



Derivative Operators on Quantum Space(3) with Two Parameters and Weyl Algebra

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Abstract. In this paper, we introduced a quantum space generated by three noncommutative coordinates with two commutation parameters. We also give a Hopf algebra structure in order to construct a bicovariant differential calculus over this quantum space. Moreover, it is shown that noncommutative derivative operators corresponding to the coordinates comprise a Weyl algebra deformed by the commutation parameters.

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1. Introduction

There has been a lot of interest in recent years in ‘noncommutative geometry’ or the principle of doing geometry on ‘coordinate rings’ which are noncommutative algebras (and hence could not be the ring of functions on any actual space). Central to most approaches, including that of Connes [8], is the notion of differential structure on a (possibly noncommutative) algebra A , which is expressed directly as the specification of an A - A -bimodule Ω^1 of 1-forms equipped with an exterior derivative operator d obeying the Leibniz rule:

$$d : A \rightarrow \Omega^1; \quad d(fg) = d(f)g + fd(g), \quad f, g \in A. \quad (1)$$

Recall that in classical geometry a differential form can be multiplied by a function; in noncommutative geometry we allow possibly different but mutually noncommuting such multiplications from the left and the right (an A - A -bimodule). The reason is that if one supposed $f(dg) = (dg)f$ for all f, g then one would find $d(fg - gf) = 0$ which would mean a large kernel for a typical noncommutative algebra. In fact, we typically ask that $\ker d$ is 1-dimensional and given by the constant function, which is a connectness condition on the noncommutative

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space. We also require that forms of the form $f(dg)$ span all of Ω^1 as in classical geometry. Higher differential forms can be formulated as a differential graded algebra (or exterior algebra) generated by Ω^1 and $\Omega^0 = A$ with d extended by $d^2 = 0$ and the graded Leibniz rule.

Quantum groups [10, 22, 21, 27, 13, 16, 17] and quantum spaces [22, 21, 28, 24] are explicit realizations of noncommutative spaces. Many studied quantum groups in the context of theory of the integrable models, conformal field theory [29, 15] and the classification of knots and links [30, 12]. In particular, the quantum spaces have been envisioned by many as a paradigm for the general programme of quantum deformed physics. The most hoped for applications include a possible role in a future quantized theory of gravity [20]. For the sake of this hope, many efforts have been accomplished in order to develop noncommutative differential structures on noncommutative spaces [26, 25, 6, 23, 19, 11, 5, 1, 14, 9, 2, 4, 3, 18, 7]. Among them, as a fundamental work, the noncommutative differential calculus on quantum groups is introduced by Woronowicz [26]. In Woronowicz's approach, differential structures on quantum groups is introduced in the context of Hopf algebra. In [18], the graded differential Hopf algebra over a Hopf algebra is constructed by following some results obtained in [26].

In this paper, we define a special quantum(3) equipped with a Hopf algebra structure. Following some ideas of Woronowicz contained in [26], we build a bicovariant differential calculus on this quantum(3) space. Based on this differential calculus, noncommutative derivative operators and the corresponding Weyl algebra are obtained.

2. Preliminary Notes

First we quote briefly some the basic definitions and statements which will be used in the paper.

An algebra is a vector space A over a field K such that the algebra multiplication $m : A \otimes A \rightarrow A$ is a bilinear map satisfying

$$m(a \otimes (b + c)) = m(a \otimes b) + m(a \otimes c) \tag{2}$$

$$m((a + b) \otimes c) = m(a \otimes c) + m(b \otimes c) \tag{3}$$

for all $a, b, c \in A$.

A coalgebra is a K -algebra A , together with linear homomorphisms $\Delta_A : A \rightarrow A \otimes A$ and $\epsilon_A : A \rightarrow K$ (the coproduct and the counit, respectively) which satisfy

$$(\Delta_A \otimes \text{id})\Delta_A(a) = (\text{id} \otimes \Delta_A)\Delta_A(a) \tag{4}$$

$$\mu((\epsilon_A \otimes \text{id})\Delta_A(a)) = \text{id}(a) = \mu'((\text{id} \otimes \epsilon_A)\Delta_A(a)), \tag{5}$$

where $\mu : K \otimes A \rightarrow A$ and $\mu' : A \otimes K \rightarrow A$ are the canonical isomorphisms, defined by

$$\mu(l \otimes a) = la = \mu'(a \otimes l), \quad \forall a \in A, \quad \forall l \in K,$$

and id denotes identity map.

A *bialgebra* is both a unital associative algebra and coalgebra, with the compatibility conditions that Δ_A and ϵ_A are both algebra maps with $\Delta(1_A) = 1_A \otimes 1_A$ and $\epsilon_A(1_A) = 1_K$.

