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Naturally Ordered Abundant Semigroups for which each Idempotent has a Greatest Inverse

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Abstract. G-regular and reflexive naturally ordered abundant semigroups each of whose idempotents has a greatest inverse are studied. In this paper, we give a construction theorem for such ordered semigroups. Our theorem extends a previous structure theorem on naturally ordered abundant semigroups of X.J. Guo and X.Y. Xie [13]. Some other results related to naturally ordered regular semigroups are amplified and strengthened.

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1. Introduction

Let E be the set of idempotents of a semigroup S, Define a relation "$\preceq$" on E by

$$e \preceq f \Leftrightarrow e = ef = fe.$$

Then, the relation "$\preceq$" is clearly an order on E which is called the *natural order* on E. An ordered semigroup $(S, \leq)$ endowed with a natural order is said to be a *naturally ordered semigroup* if the order $\leq$ extends the natural order $\preceq$ on the idempotents, i.e. if

$$(\forall e, f \in E)\ e \preceq f \ \Rightarrow\ e \leq f\ .$$

In general, in an ordered semigroup $(S, \leq)$ the order $\leq$ does not give much information of the algebraic properties of S. For instance, this is particularly true in the theory of residuated semigroups. Nevertheless, the semigroups which are regular and naturally ordered form a

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large class which has been investigated by many authors. In particular, the structure of an naturally ordered regular semigroups with a greatest idempotent was investigated by T.S. Blyth and R.B. McFadden in [1] in 1981. In particular, they established a structure theorem for these semigroups. They have also shown that each element of such a naturally ordered regular semigroup has a greatest inverse. In [2], they extended their results more generally to the naturally ordered regular semigroups for which each element has a greatest inverse. After this interesting result, a series of papers on this topic have been produced, for instance, the reader is referred to the papers [1]-[7], [19] and the references listed in [18].

It is easy to see that the relations $\mathscr{R}^*$ and $\mathscr{L}^*$ are the generalizations of the Green's relations $\mathscr{R}$ and $\mathscr{L}$, respectively. For the elements $a, b \in S$, define $a\mathscr{R}^*b$ if and only if they are related by $\mathscr{R}$ in some oversemigroup of S. The relation $\mathscr{L}^*$ can be dually defined. By an *rpp semigroup*, we mean a semigroup S in which for any $a \in S$, the system aS^1, regarded as an S^1-system, is projective. In the literature, Fountain [10] called such a semigroup *abundant* if each $\mathscr{R}^*$-class and each $\mathscr{L}^*$-class contains an idempotent. In fact, abundant semigroups are rpp and lpp semigroups.

It is known that the regular semigroups are abundant semigroups. Hence, it is natural to probe the ordered abundant semigroups. In this aspect, the first author and his collaborators (see [11]-[16]) have probed the structure of several classes of naturally ordered abundant semigroups. In this paper, we will also investigated the naturally ordered abundant semigroups for which each idempotent has a greatest inverse. Recently, Ren and the authors [12] have studied the naturally ordered rpp semigroups in which each idempotent has a greatest inverse and they obtained a structure theorem for a special class of such naturally ordered semigroups.

As a continuation of the work of Guo and Xie [13], we further investigate the structure of the naturally ordered abundant semigroups for which every idempotent has a greatest inverse. A new construction theorem for such semigroups will be presented. It is noteworthy that our construction theorem is different from the structure theorem previously given by Guo and Xie in [13]. In fact, our new theorem generalizes the previous theorem given by Guo and Xie in the following aspects:

(1) In the previous theorem of Guo-Xie, the authors only considered the case that $S^\flat$ is an adequate subsemigroup of the semigroup S. In other words, the ordered semigroup S in [13] is only an abundant semigroup with a quasi-ideal adequate transversal. In our new theorem, we consider a more general situation.

(2) In the previous theorem given by Guo-Xie in [13], the authors only studied the naturally ordered abundant semigroups with a greatest idempotent and showed how to construct a kind of naturally ordered abundant semigroups each of whose idempotents has a greatest inverse and which is G-regular and reflexive.

