences, while Quintero et al. [33] showed that engagement and anxiety can jointly contribute to mathematics achievement. Similarly, Barroso et al. [20] documented the negative association between mathematics anxiety and achievement. The model translates these interrelationships into nonlinear feedback, with changes in one state influencing the evolution of others. Convergence to a single equilibrium indicates stabilization under the baseline configuration, although the simulated trajectory should not be interpreted as an empirical developmental pathway for individual students.

7.4. Effect of the Engagement–Achievement Interaction

The parameter β was varied from 0 to 2 to examine how the strength of the engagement–achievement interaction influences the equilibrium states. Increasing β produced a marked increase in equilibrium achievement. For example,

$$A^* = 0.5164 \qquad \text{at } \beta = 0,$$

whereas

$$A^* = 0.9088 \qquad \text{at } \beta = 2.$$

Over the same interval, equilibrium engagement increased slightly, while equilibrium mathematics anxiety decreased. Thus, within the specified model and parameter configuration, stronger engagement-related enhancement of achievement was associated with higher equilibrium achievement and modestly lower equilibrium anxiety.

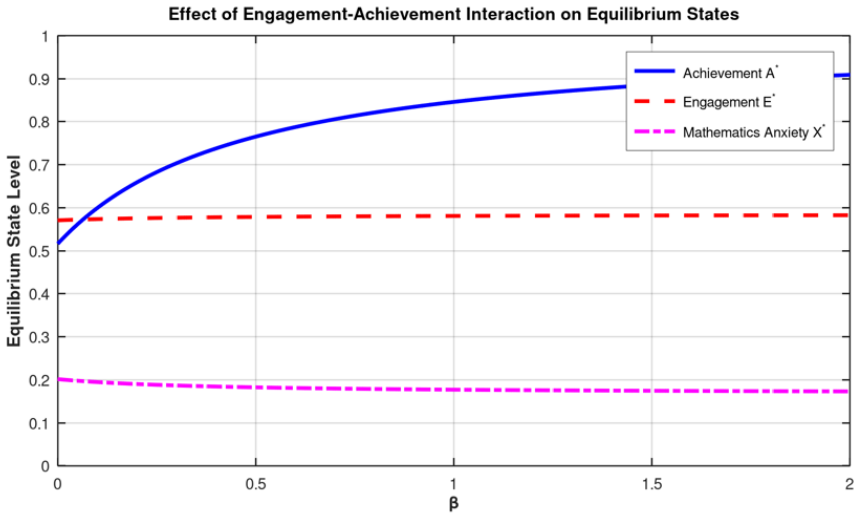

Figure 5: Equilibrium values of mathematics achievement, student engagement, and mathematics anxiety as functions of the engagement–achievement parameter β

Figure 5 illustrates the sensitivity of the equilibrium states to the engagement–achievement interaction parameter β. As β increased from 0 to 2, equilibrium achievement A^* increased from approximately 0.516 to 0.909, although the increase became smaller at higher values of β. This diminishing response reflects the bounded term $(1 - A)$, which limits further gains as achievement approaches its upper bound. Equilibrium engagement E^* increased only slightly from 0.571 to 0.583, while mathematics anxiety X^* decreased from 0.202 to 0.173. Thus, increasing β produced its largest change in equilibrium achievement, with smaller indirect changes in engagement and anxiety.

This pattern follows partly from the mathematical structure of the model because β enters directly into the achievement equation through $\beta E(1 - A)$, whereas its effects on engagement and anxiety occur indirectly through the coupled feedback system. The result should therefore be interpreted as a structural property of the specified model rather than independent empirical evidence that engagement has a stronger effect on achievement than on the other states. The positive direction of the engagement–achievement relationship was specified from the literature when constructing the model [1, 4], while the numerical analysis shows how this assumed interaction propagates through the coupled system.

