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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Hankel's Transform and Riemann's Hypothesis

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Abstract. A necessary and sufficient condition for validity of Rieman's hypothesis is given in terms of the growth of Hankel's transform of a function closely related to the classical ζ-function.

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1. Expansion of Holomorphic Functions in Series of the Polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$

It is well-known that the region of convergence of a series in Laguerre's polynomial $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}, \alpha > -1$ is, in general, the interior $\Delta(\lambda_0)$ of the parabola with equation $\Re(-z)^{1/2} = \lambda_0, 0 < \lambda_0 \leq \infty$, [11, 9.2., (5)]. A corollary of this fact is that the region of convergence of a series of the kind

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z^2), \quad \alpha > -1 \tag{1}$$

is a strip $S(\lambda_0)$ defined by the inequality $|\Im z| < \lambda_0$ [5, 1. Introduction].

Denote by $\mathscr{P}^{(\alpha)}(\lambda_0), 0 < \lambda_0 \leq \infty, \alpha > -1$ the $\mathbb{C}$-vector space of the even complex functions holomorphic in the strip $S(\lambda_0)$ and having there a representation by a series of the kind (1).

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The space $\mathscr{P}^{(0)}(\lambda_0)$ or, more precisely, the growth of the functions in it is completely characterized first by H. Pollard [5, Theorem A] by means of the function $\eta(\lambda; x, y) = \exp\{x^2/2 - |x|(\lambda^2 - y^2)^{1/2}\}$, $0 \leq \lambda < \infty$, $x + iy \in \overline{S(\lambda)}$, $S(0) := \mathbb{R}$ actually introduced in E. Hille's paper [4]. In fact, Pollard has proved that:

Theorem 1. *A complex function f, holomorphic in the strip $S(\lambda_0)$, $0 < \lambda_0 \leq \infty$, is in the space $\mathscr{P}^{(0)}(\lambda_0)$ iff for each $\lambda \in [0, \lambda_0)$ and $z = x + iy \in \overline{S(\lambda)}$,*

$$|f(z)| = |f(x + iy)| = O(\eta(\lambda; x, y)). \tag{2}$$

Pollard's theorem has been generalized by O. Százs and N. Yeardley [10, Theorem A], who proved that if $\alpha > -1$, then a function f, holomorphic in the strip $S(\lambda_0), 0 < \lambda_0 \leq \infty$, is in the class $\mathscr{P}^{(\alpha)}(\lambda_0)$ iff it satisfies (2).

2. Hankel's Transform and Series Representation by Laguerre Polynomials

Another approach to the series representation of the kind (1) is based on the integral representation of Laguerre's polynomials by means of Bessel's functions of first kind [1, 10.12., (21)] as well as on the class $G(\lambda), -\infty < \lambda \leq \infty$ of entire functions F of exponential type introduced in [6, Definition 1.] by the requirement

$$\limsup_{|w|\to\infty} (2\sqrt{|w|})^{-1}(\log|F(w)| - |w|) \leq -\lambda.$$

The corresponding proposition is announced in [6, Theorem 1.] and says that:

Proposition. *Let $0 < \lambda_0 \leq \infty$ and $\alpha > -1$. A complex function f analytic in the region $\Delta(\lambda_0)$ can be represented in this region as a series of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ if and only if the following representation holds in the region $\Delta(\lambda_0) \setminus (-\lambda_0, 0]$:*

$$f(z) = z^{-\alpha/2} e^z \int_0^\infty t^{\alpha/2} e^t F(t) J_\alpha(2\sqrt{zt})\, dt$$

where J_α is the Bessel function of the first kind of order α and the function $F \in A(\lambda_0)$.

Remark. *A proof can be found in [7] as well as in [8, Chapter VI, 1].*

Let f be an even complex function holomorphic in the strip $S(\lambda_0)$, $0 < \lambda_0 \leq \infty$. Then, the function $f(\sqrt{z})$ is holomorphic in the region $\Delta(\lambda_0) \setminus (-\lambda_0^2, 0]$. Since f is even, $\lim_{z\to x, \Im z>0} f(\sqrt{z}) = \lim_{z\to x, \Im z<0} f(\sqrt{z})$ for each $x \in (-\lambda_0^2, 0)$, i.e. f has a continuous extension in the region $\Delta(\lambda_0)$. In fact, f is holomorphic there and this can be proved e.g. by an usual use of Morera's theorem [9, (8.1)].