In this paper, we shall prove that any naturally ordered abundant semigroups each of whose idempotents has a greatest inverse and which is G-regular and reflexive is orderly isomorphic to some ordered semigroup constructed in [13]. So, we give a construction theorem of such naturally ordered semigroups.

2. Main Result

Throughout this paper, the notations and terminologies are taken from [9]-[12]. For other undefined terminologies and definitions, the reader is referred to the text of Howie [17].

We first recall some known concepts.

A nonempty set M is called a *groupoid* if there is a partial operation on M. We call a groupoid $(M,\circ)$ a *partial semigroup* if for the elements $x,y,z \in S$, $(xy)z$ and $x(yz)$ are defined, then the other one of $(xy)z$ and $x(yz)$ is defined and $x(yz) = (xy)z$. A partial semigroup M with a partial order $\leq$ is called an *ordered partial semigroup* provided that for all $x,y,u,v \in M$ with $x \leq y$ and $u \leq v$, if xu and yv are defined, then $xu \leq yv$. Moreover,an ordered partial semigroup $(M,\leq)$ is called *naturally ordered* if $\leq$ extends the natural order $\preceq$ on M.

We now assume that M is a partial semigroup. As in [12], M is called a *partial semilattice* if there exists a unipotent semigroup S (that is, a semigroup in which each $\mathscr{L}$-class and each $\mathscr{R}$-class contains at most one idempotent) such that $E(S) = M$. And, M is called a *left (right) regular partial band* if M is the disjoint union of left (right) zero rectangular bands M_α with $\alpha \in Y$, where Y is a partial semilattice, satisfying the following conditions:

(PB1) For all $\alpha,\beta \in Y$ and $x \in M_\alpha, y \in M_\beta$, if $\alpha \preceq \beta$, then xy and yx are defined for all $x \in M_\alpha, y \in M_\beta$.

(PB2) For all $x \in M$ and $y \in M_\alpha$, if xy and yx are defined and such that $xy = x$ and $yx = y$ ($xy = y$ and $yx = x$), then $x \in M_\alpha$.

In this case, we call Y the *structure partial semilattice* of the left (right) regular partial band M. Moreover, a subset N of the left (right) regular partial band $(M,\circ) = \cup_{\alpha\in Y} M_\alpha$ with a structure partial semilattice Y is called a *skeleton* if $(N,\circ)$ is partial semilattice isomorphic to Y and $|N \cap M_\alpha| = 1$, for all $\alpha \in Y$.

A semigroup is called *left *-unipotent* if each $\mathscr{L}^*$-class contains at most one idempotent. Right $*$-unipotent semigroups can be dually defined. And, we call a semigroup S *$*$-unipotent* if S is both left $*$-unipotent and right $*$-unipotent.

We now introduce the concept of a *GI-system*. By a *GI-system*, we mean a quadruple $(T;L,R;[,])$, where

- T is a *-unipotent abundant semigroup with a set of idempotents Y.
- $L = \bigcup_{\alpha\in Y} L_\alpha$ is a left regular partial band with Y as its skeleton.
- $R = \bigcup_{\alpha\in Y} R_\alpha$ is a right regular partial band with Y as its skeleton.
- $[,]$ is a mapping from $R \times L$ into T,

if the following conditions hold:

(GI1) $[xy,uv] = [x,\alpha][y,uv] = [xy,u][\beta,v]$ for all $x \in R$, $y \in R_\alpha$, $u \in L_\beta$ and $v \in L$.

(GI2) $[\alpha,\beta] = \alpha\beta$ for all $\alpha,\beta \in Y$.

(GI3) $[u,\alpha]=\alpha=[\alpha,v]$ for all $u\in R_\alpha$ and $v\in L_\alpha$.

Given a GI-system $(T;L,R;[,])$, we put the set

$$GI=GI(T;L,R;[,])=\{(x,s,u)\in L\times T\times R : x\in L_{s^\dagger} \text{ and } R_{s^*}\}.$$

Now, we define

$$(x,s,u)\circ(y,t,v)=(xa^\dagger,a,a^*v),$$

where $a=s[u,y]t$. Then, it is easy to verify that the system $(GI,\circ)$ forms an abundant semigroup [see 13]. We denote this semigroup by $GI(T;L,R;[,])$.

