EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 2, 2012, 108-115
ISSN 1307-5543 – www.ejpam.com

A Generalization of ⊕-Supplemented Modules

Tayyebeh Amouzegar[1], Yahya Talebi [2,*]

[1] *Department of Mathematics, Quchan Institute of Engineering and Technology, Quchan, Iran*
[2] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*

Abstract. Let M and X be R-modules. We define the X-⊕-supplemented modules via the class $\mathscr{B}(M,X)$ as a generalization of ⊕-supplemented modules. We show that any finite direct sum of X-⊕-supplemented modules is X-⊕-supplemented. It is given a number of necessary and sufficient conditions for every direct summand of an X-⊕-supplemented module to be X-⊕-supplemented.

2010 Mathematics Subject Classifications: 16D90, 16D99

Key Words and Phrases: X-⊕-supplemented module, Completely X-⊕-supplemented module, Hollow module

---

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and M a unitary R-module. A submodule N of M is called *small* in M (notation $N \ll M$) if $\forall L \lneq M, L + N \neq M$. A non-zero module M is called *hollow* if every proper submodule is small in M. Let K and N be submodules of M. K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$. A submodule K of M is called a *supplement* in M provided there exists a submodule N of M such that K is a supplement of N in M. Following [9], a module M is called *supplemented* if every submodule of M has a supplement in M. According to [6], a module M is called ⊕-*supplemented* if every submodule of M has a supplement that is a direct summand of M. A module M is called *completely* ⊕-*supplemented* if every direct summand of M is ⊕-supplemented [see 4].

Let M and X be R-modules. In [5], Keskin Tütüncü and Harmancı defined the family $\mathscr{B}(M,X) = \{A \leq M \mid \exists Y \leq X, \exists f \in Hom(M,X/Y), \text{Ker} f/A \ll M/A\}$ and used this class to define $\mathscr{B}(M,X)$-projective modules as a generalization of projective modules. In this paper we define X-⊕-supplemented modules and completely X-⊕-supplemented modules via the class

*Corresponding author.

Email addresses: `t.amoozegar@yahoo.com` (T. Amouzegar), `talebi@umz.ac.ir` (Y. Talebi)

$\mathscr{B}(M,X)$ as generalizations of ⊕-supplemented modules and completely ⊕-supplemented modules respectively.

Let A and P be submodules of M with $P \in \mathscr{B}(M,X)$. Following [7], P is called an *X-supplement* of A in M if it is minimal with the property $M = A + P$. Equivalently, if $M = A + P$ and $A \cap P \ll P$. A module M is called *X-supplemented* if every submodule N of M with $N \in \mathscr{B}(M,X)$ has an X-supplement in M. We say that a module M is *X-⊕-supplemented* if every submodule N of M with $N \in \mathscr{B}(M,X)$, has an X-supplement that is a direct summand of M.

We prove some results on these classes of modules. In Section 2, we recall some notions and results that they are used in this paper. In Section 3, we give a characterization of X-⊕-supplemented modules. It is shown that any finite direct sum of X-⊕-supplemented modules is X-⊕-supplemented. We give a number of necessary and sufficient conditions for every direct summand of an X-⊕-supplemented module to be X-⊕-supplemented. We show that the direct sum of any finite family M_i of relatively $\mathscr{B}$-projective modules is X-⊕-supplemented if and only if every M_i is X-⊕-supplemented. In Section 4, we prove the equivalence of two conditions for a module with finite Goldie dimension: One saying that every direct summand N of M with $N \in \mathscr{B}(M,X)$ is a finite direct sum of X-hollow modules, and the other stating that M is a completely X-⊕-supplemented module.

2. Preliminaries

Let M be a module and $N \leq M$. N is called a *coclosed submodule* in M if whenever $N/K \ll M/K$ then $N = K$. Let M be a module and $B \leq A \leq M$. If B is coclosed in M and $A/B \ll M/B$, then B is called an *co-closure* of A in M. A non-zero module M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Every local module is hollow and hollow modules are ⊕-supplemented. A submodule K of M is called *essential* in M (notation $K \leq_e M$) if $K \cap A \neq 0$ for any nonzero submodule A of M. Recall that a module M is said to have the *summand sum property* (SSP) if the sum of two direct summands is again a direct summand. A module M is said to have the *(finite) internal exchange property* if for every (finite) index set I, whenever $M = \oplus_{i\in I} A_i$ for modules A_i, then for every direct summand K of M there exist submodules B_i of A_i such that $M = K \oplus (\oplus_{i\in I} B_i)$. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \lhd M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M), \phi(N) \subseteq N$).