7.5. Stability and Bifurcation Behavior Across β

The local stability of the positive equilibrium was further examined as the engagement–achievement interaction parameter β varied over the interval $0 \leq \beta \leq 2$. For each value of β, the equilibrium was recomputed and the corresponding Routh–Hurwitz quantities were evaluated. Particular attention was given to the stability function

$$H(\beta) = a_1(\beta)a_2(\beta) - a_3(\beta),$$

because $H(\beta) = 0$, together with the positivity of the remaining Routh–Hurwitz quantities, defines the candidate stability boundary considered in the bifurcation analysis.

Figure 6 shows that $H(\beta)$ remained positive throughout the investigated interval, increasing from approximately 0.4156 at $\beta = 0$ to 3.3134 at $\beta = 2$. At the baseline value $\beta = 0.40$,

$$H(0.40) = 0.79697 > 0.$$

Thus, the candidate boundary $H(\beta) = 0$ was not encountered within the investigated range. Together with satisfaction of the remaining Routh–Hurwitz conditions, this indicates that the positive equilibrium remained locally asymptotically stable and that no Hopf bifurcation with respect to β was detected for $0 \leq \beta \leq 2$ under the baseline values of the remaining parameters.

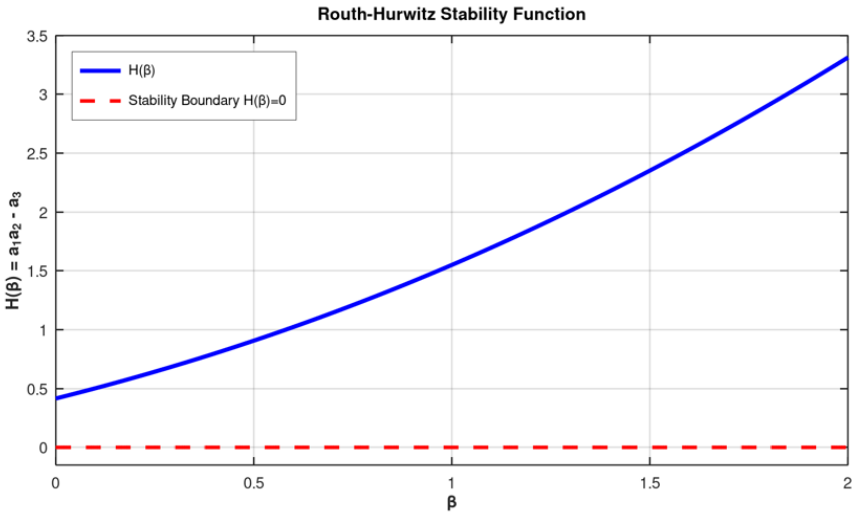

Figure 6: Variation of the Routh–Hurwitz stability function $H(\beta)$ with the engagement–achievement interaction parameter β. The dashed horizontal line represents the stability boundary $H(\beta) = 0$.

The stability result complements the equilibrium sensitivity analysis in Figure 5. Within the specified model, increasing the engagement–achievement interaction is associated with a higher equilibrium achievement state while the equilibrium remains locally asymptotically stable throughout the investigated interval. This behavior reflects the positive engagement–achievement pathway specified in the model and illustrates its consequences within the coupled dynamical system. Because the direction of this relationship was incorporated during model construction, the numerical result should not be regarded as independent confirmation of the empirical relationship reported by Fung et al. [4].

However, β is a theoretical coupling parameter rather than an empirically estimated intervention effect. Accordingly, the result does not imply that increasing students' engagement by a specified amount will produce the numerical changes in achievement generated by the model. Moreover, the absence of a detected Hopf bifurcation is specific to β and the selected values of the remaining parameters; bifurcation behavior under other parameter

configurations is not ruled out.

7.6. Parameter–Time Response Surfaces

To complement the equilibrium analysis, parameter–time response surfaces were generated to examine how variation in β influences the transient as well as long-run behavior of the three state variables. Unlike Figure 5, which summarizes only equilibrium values, the response surfaces show how differences associated with β emerge during the temporal evolution of the system.