Assume that S is an ordered abundant semigroup in which each regular element has a maximum inverse. Throughout this paper, we use x° to denote the greatest inverse of x if x has the greatest inverse. Similar as in [13], we call the semigroup S *G-regular* if for any $x,y\in S$, $x\le y$ implies $x^{*\circ}x^*\le y^{*\circ}y^*$ and $x^\dagger x^{\dagger\circ}\le y^\dagger y^{\dagger\circ}$. Also, we call S *reflexive* if for all $x,y\in \operatorname{Reg}S$, $x\le y$ implies that $x^\circ\le y^\circ$.

In the following theorem, we establish a construction theorem for the G-regular and reflexive naturally ordered abundant semigroup S in which each idempotent of S has a greatest inverse.

Theorem 1. *Let $(T;L,R;[,])$ be a GI-system. Assume that the following GI-conditions are satisfied:*

1. *T is a *-unipotent naturally ordered abundant semigroup in which each idempotent has a greatest inverse and which is G-regular and reflexive.*
2. *L is a naturally ordered left regular partial band in which α is the greatest element in L_α for all $\alpha\in Y$.*
3. *R is a naturally ordered right regular partial band in which β is the greatest element in R_β for all $\beta\in Y$.*
4. *For all $x,y\in L$ and $u,v\in R$, if $x\le y$ and $u\le v$, then $[u,x]\le[v,y]$.*
5. *For any $(x,t,u),(y,s,v)\in LGI$, if (x,s,v) is an inverse of (y,t,u), then $t\in RegT$ and $s\le t^\circ$.*

Then, with respect to the Cartesian order, $GI(T;L,R;[,])$ forms a G-regular and reflexive naturally ordered abundant semigroup in which each idempotent has a greatest inverse. Conversely, any G-regular and reflexive naturally ordered abundant semigroup in which each idempotent has a greatest inverse can be constructed in the above manner.

3. Proofs

In this section, we give the proof of Theorem 1.

To begin with the proof, we list the following known results which will be useful in the sequel.

Lemma 1 (13). *Let S be an ordered semigroup. Then the following statements are equivalent:*

1. *Every idempotent of S has a greatest inverse.*
2. *Every regular element of S has a greatest inverse.*

Lemma 2 (13, Lemmas 3.2 and 3.3). *Let S be an ordered semigroup and $e, f \in E$. If any idempotent of S has a greatest inverse, then*

1. $(xx^{\circ})^{\circ} = (x^{\circ})^{\circ}x^{\circ}$ *and* $(x^{\circ}x)^{\circ} = x^{\circ}(x^{\circ})^{\circ}$ *for all* $x \in \operatorname{Reg} S$.
2. xx° *[resp.* $x^{\circ}x$*] is the greatest idempotent of* R_x *[resp.* L_x*] for all* $x \in \operatorname{Reg} S$.
3. $x^{\circ\circ\circ} = x^{\circ}$ *for all* $x \in \operatorname{Reg} S$.

Let S be an ordered abundant semigroup in which each regular element has a maximum inverse. If $x \in S$, the greatest idempotent of the $\mathscr{L}$-class [resp. $\mathscr{R}$-class] of S containing the greatest idempotent of R^*_x [resp. L^*_x]is denoted by e_x [resp. f_x]. Clearly, $e_x = e_{x^{\dagger}}$ and $f_x = f_{x^*}$. By Lemma 1, we have $e_x = (x^{\dagger}x^{\dagger^{\circ}})^{\circ}(x^{\dagger}x^{\dagger^{\circ}})$ and $f_x = (x^{*^{\circ}}x^*)(x^{*^{\circ}}x^*)^{\circ}$. Putting $x^{\flat} = e_x x f_x$. Then, it is easy to see that $x^{\flat}$ is the greatest element of the set

$$\{exf : e, f \in E(S) \text{ such that } e\mathscr{L}x^{\dagger}x^{\dagger^{\circ}} \text{ and } f\mathscr{R}x^{*^{\circ}}x^*\}.$$

In what follows, we use the set $S^{\flat}$ to denote the set $\{x^{\flat} : x \in S\}$.