Lemma 1. *Let M, N and X be R-modules. Then the following hold:*

(1) If $A \in \mathscr{B}(M,X)$ and $B \leq A$ with $A/B \ll M/B$, then $B \in \mathscr{B}(M,X)$.

(2) Let $h : M \to N$ be an epimorphism and $A \in \mathscr{B}(M,X)$ with $\operatorname{Ker} h \leq A$. Then $h(A) \in \mathscr{B}(N,X)$. Conversely, if $h(A) \in \mathscr{B}(N,X)$ and $\operatorname{Ker} h \leq A$, then $A \in \mathscr{B}(M,X)$.

(3) Let $B \leq A \leq M$. Then $A \in \mathscr{B}(M,X)$ if and only if $A/B \in \mathscr{B}(M/B,X)$.

(4) Let $h : N \to M$ be an epimorphism and $A \in \mathscr{B}(M,X)$. Then $h^{-1}(A) \in \mathscr{B}(N,X)$.

Proof. See [5, Lemma 2.2].

Lemma 2. *Let M and X be R-modules. Then the following hold:*

(1) Let $M = A + B$. If $B \in \mathscr{B}(M,X)$, then $A \cap B \in \mathscr{B}(M,X)$.

(2) Let $M = \bigoplus_{i\in I} M_i$. If $N_i \in \mathscr{B}(M_i,X)$, for every $i \in I$. Then $\oplus_{i\in I} N_i \in \mathscr{B}(M,X)$.

(3) Let $M = M_1 \oplus M_2$. If $A \in \mathscr{B}(M,X)$, then $A + M_i \in \mathscr{B}(M,X)$ for $i = 1,2$.

Proof.

(1) Let $M = A + B$ and $B \in \mathscr{B}(M,X)$. There exist $Y \le X$ and $f : M \to X/Y$ such that $\operatorname{Ker} f/B \ll M/B$. Consider the isomorphism $\alpha : M/B \to A/(A\cap B)$. Then $\alpha(\operatorname{Ker} f/B) = \operatorname{Ker} f/(A\cap B)$. Hence $\operatorname{Ker} f/(A\cap B) \ll M/(A\cap B)$. Therefore $A\cap B \in \mathscr{B}(M,X)$.

(2) Since $N_i \in \mathscr{B}(M_i,X)$, there exist a submodule Y of X and a homomorphism $f_i : M_i \to X/Y$ such that $\operatorname{Ker} f_i/N_i \ll M_i/N_i$. Put $f = \oplus_{i\in I} f_i$. Then $f : M \to X/Y$ such that $\operatorname{Ker} f/\oplus_{i\in I} N_i \ll M/\oplus_{i\in I} N_i$. Thus $\oplus_{i\in I} N_i \in \mathscr{B}(M,X)$.

(3) By Lemma 1 and [5, Lemma 3.5].

3. X-$\oplus$-Supplemented Modules

Let X and M be R-modules. We recall that a module M is *X-$\oplus$-supplemented* if every submodule N of M with $N \in \mathscr{B}(M,X)$, has an X-supplement that is a direct summand of M. Clearly X-hollow modules are X-$\oplus$-supplemented. It is obvious that X-$\oplus$-supplemented modules are X-supplemented.

Proposition 1. *Let M be a module such that every submodule A of M with $A \in \mathscr{B}(M,X)$ has a co-closure in M. Then the following statements are equivalent:*

(1) M is X-$\oplus$-supplemented.

(2) Any coclosed submodule H of M with $H \in \mathscr{B}(M,X)$, has an X-supplement that is a direct summand of M.

(3) For any submodule N of M with $N \in \mathscr{B}(M,X)$, there exists a direct summand K of M with $K \in \mathscr{B}(M,X)$ such that $M = N + K$ and $N \cap K \ll M$.