Suppose now that $0 < \lambda_0 \leq \infty, \alpha > -1$ and that an even complex function f, holomorphic in the strip $S(\lambda_0)$, has representation by the series (1) in this strip. Then, the function $f(\sqrt{z})$

admits representation in the region $\Delta(\lambda_0)$ by series in the polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ and, hence, the representation

$$z^{\alpha/2}\exp(-z)f(\sqrt{z}) = \int_0^\infty t^{\alpha/2}\exp(-t)F(t)J_\alpha(2\sqrt{zt})\,dt$$

holds in the region $\Delta(\lambda_0)\setminus(-\lambda_0^2, 0]$. Replacing z by z^2, we obtain that the representation

$$z^{\alpha}\exp(-z^2)f(z) = \int_0^\infty t^{\alpha/2}\exp(-t)F(t)J_\alpha(2z\sqrt{t})\,dt$$

holds in the half-strip $S^+(\lambda_0) := \{\, z \in S(\lambda_0) : \Re z > 0\}$. The converse is also true, i.e. if the above representation holds for an even function f holomorphic in the strip $S(\lambda_0)$, then it has a representation there by a series in the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$. Further, replacing z by $z/\sqrt{2}$ and changing t by $t^2/2$,we come to the following assertion:

Assertion. *An even complex function f, holomorphic in the strip $S(\lambda_0), 0 < \lambda_0 \le \infty$ is in the space $\mathscr{P}^{(\alpha)}(\lambda_0), \alpha > -1$ iff the representation*

$$\begin{aligned} &z^{\alpha+1/2}\exp(-z^2)f(z/\sqrt{2}) \\ &= \int_0^\infty t^{\alpha+1/2}\exp(-t^2)f(t^2/2)(zt)^{1/2}J_\alpha(zt)\,dt \end{aligned}$$

holds in the half-strip $S^+(\lambda_0)$ with a function $F \in G(\lambda_0)$.

3. The Main Results

Since the Riemann function $\zeta(s), s = \sigma + it$ does not vanish on the closed half-plane $\sigma \ge 1$, there exists a region B containing this half-plane and such that $\zeta(s) \ne 0$ for $s \in B$. Hence, the function

$$\Phi(s) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}$$

is holomorphic in the region B. Moreover, the integral representation

$$\Phi(s) = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\,dx \tag{3}$$

holds on the closed half-plane $\sigma \ge 1$, where ψ is one of the Chebisheff functions [3, ChapterXI, Section 3]. A corollary of (3) is that the function Φ is bounded in this half-plane. Indeed, since $\psi(x) - x = O(x\exp(-c(\log x)^{1/2}))$, $c > 0$ as $x \to \infty$ [3, Section 18, (1)], we have that for $\sigma \ge 1$ and $-\infty < t < \infty$,

$$|\Phi(s)| \le \int_1^\infty \frac{|\psi(x) - x|}{x^{\sigma+1}}\,dx = O\left(\int_1^\infty x^{-1}\exp(-c(\log x)^{1/2})\,dx\right)$$