An abundant subsemigroup U of S is said to be a *left* [*right*] **-subsemigroup*[8] of S if for all $a \in U$, there exists $e \in U \bigcap E$ such that $a\mathscr{L}^*(S)e$ [$a\mathscr{R}^*(S)e$]. Furthermore, if U is both a left *-subsemigroup and a right *-subsemigroup, then we call U a **-subsemigroup.*

Lemma 3. *Let S be an ordered abundant semigroup in which each idempotent has a greatest idempotent.*

1. *[13, Lemma 3.4] If* $x \in RegS$*, then* $x^{\flat} = x^{\circ\circ}$.
2. *[13, Theorem 3.6]* $S^{\flat}$ *is a $*$-subsemigroup of S, which is $*$-unipotent, and is a quasi-ideal of S.*
3. *[13, Lemma 5.1]* $M = \{e \in E(S) : e\mathscr{L}\alpha,\ \alpha \in E(S^{\flat})\}$ *is a left regular partial band with skeleton* $E(S^{\flat})$ *while* $N = \{e \in E(S) : e\mathscr{R}\beta,\ \beta \in E(S^{\flat})\}$ *is a right regular partial band with skeleton* $E(S^{\flat})$.

We now state a lemma related to the *-subsemigroup of an abundant semigroup S.

Lemma 4 (8). *Let S be an abundant semigroup and T a *-subsemigroup of S. If $x \in S$ and $a \in T$ such that $x = eaf$ with $e, f \in E(S), e\mathscr{L}a^{\dagger}, f\mathscr{R}a^*$ for $a^{\dagger}, a^* \in T$, then $e\mathscr{R}^*x\mathscr{L}^*f$.*

Lemma 5. *Let S be a naturally ordered abundant semigroup in which each idempotent has a greatest inverse and which is both G-regular and reflexive. Then, the following properties hold:*

1. *For every* $s \in S$, $s^{\circ\flat}$ *is an inverse of* $s^{\flat}$ *satisfying* $t^{\flat} \le s^{\circ\flat}$ *for all inverse t of s.*

2. *For any $x \in S$, there exists a unique $b \in S^\flat$ such that $x = ebf$, where $e, f \in E$, $f \mathscr{R} b^*$ and $e \mathscr{L} b^\dagger$ for $b^\dagger, b^* \in E(S^\flat)$.*

Proof.

1. By Lemma 3, $s^\flat = s^{\circ\circ}$ and by Lemma 2, we obtain the following equalities:

$$\begin{aligned} s^{\circ\flat} &= (s^\circ s^{\circ\circ})^\circ (s^\circ s^{\circ\circ}) s^\circ (s^{\circ\circ} s^\circ)(s^{\circ\circ} s^\circ)^\circ \\ &= (s^{\circ\circ\circ} s^{\circ\circ})(s^\circ s^{\circ\circ}) s^\circ (s^{\circ\circ} s^\circ)(s^{\circ\circ} s^{\circ\circ\circ}) \\ &= s^\circ s^{\circ\circ} s^\circ s^{\circ\circ} s^\circ s^{\circ\circ} s^\circ \\ &= s^\circ s^{\circ\circ} s^\circ = s^\circ, \end{aligned}$$

Hence, $s^\flat s^{\circ\flat} s^\flat = s^{\circ\circ} s^\circ s^{\circ\circ} = s^{\circ\circ} = s^\flat$ and $s^{\circ\flat} s^\flat s^{\circ\flat} = s^\circ s^{\circ\circ} s^\circ = s^\circ = s^{\circ\flat}$, whence $s^{\circ\flat}$ is an inverse of $s^\flat$.