(4) For any coclosed submodule H of M with $H \in \mathscr{B}(M,X)$, there exists a direct summand K of M with $K \in \mathscr{B}(M,X)$ such that $M = H + K$ and $H \cap K \ll M$.

Proof. (1) $\Leftrightarrow$ (3), (2) $\Leftrightarrow$ (4), (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Let $A \in \mathscr{B}(M,X)$. By assumption, there exists a coclosed submodule B of M such that $B \le A$ and $A/B \ll M/B$. By Lemma 1, $B \in \mathscr{B}(M,X)$. Therefore there exists a direct summand K of M with $K \in \mathscr{B}(M,X)$ such that $M = B + K$ and $B \cap K \ll M$. Hence K is an X-supplement of B in M. Note that $M = A + K$. Assume that $K' < K$ and $M = A + K'$. Then $M \ne B + K'$ and so $M \ne A + K'$ since $A/B \ll M/B$. Thus K is an X-supplement of A in M.

Theorem 1. *Any finite direct sum of X-$\oplus$-supplemented modules is X-$\oplus$-supplemented.*

Proof. Let $M = M_1 \oplus M_2$ where M_1 and M_2 are X-$\oplus$-supplemented modules. Let N be any submodule of M with $N \in \mathscr{B}(M,X)$. We have $N + M_2 = M_2 \oplus [(N + M_2) \cap M_1]$. Since $N \in \mathscr{B}(M,X)$, $N + M_2 \in \mathscr{B}(M,X)$ by Lemma 2. From [7, Lemma 3.1], $(N+M_2)\cap M_1 \in \mathscr{B}(M_1,X)$. Since M_1 is X-$\oplus$-supplemented, there exists a direct summand K_1 of M_1 with $K_1 \in \mathscr{B}(M_1,X)$ such that $[(N+M_2)\cap M_1]+K_1 = M_1$ and $(N+M_2)\cap K_1 \ll K_1$. By Lemma 2 and [7, Lemma 3.1], $(N+K_1)\cap M_2 \in \mathscr{B}(M_2,X)$. Thus there exists a direct summand K_2 of M_2 with $K_2 \in \mathscr{B}(M_2,X)$ such that $[(N+K_1)\cap M_2]+K_2 = M_2$ and $(N+K_1)\cap K_2 \ll K_2$. Let $K = K_1 \oplus K_2$, then K is a direct summand of M and $K \in \mathscr{B}(M,X)$ (Lemma 2). Moreover, $M_1 \leq N + M_2 + K_1$ and $M_2 \leq N + K_1 + K_2$. Hence $M = N + K_1 + K_2 = N + K$. Since $N\cap(K_1+K_2) \leq (N+K_1)\cap K_2+(N+K_2)\cap K_1$, $N\cap(K_1+K_2) \leq (N+K_1)\cap K_2+(N+M_2)\cap K_1$. As $(N+M_2)\cap K_1 \ll K_1$ and $(N+K_1)\cap K_2 \ll K_2, (N\cap K) \ll K$. Thus M is X-$\oplus$-supplemented.

Corollary 1. *Any finite direct sum of X-hollow modules is X-$\oplus$-supplemented.*

Lemma 3. *Let $M = N \oplus N'$ be a module. Assume that A is a submodule of N and K a submodule of M. If $K \cap (A \oplus N') \ll K$, then $A \cap (K + N') \ll N \cap (K + N')$.*

Proof. Let π be the projection $N \oplus N' \to N$. Since $K \cap \pi^{-1}(A) = K \cap (A \oplus N') \ll K$, $\pi(K\cap\pi^{-1}(A)) = \pi(K)\cap A \ll \pi(K)$. But $\pi(K) = N\cap(K+N')$. Hence $A\cap(K+N') \ll N\cap(K+N')$.

Following [5], an R-module N is called *$\mathscr{B}(M,X)$-projective* if for any submodule A of M with $A \in \mathscr{B}(M,X)$, any homomorphism $\phi : N \to M/A$ can be lifted to a homomorphism $\psi : N \to M$. Two R-modules M_1 and M_2 are called *relatively $\mathscr{B}$-projective* if M_1 is $\mathscr{B}(M_2,X)$-projective and M_2 is $\mathscr{B}(M_1,X)$-projective.