$$= O\left(\int_0^\infty \exp(-cx^{1/2})\,dx\right) = O(1).$$

Suppose now that the function ζ has no zeros in the half-plane $\sigma > \theta, 1/2 \leq \theta < 1$. Then, $\psi(x) = x + O(x^\theta \log^2 x)$ as $x \to \infty$ [3, Section 18], i.e. whatever $\varepsilon > 0$ may be, $\psi(x) = x + O(x^{\theta+\varepsilon})$ as $x \to \infty$. Then, the integral in (3) is uniformly convergent on each closed half-plane $\sigma \geq \theta + \varepsilon$. That means the function $\Phi(s)$ is analytically continuable in each half-plane $\sigma > \theta + \varepsilon$ and, moreover, it is bounded when $\sigma \geq \theta + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that, in fact, Φ is holomortphic in the half-plane $\sigma > \theta$ and bounded in each half-plane $\sigma \geq \theta + \varepsilon, \varepsilon > 0$. Hence, the function $\tilde{\Phi}(s) = \Phi(s) + \Phi(2-s)$ is holomorphic in the strip $\theta < \sigma < 2 - \theta$ and is bounded in each closed strip $\theta + \varepsilon \leq \sigma \leq 2 - \theta - \varepsilon$ provided $0 < \varepsilon < 1 - \theta$. Therefore, the even function $\Phi^*(z) = \tilde{\Phi}(1+iz) = \Phi(1+iz) + \Phi(1-iz)$ is holomorphic in the strip $S(1-\theta)$ and, moreover, it is bounded on each closed strip $\overline{S(1-\theta-\varepsilon)}$ with $\varepsilon \in (0, 1-\theta)$. That means the function Φ^* is in the space $\mathscr{P}^{(\alpha)}(1-\theta)$ for each $\alpha > -1$, i.e. there is a function $F \in G(1-\theta)$ such that

$$z^{\alpha+1/2}\exp(-z^2/2)\Phi^*(z/\sqrt{2}) = \int_0^\infty t^{\alpha+1/2}\exp(-t^2/2)F(t^2/2)(zt)^{1/2}J_\alpha(zt)\,dt \tag{4}$$

for $z \in S^+(1-\theta)$ and, hence,

$$t^{\alpha+1/2}\exp(-t^2/2)F(t^2/2) = \int_0^\infty x^{\alpha+1/2}\exp(-x^2/2)\Phi^*(x/\sqrt{2})(tx)^{1/2}J_\alpha(tx)\,dx.$$

We have just proved that if $\zeta(s) \neq 0$ for $\sigma > \theta, 1/2 \leq \theta < 1$, then the Hankel transform with kernel $w^{1/2}J_\alpha(w), \alpha > -1$ of the function

$$x^{\alpha+1/2}\exp(-x^2/2)\Phi^*(x/\sqrt{2}), 0 < x < \infty \tag{5}$$

is of the form

$$t^{\alpha+1/2}\exp(-t^2/2)F(t^2/2), \quad 0 < t < \infty \tag{6}$$

with function $F \in G(1-\theta)$.

The converse is also true. Indeed, suppose the Hankel transform with kernel $w^{1/2}J_\alpha(w)$, $\alpha > -1$ of the function (5) is of the form (6) with function F in the class $G(1-\theta)$, $1/2 \leq \theta < 1$, i.e. (4) holds for $z = x \in (0, \infty)$. By means of the asymptotic formula [1, 7.13, (3)] for the function $J_\alpha(z)$ it can be proved that whatever $\varepsilon \in (0, 1-\theta)$ may be, the integral in (4) is uniformly convergent in the strip $S(\sqrt{2}(1-\theta-\varepsilon))$. That means the function $\Phi^*(x/\sqrt{2}), 0 < x < \infty$ has a holomorphic extension in the half-strip $S^+(\sqrt{2}(1-\theta))$, i.e. the function $\Phi^*(x)$ has a holomorphic extension in the strip $S(1-\theta)$. Therefore, the function $\Phi(s)$ is analytically continuable in the half-plane $\sigma > \theta$ and, hence, $\zeta(s) \neq 0$ in this half-plane. Thus we have proved that:

Theorem 2. *A necessary and sufficient condition the function $\zeta(s)$ to have no zeros in the half-plane $\sigma > \theta, 1/2 \leq \theta < 1$ is the Hankel transform with kernel $w^{1/2}J_\alpha(w)$ of the function (5) to be of the form (6).*

A corollary of the above assertions is the following criterion:

Corollary. *Riemann's hypothesis is true iff the Hankel transform with kernel* $w^{1/2}J_\alpha(w)$ *of the function (5) is of the form (6) with a function* $F \in G(1/2)$.

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