On the other hand, since S is G-reflexive, $e_t \le e_{s^\circ}$ and $f_t \le f_{s^\circ}$ for any inverse t of s. This shows that $t^\flat = e_t t f_t \le e_{s^\circ} s^\circ f_{s^\circ} = s^{\circ\flat}$, as required.

2. For all $x \in S$, we have

$$x = (x^\dagger x^{\dagger\circ})[(x^\dagger x^{\dagger\circ})^\circ (x^\dagger x^{\dagger\circ})] x [(x^{*\circ} x^*)(x^{*\circ} x^*)^\circ](x^{*\circ} x^*),$$

that is, $x = (x^\dagger x^{\dagger\circ}) x^\flat (x^{*\circ} x^*)$. It is easy to see that $x^\dagger x^{\dagger\circ} \mathscr{L} e_x$ and $x^{*\circ} x^* \mathscr{R} f_x$. Now let $x = gbh$ with $g, h \in E(S)$, $g \mathscr{L} b^\dagger$ and $h \mathscr{R} b^*$ for $b^\dagger, b^* \in E(S^\flat)$. Denote $b = y^\flat$. Then, by Lemma 4, $e_y \mathscr{R}^* y^\flat$ and, by [13, Lemma 3.5(3)], $e_y \in S^\flat$. Furthermore, by [13, Lemma 3.5(4)], we have $e_{y^\flat} = e_y = y^{\flat\dagger} = b^\dagger$ and $f_{y^\flat} = f_y = b^*$. By $x = g y^\flat h$, we have $x \le y^{\flat\flat} = y^\flat$, and so $x^\flat \le y^{\flat\flat} = y^\flat$, that is, $x^\flat \le b$. By Lemma 2 (2), $g \le g g^\circ$, hence $g^\circ \le (g g^\circ)^\circ$. Thus, $g^\circ g \le (g g^\circ)(g g^\circ)^\circ$. Again, by Lemma 2 (2), we know that $g^\circ g$ is the greatest idempotent in $L_g = L_{b^\dagger}$. This shows that $b^\dagger \le g^\circ g$ $(\le (g g^\circ)(g g^\circ)^\circ)$. Therefore, $b^\dagger \le (x^\dagger x^{\dagger\circ})^\circ (x^\dagger x^{\dagger\circ})$, that is, $b^\dagger \le e_x$.Dually, $b^* \le f_x$. Thus,

$$b = b^\dagger b b^* = (b^\dagger g) b (h b^*) = b^\dagger x b^* \le e_x x f_x = x^\flat.$$

We have now proved that $b = x^\flat$.

Proof. [Theorem 1]
($\Rightarrow$) Assume that $(T; L, R; [,])$ is a GI-system satisfying the conditions in Theorem 1. By [13, Theorem 4.7], $GI = GI(T; L, R; [,])$ is a naturally ordered abundant semigroup. Now let (x, s, u) be a regular element of GI. Then, there exists $(y, t, v) \in GI$ such that

$$(x, s, u) = (x, s, u)(y, t, v)(x, s, u).$$

By comparing components,

$$s = s[u, y] t [(s[u, y] t)^* v, x] s$$

and s is a regular element of T.

Note that

$$s[u,s^\circ s]s^\circ[ss^\circ,x]s = ss^\circ ss^\circ ss^\circ s = ss^\circ s = s,$$

we have,

$$\begin{aligned}(x,s,u)&(s^\circ s,s^\circ,ss^\circ)(x,s,u)\\ &= \left(x(s[u,s^\circ s]s^\circ[ss^\circ,x]s)^\dagger,\ s[u,s^\circ s]s^\circ[ss^\circ,x]s, (s[u,s^\circ s]s^\circ[ss^\circ,x]s)^* u \right)\\ &= (xs^\dagger,s,s^*u) = (x,s,u)\end{aligned}$$

and similarly,

$$(s^\circ s,s^\circ,ss^\circ)(x,s,u)(s^\circ s,s^\circ,ss^\circ) = (s^\circ s,s^\circ,ss^\circ)\ .$$

Hence, $(s^\circ s,s^\circ,ss^\circ)$ is an inverse of (x,s,u). If (z,p,w) is an inverse of (x,s,u), then by the condition (5), $p \le s^\circ$ and $(z,p,w) \le (s^\circ s,s^\circ,ss^\circ)$. This means that $(s^\circ s,s^\circ,ss^\circ)$ is a greatest inverse of (x,s,u). Thus, GI becomes a naturally ordered abundant semigroup in which each idempotent has a greatest inverse.