Theorem 2. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum of relatively $\mathscr{B}$-projective modules M_i and let M have the summand sum property. Then the module M is X-$\oplus$-supplemented if and only if M_i is X-$\oplus$-supplemented for all $1 \leq i \leq n$.*

Proof. The sufficiency is proved in Theorem 1. Conversely, we only prove M_1 is X-$\oplus$-supplemented. Let $A \in \mathscr{B}(M_1,X)$. By Lemma 1, $A \oplus M_2 \in \mathscr{B}(M,X)$. Since M is X-$\oplus$-supplemented, there exists $B \in \mathscr{B}(M,X)$ such that $M = (A \oplus M_2) + B$, $(A \oplus M_2) \cap B \ll B$ and B is a direct summand of M. By Lemma 2, $M_2 + B \in \mathscr{B}(M,X)$. Clearly $M = M_1 + M_2 + B$. By [5, Proposition 2.5], there exists $T \leq M_2 + B$ such that $M = M_1 \oplus T$. Thus $B + M_2 = (M_1 \cap (B + M_2)) \oplus T$. Now $M_1 = A + ((B + M_2) \cap M_1)$ and since $(A \oplus M_2) \cap B \ll B$, by Lemma 3, $A \cap (M_1 \cap (B + M_2)) \ll M_1 \cap (B + M_2)$. As M has the summand sum property, $B + M_2$ is a direct summand of M. Thus $(B + M_2) \cap M_1 \leq^{\oplus} M$ and so $(B + M_2) \cap M_1$ is a direct summand of M_1. By [7, Lemma 3.1 (1)], $(B + M_2) \cap M_1 \in \mathscr{B}(M_1,X)$. Hence M_1 is X-$\oplus$-supplemented.

Proposition 2. *Let M and N be R-modules and $h : M \to N$ be an epimorphism such that* $\operatorname{Ker} h \lhd M$. *If M is X-$\oplus$-supplemented, then N is X-$\oplus$-supplemented.*

Proof. Let $A \in \mathscr{B}(N,X)$. By Lemma 1, $h^{-1}(A) \in \mathscr{B}(M,X)$. Since M is X-$\oplus$-supplemented, there exist submodules H and H' of M such that $M = H \oplus H'$, $M = h^{-1}(A) + H$ and $h^{-1}(A) \cap H \ll H$. Now $N = A + h(H)$ and since $h^{-1}(A) \cap H \ll H$, $h(h^{-1}(A) \cap H) = A \cap h(H) \ll h(H)$. Moreover, since $\operatorname{Ker} h \lhd M$, $N = h(H) \oplus h(H')$. Therefore $h(H)$ is an X-supplement of A in N and it is a direct summand of N. Hence N is X-$\oplus$-supplemented.

Corollary 2. *Let M be an R-module and N be a fully invariant submodule of M. If M is X-$\oplus$-supplemented, then M/N is X-$\oplus$-supplemented.*

Proof. By Proposition 2.

Recall that a module M is a *duo module*, if every submodule of M is a fully invariant submodule of M.

Corollary 3. *Let M be an X-$\oplus$-supplemented duo module, then every direct summand of M is X-$\oplus$-supplemented.*

Proof. By Corollary 2.

Definition 1. *A module M is said to have the* (finite) strong internal exchange property *if for every (finite) index set I, whenever $M = K + (\oplus_{i \in I} A_i)$ for a direct summand K of M and modules A_i, then $M = K \oplus (\oplus_{i \in I} B_i)$ for submodules B_i of A_i.*

It is clear that if a module M has the (finite) strong internal exchange property, then M has the (finite) internal exchange property.

Theorem 3. *Let M be an X-$\oplus$-supplemented module with the finite strong internal exchange property. Then any direct summand of M is X-$\oplus$-supplemented.*