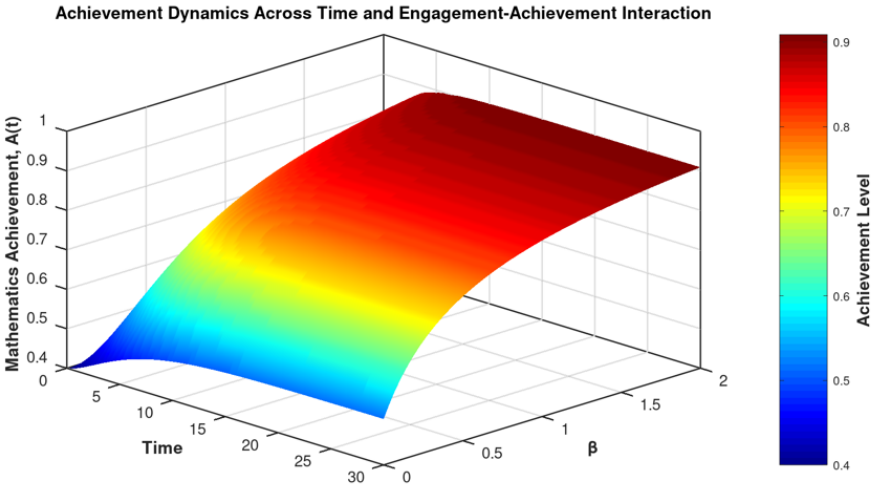

Figure 7: Three-dimensional response surface showing the evolution of mathematics achievement $A(t)$across time and the engagement–achievement interaction parameter β. The color scale represents the normalized achievement level

Figure 7 shows the joint effects of time and β on mathematics achievement. From the common initial value $A(0) = 0.40$, achievement increases toward a β-dependent equilibrium, with larger β producing higher trajectories and long-term achievement. The diminishing response at higher levels reflects the bounded term $(1 - A)$. Educationally, this pattern accords with Fung et al. [4], who reported a positive relationship between student engagement and mathematics achievement. Fredricks et al. [1] similarly emphasized engagement as an important component of productive learning. Within the model, β represents the strength of this engagement-related contribution rather than an empirically estimated effect.

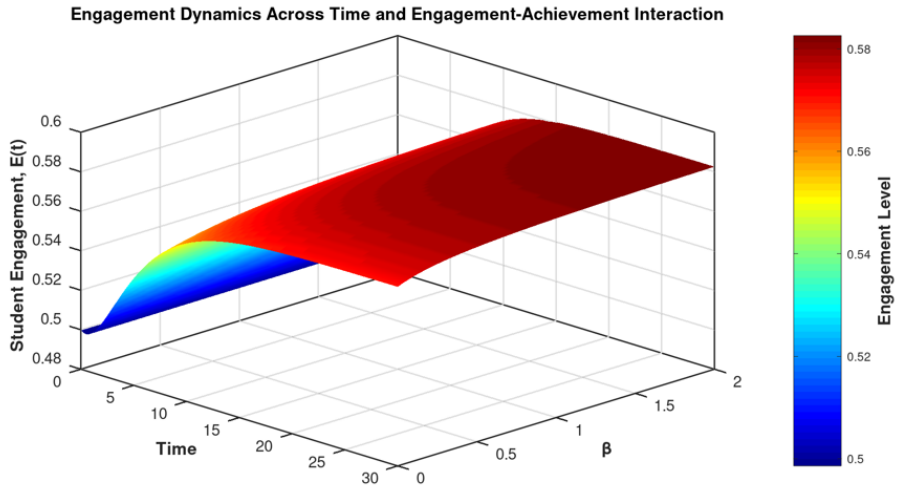

Figure 8: Three-dimensional response surface of student engagement $E(t)$ across time and the engagement-achievement interaction parameter β. The color scale represents the normalized engagement level.