If $(x,s,u),(y,t,v) \in GI$ and $(x,s,u) \le (y,t,v)$, then $s \le t$. Hence, $s^\circ \le t^\circ$, so that $ss^\circ \le tt^\circ$ and $s^\circ s \le t^\circ t$. Thus, $(s^\circ s,s^\circ,ss^\circ) \le (t^\circ t,t^\circ,tt^\circ)$, that is, $(x,s,u)^\circ \le (y,t,v)^\circ$. Therefore, GI is reflexive and

$$(x,s,u)(s^\circ s,s^\circ,ss^\circ) \le (y,t,v)(t^\circ t,t^\circ,tt^\circ).$$

From this result and its dual, one can see immediately that GI is a G-regular semigroup.

($\Leftarrow$) Suppose that S is a naturally ordered abundant semigroup in which each idempotent has a greatest inverse which is both G-regular and reflexive. Let M,N and $S^\flat$ have the same meanings as Lemma 3.

Define a mapping

$$\langle,\rangle : N \times M \to S^\flat;\ (u,x) \mapsto \langle u,x\rangle = ux.$$

By routine checking, we can see that $(S^\flat;M,N;\langle,\rangle)$ is a GI-system.

We next prove that the mapping

$$\theta : S \to GI(S^\flat;M,N;\langle,\rangle);\quad s \mapsto (s^\dagger s^{\dagger\circ},s^\flat,s^{*\circ}s^*)$$

is a semigroup isomorphism. By Lemma 4, we have $s^{\flat\dagger} = e_s = (s^\dagger s^{\dagger\circ})^\circ(s^\dagger s^{\dagger\circ})$ and clearly $s^{\flat\dagger}\mathscr{L}s^\dagger s^{\dagger\circ}$. Dually, $s^{\flat *}\mathscr{R}s^{*\circ}s^*$. This means that $(s^\dagger s^{\dagger\circ},s^\flat,s^{*\circ}s^*) \in GI$. In other words, θ is well defined.

- Let $(x,t,u) \in GI(S^\flat;M,N;\langle,\rangle)$. Then, by Lemma 4, $x\mathscr{R}^*xtu\mathscr{L}^*u$. Put $a = xtu$. Again, by Lemma 5(2), $a^\flat = t, a^\dagger a^{\dagger\circ} = x$ and $a^{*\circ}a^* = u$. Hence, $a\theta = (x,t,u)$ and θ is a surjective mapping.

- By Lemma 5(2), we can easily see that θ is an injective mapping.

- If $a,b\in S$, then $a=(a^{\dagger}a^{\dagger^{\circ}})a^{\flat}(a^{*\circ}a^{*})$ and $b=(b^{\dagger}b^{\dagger^{\circ}})b^{\flat}(b^{*\circ}b^{*})$, which imply that

$$\begin{aligned}ab&=(a^{\dagger}a^{\dagger^{\circ}})a^{\flat}(a^{*\circ}a^{*})(b^{\dagger}b^{\dagger^{\circ}})b^{\flat}(b^{*\circ}b^{*})\\&=(a^{\dagger}a^{\dagger^{\circ}})a^{\flat}\langle(a^{*\circ}a^{*}),(b^{\dagger}b^{\dagger^{\circ}})\rangle b^{\flat}(b^{*\circ}b^{*})=(a^{\dagger}a^{\dagger^{\circ}})e_s s f_s(b^{*\circ}b^{*}),\end{aligned}$$