Proof. Let N be a direct summand of M. Thus $M = N \oplus N'$ for some submodule $N'$of M. Let $A \in \mathscr{B}(N,X)$. By Lemma 1, $A \oplus N' \in \mathscr{B}(M,X)$. Since M is X-$\oplus$-supplemented, there exists a direct summand K of M with $K \in \mathscr{B}(M,X)$ such that $M = K + (A \oplus N')$ and $(A \oplus N') \cap K \ll K$. Since M has the finite strong internal exchange property, $M = K \oplus N_1 \oplus N_1'$ such that $N_1 \subseteq A$ and $N_1' \subseteq N'$. By modularity, $N = N_1 \oplus (N \cap (K \oplus N_1'))$. By Lemma 2 and [7, Lemma 3.1], $N \cap (K \oplus N_1') \in \mathscr{B}(N,X)$. As $M = A + (K \oplus N_1')$, $N = A + (N \cap (K' \oplus N_1')$. Since $(A \oplus N') \cap K \ll K$, by Lemma 3, $A \cap (K \oplus N') \ll N \cap (K \oplus N')$. Thus $A \cap (K \oplus N_1') \ll N \cap (K \oplus N')$. Since $N \cap (K \oplus N_1') \leq^{\oplus} M$, $A \cap (K \oplus N_1') \ll N \cap (K \oplus N_1')$. Hence N is X-$\oplus$-supplemented.

If in set $\mathscr{B}(M,X)$, we take $X = M$, then $\mathscr{B}(M,X)$ coincides with the set of all submodules of M. Therefore we obtain the following corollary:

Corollary 4. *Let M be a $\oplus$-supplemented module with the finite strong internal exchange property. Then any direct summand of M is $\oplus$-supplemented.*

4. Completely X-⊕-Supplemented Modules

Let X and M be R-modules. We call a module M *completely X-⊕-supplemented* if every direct summand N of M with $N \in \mathscr{B}(M,X)$ is X-⊕-supplemented.

Recall that a module M has $\mathscr{B}(M,X)$-(D_3) condition if for all $A \in \mathscr{B}(M,X)$ and direct summand B of M, if A is a direct summand of M and $M = A + B$ then $A \cap B$ is a direct summand of M [5].

Proposition 3. *Let M be an X-⊕-supplemented module with $\mathscr{B}(M,X)$-$(D3)$. Then M is completely X-⊕-supplemented.*

Proof. Let N be a direct summand of M and A a submodule of N such that $N \in \mathscr{B}(M,X)$ and $A \in \mathscr{B}(N,X)$. We show that A has an X-supplement in N that is a direct summand of N. We have $M = N \oplus N'$ for some submodule N' of M. Let $\pi : M \to N$ be the projection along N'. Since $A \in \mathscr{B}(N,X)$, by Lemma 1(4), $A \oplus N' = \pi^{-1}(A) \in \mathscr{B}(M,X)$. Since $M = A + N + N'$, $A = (A \oplus N') \cap N \in \mathscr{B}(M,X)$ (Lemma 2). Since M is X-⊕-supplemented, there exists a direct summand B of M with $B \in \mathscr{B}(M,X)$ such that $M = A + B$ and $A \cap B \ll B$. Then $N = A + (N \cap B)$. Again by Lemma 2, $N \cap B \in \mathscr{B}(M,X)$. Furthermore $N \cap B$ is a direct summand of M because M has $\mathscr{B}(M,X)$-(D_3). Then $A \cap (N \cap B) = A \cap B$ is small in $N \cap B$ and by [7, Lemma 3.1], $N \cap B \in \mathscr{B}(N,X)$.

Let X and M be R-modules. We say $N \in \mathscr{B}(M,X)$ is *semisimple relative to the class* $\mathscr{B}(M,X)$ if, for every submodule K of N with $K \in \mathscr{B}(N,X)$, there exists a submodule K' of N with $K' \in \mathscr{B}(N,X)$ such that $N = K \oplus K'$. It is clear that every semisimple module relative to the class $\mathscr{B}(M,X)$ is X-⊕-supplemented.