Figure 8 shows that engagement exhibits a slight initial decline from $E(0) = 0.50$ before increasing and converging toward equilibrium, with only modest variation across β. Equilibrium engagement remains approximately 0.571–0.583 because β does not enter the engagement equation directly; its influence occurs indirectly through the coupled system. Quintero et al. [33] likewise highlighted the interconnected roles of engagement, anxiety, and achievement in mathematics learning. The comparatively small change in E^* therefore illustrates an important feature of the model: strengthening the engagement–achievement interaction primarily affects achievement rather than engagement itself.

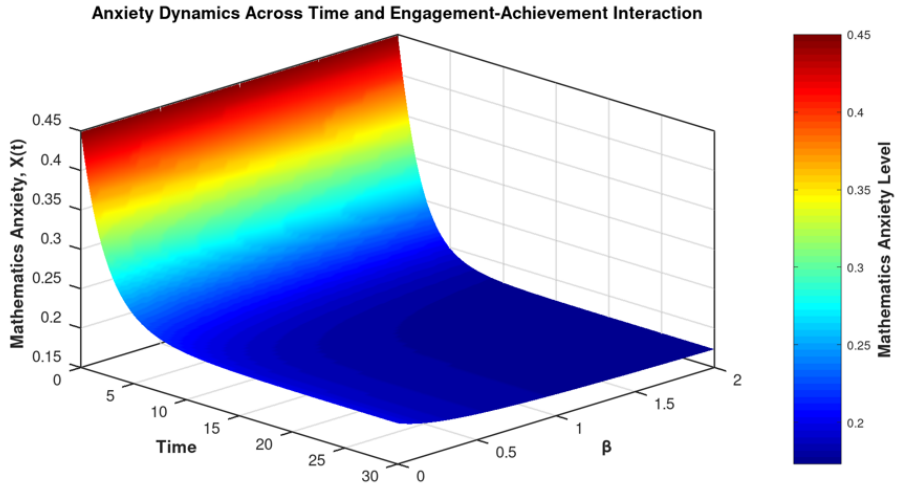

Figure 9: Three-dimensional response surface of mathematics anxiety $X(t)$ across time and the engagement-achievement interaction parameter β. The color scale represents the normalized mathematics anxiety level.

Figure 9 shows mathematics anxiety declining from $X(0) = 0.45$ toward a stable equilibrium, with X^* decreasing from approximately 0.202 to 0.173 as β increases. This effect is indirect because β does not appear in the anxiety equation. Instead, stronger engagement-related achievement enhancement raises achievement, which contributes to anxiety reduction through the term $-\delta AX$. The direction of this relationship is compatible with the negative association between mathematics anxiety and performance documented by Barroso et al. [20] and Namkung et al. [9]. Sorvo et al. [13] also found that earlier arithmetic achievement predicted later anxiety about failure, although anxiety did not predict later achievement in their sample. Thus, Figure 9 illustrates a theoretically plausible indirect pathway rather than an empirically established causal effect of β on anxiety.

8. Conclusion

This study developed a nonlinear dynamical model describing the interactions among mathematics achievement, student engagement, and mathematics anxiety. The model established a positively invariant feasible region, the existence of a positive equilibrium, and conditions for local asymptotic stability. Under the baseline parameter configuration, the positive equilibrium was locally asymptotically stable, with numerical trajectories showing increasing achievement and engagement alongside declining mathematics anxiety. Sensitivity analysis further showed that changes in the engagement–achievement interaction parameter β substantially influenced equilibrium achievement, while the Routh–Hurwitz stability conditions remained satisfied over the investigated range of β, with no Hopf bifurcation detected.

The principal contribution of the study is the representation of achievement, engagement, and mathematics anxiety as a coupled nonlinear feedback system rather than as isolated educational relationships. The findings describe the behavior of the proposed theoretical system and should not be interpreted as empirical predictions of individual students. Future research should estimate the model parameters using longitudinal or intensive repeated-measures data and examine the model across different educational contexts. Further mathematical investigations could consider global stability, broader sensitivity and uncertainty analyses, alternative bifurcation parameters, time delays, stochastic effects, and additional educational variables. These extensions could strengthen the connection between the theoretical model and observed dynamics in mathematics learning.

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