where $s=a^{\flat}\langle(a^{*\circ}a^{*}),(b^{\dagger}b^{\dagger^{\circ}})\rangle b^{\flat}$. On the other hand, by Lemma 4, $e_a\mathscr{R}^*a^{\flat}$ and $e_a s=s$. Hence $e_a s^{\dagger}=s^{\dagger}$, that is, $(a^{\dagger}a^{\dagger^{\circ}})^{\circ}a^{\dagger}a^{\dagger^{\circ}}s^{\dagger}=s^{\dagger}$ and $a^{\dagger}a^{\dagger^{\circ}}s^{\dagger}\mathscr{L}s^{\dagger}$ and dually $s^{*}b^{*\circ}b^{*}\mathscr{R}s^{*}$. Thus, by Lemma 5(2), $(ab)^{\flat}=s,(ab)^{\dagger}(ab)^{\dagger^{\circ}}=a^{\dagger}a^{\dagger^{\circ}}e_s$ and $(ab)^{*\circ}(ab)^{*}=f_s b^{*\circ}b^{*}$. This proves that

$$\begin{aligned}(ab)\theta&=(a^{\dagger}a^{\dagger^{\circ}}e_s,(ab)^{\flat},f_s b^{*\circ}b^{*})\\&=(a^{\dagger}a^{\dagger^{\circ}},a^{\flat},a^{*\circ}a^{*})(b^{\dagger}b^{\dagger^{\circ}},b^{\flat},b^{*\circ}b^{*})=(a\theta)(b\theta).\end{aligned}$$

Thus, we have proved that θ is a semigroup isomorphism, as required.

It remains to show that $(S^{\flat};M,N;\langle,\rangle)$ satisfying the condition (E) in Theorem 1. For this purpose, we need only verify that for any $s\in S$ and for all inverse t of s, $t^{\flat}\le s^{\flat^{\circ}}$ (since θ is an order isomorphism). In fact, by Lemma 5(1), $s^{\circ\flat}$ is an inverse of $s^{\flat}$ satisfying that $t^{\flat}\le s^{\circ\flat}$ for all inverse t of s. But clearly, $s^{\circ\flat}\le s^{\flat^{\circ}}$. Now, we have $t^{\flat}\le s^{\flat^{\circ}}$, as required. Thus, the proof is completed.

4. Examples

In this section, we give some examples of a *-unipotent naturally ordered abundant semigroup in which each idempotent has a greatest inverse and the semigroup is G-regular and reflexive.

Example 1 (9, Example 1.4). *Let* $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $B=\begin{pmatrix}1&0\\1&0\end{pmatrix}$, $C=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $D=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ *and put*

$$T=\{2^mA,2^nB,2^nC,2^nD:\ m\ge 1,\ n\ge 0\}.$$

It is easy to see that T is a semigroup under matrix multiplication. Further, T is generated by B and C and these elements are the only idempotents in T. It is routine to check that the $\mathscr{L}^*$-classes of T are

$$\{2^mA,\ 2^nB:\ m\ge 1,n\ge 0\},\quad \{2^nC,2^nD:\ n\ge 0\}$$

and that the $\mathscr{R}^*$-classes of T are

$$\{2^mA,2^nC:\ m\ge 1,n\ge 0\},\quad \{2^nB,2^nD:\ n\ge 0\}.$$

*Thus, each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class of T contains a unique idempotent but T is not adequate since $BC\ne CB$. On the other hand, $Reg(T)=\{B,C\}$ and obviously, with respective to the trivial order, T is a *-unipotent naturally ordered abundant semigroup in which each idempotent has a greatest inverse and which is G-regular and reflexive. And $E(T)=\{B,C\}$ is a partial semilattice.*

Example 1 shows that Theorem 1 is an extension of the construction theorem previously given by Guo and Xie in [13] on naturally ordered abundant semigroups in which every idempotent has a greatest inverse.

Obviously, the left regular bands are left regular partial bands and right regular bands are right regular partial bands. The following example illustrates that there exist left (right) regular partial bands being not left (right) regular bands and that there exist left (right) regular partial bands being also right (left) regular partial bands.