Lemma 4. *Let M be an X-supplemented module and let N be a submodule of M such that $N \cap Rad(M) = 0$ and $N \in \mathscr{B}(M,X)$. Then N is semisimple relative to the class $\mathscr{B}(M,X)$.*

Proof. We have to prove that $M/Rad(M)$ contains no non-zero small submodule $K/Rad(M)$ with $K/Rad(M) \in \mathscr{B}(M/Rad(M),X)$. Let $K/Rad(M) \ll M/Rad(M)$ and $K/Rad(M) \in \mathscr{B}(M/Rad(M),X)$. From Lemma 1, $K \in \mathscr{B}(M,X)$. By hypothesis, there exists a submodule B of M with $B \in \mathscr{B}(M,X)$ such that $M = K + B$ and $K \cap B \ll B$. As $K/Rad(M) \ll M/Rad(M)$, $Rad(M) = K$. Thus every submodule $K/Rad(M)$ of $M/Rad(M)$ with $K/Rad(M) \in \mathscr{B}(M/Rad(M),X)$ is a direct summand of $M/Rad(M)$. Hence $M/Rad(M)$ is semisimple relative to the class $\mathscr{B}(M/Rad(M),X)$. Hence N is semisimple relative to the class $\mathscr{B}(M,X)$.

Proposition 4. *Let M be an X-supplemented module and suppose that for every submodule N of M such that $N \cap Rad(M) = 0$ we have $N \in \mathscr{B}(M,X)$. Then $M = M_1 \oplus M_2$, where M_1 is a semisimple module relative to the class $\mathscr{B}(M,X)$ and $Rad(M_2)$ essential in M_2.*

Proof. Let M_1 be a complement of $Rad(M)$ in M, hence $Rad(M) \oplus M_1$ is essential in M. Since M is X-supplemented, there exists a submodule M_2 of M such that $M = M_1 + M_2, M_1 \cap M_2 \ll M_2$ and $M_2 \in \mathscr{B}(M,X)$. Then $M_1 \cap M_2$ is a submodule of both

$Rad(M)$ and M_1. It follows that $M = M_1 \oplus M_2$, $Rad(M) = Rad(M_2)$ is essential in M_2, and by Lemma 4, M_1 is semisimple relative to the class $\mathscr{B}(M,X)$.

A module M is said to be *finite Goldie-dimensional* provided M contains no infinite independent families of nonzero submodules.

Theorem 4. *Consider the following conditions for a projective module M:*

(i) M is a direct sum of X-$\oplus$-supplemented modules and $Rad(M)$ has finite Goldie dimension.

(ii) $M = M_1 \oplus M_2$ such that M_1 is semisimple relative to the class $\mathscr{B}(M,X)$ and M_2 has finite Goldie dimension and M_2 is a (finite) direct sum of local modules.

If for every submodule N of a direct summand M_i of M such that $N \cap Rad(M_i) = 0$ we have $N \in \mathscr{B}(M_i,X)$, then $(i) \Rightarrow (ii)$ holds and if for every small submodule N of M_1 we have $N \in \mathscr{B}(M_1,X)$, then $(ii) \Rightarrow (i)$ holds.

Proof. $(i) \Rightarrow (ii)$ Let $M = \bigoplus_{i\in I} M_i$ and M_i is X-$\oplus$-supplemented for every $i \in I$. Since $Rad(M) = \oplus_{i\in I} Rad(M_i)$, then there is a finite subset J of I such that $Rad(M_i) = 0$ for all $i \in I \setminus J$. Therefore M_i is semisimple relative to $\mathscr{B}(M,X)$ for all $i \in I \setminus J$. Hence there is a submodule M_1 semisimple relative to $\mathscr{B}(M,X)$ such that $M = M_1 \oplus (\oplus_{j\in J} M_j)$. By Proposition 4, without loss of generality, we may assume $Rad(M_j)$ is essential in $M_j (j \in J)$. Then $M_j (j \in J)$ has finite Goldie dimension by [3, Proposition 3.20]. Next we prove that each M_j, for $j \in J$, is local or a finite direct sum of local modules. Set $H = M_j$ for any $j \in J$. First, note that $Rad(H) \neq H$ because H is projective [1, Proposition 17.14]. Assume that H has Goldie dimension 1, and take some $x \in H \setminus Rad(H)$. Since H is X-$\oplus$-supplemented, there is a submodule K of H with $K \in \mathscr{B}(H,X)$ such that $H = xR + K$, $xR \cap K \ll K$ and $H = K \oplus K_1$ for some submodule K_1 of M. Then $K = 0$ or $K_1 = 0$. If $K_1 = 0$, then $xR \subseteq Rad(H)$ which is a contradiction. Hence $K = 0$ and $H = xR$. It follows that H is local. Let $n > 1$ be a positive integer and assume that each M_j having Goldie dimension k $(1 \leq k < n)$ is local or a finite direct sum of local submodules. Let $j \in J$ and $H = M_j$ and assume H has Goldie dimension n. Suppose that H is not local. Let $x \in H \setminus Rad(H)$ such that $H \neq xR$. Since H is X-$\oplus$-supplemented, there exist submodules K, K_1 of H with $K \in \mathscr{B}(H,X)$ such that $H = xR + K = K \oplus K_1$ and $xR \cap K \ll K$. It is clear that $K_1 \neq 0$. Also $K \neq 0$. Since projective modules satisfy (D_3), and so they satisfy $\mathscr{B}(M,X)$-(D_3). By Proposition 3, we obtain that any direct summand of M is X-$\oplus$-supplemented. Thus K and K_1 are X-$\oplus$-supplemented. By induction, K and K_1 are local or finite direct sum of local submodules. This completes the proof of $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ It is clear.

Lemma 5. *Let M be an indecomposable module. Then M is X-hollow if and only if M is completely X-$\oplus$-supplemented.*

Proof. Let M be completely X-$\oplus$-supplemented. If $N \in \mathscr{B}(M,X)$ is a proper submodule of M, then there exists an X-supplement A of M such that A is direct summand of M. By hypothesis we have $A = M$. Thus $N = N \cap M = N \cap A \ll M$. Therefore M is X-hollow. Conversely, if M is X-hollow and $N \in \mathscr{B}(M,X)$ then $N \ll M$. Since $M \in \mathscr{B}(M,X)$, M is an X-supplement of N in M.

Proposition 5. *Let $M = U \oplus V$ such that U and V have local endomorphism rings. Then M is completely X-$\oplus$-supplemented if and only if U and V are X-hollow modules.*

Proof. The necessity is clear from Lemma 5. Conversely, let $K \in \mathscr{B}(M,X)$ be a direct summand of M. If $K = M$ then by Corollary 1, K is X-$\oplus$-supplemented. Assume $K \neq M$. Then either $K \cong U$ or $K \cong V$ [1, Corollary 12.7]. In either case K is X-$\oplus$-supplemented. Thus M is completely X-$\oplus$-supplemented.

Theorem 5. *Let M be a non-zero module with finite Goldie dimension. Then the following statements are equivalent:*

(i) Every direct summand N of M with $N \in \mathscr{B}(M,X)$ is a finite direct sum of X-hollow modules.

(ii) M is a completely X-$\oplus$-supplemented module.

Proof. $(i) \Rightarrow (ii)$ It is clear by Corollary 1.

$(ii) \Rightarrow (i)$ Let N be a direct summand of M with $N \in \mathscr{B}(M,X)$. Since N has finite Goldie dimension, N has a decomposition $N = L_1 \oplus \ldots \oplus L_n$, where each L_i is indecomposable for $1 \leq i \leq n$. Thus each L_i $(1 \leq i \leq n)$ is X-hollow from Lemma 5.

References

[1] F Anderson and K Fuller. *Rings and Categories of Modules*. Springer-Verlog, New York, 1992.

[2] C Chang. X-Lifting Modules over Right Perfect Rings. *Bull. Korean Math. Soc*, 45(1):59-66, 2008.

[3] K Goodearl. *Ring Theory, Nonsingular Rings and Modules*. Marcel Dekker, Inc., New York and Basel, 1976.

[4] A Harmancı, D Tütüncü and P Smith. On $\oplus$-Supplemented Modules. *Acta Math. Hungar*, 83:161-169, 1999.

[5] D Tütüncü and A Harmancı. A Relative Version of the Lifting Property of Modules. *Algebra Colloq*, 11(3):361-370, 2004.

[6] S Mohamed and B Müller. *Continuous and Discrete Modules*. London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, 1990.

[7] N Orhan and D Tütüncü. Characterizations of Lifting Modules in Terms of Cojective Modules And The Class of $\mathscr{B}(M,X)$, *International J. Mathematics* 6:647-660, 2005.

[8] P Smith. Modules for which every submodules has a unique closure. *Ring Theory, World Sci.*, pages 302-313, Singapore, 1993.

[9] R Wisbauer. *Foundations of module and ring theory*. Gordon and Breach, Reading, 1991.