Example 2. *With the notations of Example 1, it is easy to see that the set* $X = \{A, B, C\}$ *forms a partial semigroup under the matrix multiplication. Putting* $L_B = \{A, B\}$ *and* $L_C = \{C\}$*. Then,* L_B *and* L_C *are both left zero rectangular bands. Since on the set* $E(T)$*,* $\preceq$ *is the identity relation. Thus, we can easily see that* X *satisfies the Condition* $(PB1)$*. Note that in* L*,* $AB = A$*,* $BA = B$*,* $AC = C$ *but* $CA = A$*. We observe that* $X = L_B \cup L_C$ *also satisfies the Condition* $(PB2)$*. Thus, the band* X *is a left regular partial band. Also,* $E(T)$ *is a skeleton of the left regular partial band* X*. If* $R_B = \{B\}$ *and* $R_C = \{A, C\}$*, then by applying the arguments similar to the above, we know the set* $X = R_B \cup R_C$ *is a right regular partial band with* $E(T)$ *as its skeleton.*

Example 3. *Let* $F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ *and denote* $R = \{B, C, F\}$*. Then* R *forms a partial semigroup under matrix multiplication. Set* $R_B = \{B, F\}$ *and* $R_C = \{C\}$*. Now, by computation, we see that* R_B *and* R_C *are both right zero rectangular bands. Note that in* R*, only* B *and* F*, and* F *and* B *are defined. It is easy to see that* R *satisfies the Conditions* $(PB1)$ *and* $(PB2)$*. This shows that* R *is a right regular partial band. On the other hand, it is obvious that* R *is not a left regular partial band.*

Define $[,] : R \times E(T) \to T$ by the rule that

$$[F, B] = B,\ [F, C] = BC,\ [B, B] = B,\ [B, C] = BC,\ [C, B] = CB,\ [C, C] = C.$$

Then, it is trivial to see that $[,]$ satisfies the Conditions $(GI2)$ and $(GI3)$. Compute

$$\begin{aligned} [FB, B] &= [B, B] = B = BB = [F, B][B, B], \\ [FB, C] &= [B, C] = BC = [F, B][F, C], \\ [BF, B] &= [F, B] = B = [B, B][F, B], \\ [BF, C] &= [F, C] = BC = [B, B][F, C]. \end{aligned}$$

But in R, only B and F, and F and B are defined. Thus, $[,]$ satisfies the Conditions $(GI1)$. Therefore, $(T; E(T), R; [,])$ is indeed a GI-system.

Now, define an order on R by the rule that

$$B \le B,\ C \le C,\ F \le F \text{ and } F \le B.$$

Evidently, $\le$ is well defined, and $(R, \le)$ is a naturally ordered right regular partial band satisfying Condition (C) in Theorem 1.

Proposition 1. *If we endow T with the trivial order, then $GI(T;E(T),R;[,])$ forms a naturally ordered abundant semigroup in which each idempotent has a greatest inverse and which is G-regular and reflexive.*

Proof. Obviously, $E(T)$ satisfies the Condition (B) in Theorem 1, under the trivial order. Since $[F,B] = B = [B,B]$ and $[F,C] = BC = [B,C]$, $(T;E(T),R;[,])$ satisfies the Condition (D) in Theorem 1. On the other hand, if $(U,x,V) \in GI(T;E(T),R;[,])$ is regular and (M,y,N) is an inverse of (U,x,V), then

$$(U,x,V)(M,y,N)(U,x,V) = (U,x,V),$$

and by comparing the components, we have $x[V,M]y[W,U]x = x$, where $W = (x[V,M]y)^*N$ so that $x = y = [V,M] = [W,U] \in E(T)$ since $Reg(T) = \{B,C\}$. By using these equalities, we can derive that $U = x, M = y$ and $N,V \in R_x$, and whence $(T;E(T),R;[,])$ satisfies the Condition (E) in Theorem 1. Consequently, the system $GI(T;E(T),R;[,])$ forms a naturally ordered abundant semigroup which is G-regular and reflexive, for which, each idempotent has a greatest inverse. This completes the proof.

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