A *Hopf algebra* is a bialgebra A together with a linear map $S_A : A \rightarrow A$, the antipode, which satisfies

$$m((S_A \otimes \text{id})\Delta_A(a)) = \epsilon_A(a)1_A = m((\text{id} \otimes S_A)\Delta_A(a)). \tag{6}$$

Let Ω be a bimodule over any Hopf algebra A and $\Delta_R : \Omega \rightarrow \Omega \otimes A$ be a linear homomorphism. One says that (Ω, Δ_R) is a *right-covariant bimodule* if

$$\Delta_R(ap + p'a') = \Delta_A(a)\Delta_R(p) + \Delta_R(p')\Delta_A(a') \tag{7}$$

for all $a, a' \in A$ and $p, p' \in \Omega$ and

$$(\Delta_R \otimes \text{id}) \circ \Delta_R = (\text{id} \otimes \Delta_A) \circ \Delta_R, \quad m \circ (\text{id} \otimes \epsilon_A) \circ \Delta_R = \text{id}. \tag{8}$$

Let $\Delta_L : \Omega \rightarrow A \otimes \Omega$ be a linear homomorphism. One says that (Ω, Δ_L) is a *left-covariant bimodule* if

$$\Delta_L(ap + p'a') = \Delta_A(a)\Delta_L(p) + \Delta_L(p')\Delta_A(a')$$

for all $a, a' \in A$ and $p, p' \in \Omega$ and

$$(\text{id} \otimes \Delta_L) \circ \Delta_L = (\Delta_A \otimes \text{id}) \circ \Delta_L, \quad m \circ (\epsilon_A \otimes \text{id}) \circ \Delta_L = \text{id}.$$

3. Quantum (3) Space with Two Parameters and its Hopf algebra

According to Manin's terminology [22], we define the quantum(3) space with two parameters as a finitely generated quadratic algebra

$$A = \mathbb{C} \langle x, y, z \rangle / I, \tag{9}$$

where I is an ideal generated by the following relations

$$xy = pyx, \quad xz = qzx, \quad yz = p^{-n}q^mzy, \quad m, n \in \mathbb{Z}, \tag{10}$$

where p and q are non-zero complex parameters. If we require that x is invertible, then we could provide a Hopf algebra on A by the following maps

$$\Delta(x) = x \otimes x, \quad \Delta(y) = x^m \otimes y + y \otimes x^m, \quad \Delta(z) = x^n \otimes z + z \otimes x^n \tag{11}$$

$$\epsilon(x) = 1, \quad \epsilon(y) = 0, \quad \epsilon(z) = 0 \tag{12}$$

$$S(x) = x^{-1}, \quad S(y) = x^{-m}yx^{-m}, \quad S(z) = x^{-n}zx^{-n}, \tag{13}$$

which satisfy the axioms (4), (5) and (6).

4. Differential algebra over A

To construct exterior algebra of differential n -forms, we need commutation relations between x, y, z and dx, dy, dz . For this, we first assume the following noncommutative relations:

$$x dx = a dx x, \quad x dy = f_{11} dy x + f_{12} dx y, \quad x dz = g_{11} dz x + g_{12} dx z \tag{14}$$

$$y dx = f_{21} dx y + f_{22} dy x, \quad y dy = b dy y, \quad y dz = h_{11} dz y + h_{12} dy z \tag{15}$$

$$z dx = g_{21} dx z + g_{22} dz x, \quad z dy = h_{21} dy z + h_{22} dz y, \quad z dz = c dz z, \tag{16}$$

where $a, b, c, f_{ij}, g_{ij}, h_{ij} \in \mathbb{C}/\{0\}$ for $i, j = 1, 2$. It is natural to consider that the commutation coefficients must depend on the deformation parameters p, q . To see this, one gives the bicovariant structure [26] on the Hopf algebra A ; (Ω_1, Δ_R) is a *right-covariant bimodule* on the Hopf algebra A under the following linear homomorphism Δ_R defined as

$$\Delta_R : \Omega_1 \longrightarrow \Omega_1 \otimes A \tag{17}$$

$$\Delta_R(da) = (d \otimes \text{id})\Delta(a), \forall a \in A. \tag{18}$$

Thus, it acts on dx, dy, dz in the following form:

$$\Delta_R(dx) = dx \otimes x \tag{19}$$

$$\Delta_R(dy) = \sum_{k=0}^{m-1} a^k dx x^{m-1} \otimes y + dy \otimes x^m \tag{20}$$

$$\Delta_R(dz) = \sum_{k=0}^{n-1} a^k dx x^{n-1} \otimes z + dz \otimes x^n, \tag{21}$$

where Δ_R acts on A as Δ . In similar way, a *left-covariant bimodule* structure on the Hopf algebra A is defined in the following way

$$\Delta_L : \Omega_1 \longrightarrow A \otimes \Omega_1 \tag{22}$$

$$\Delta_L(da) = (\text{id} \otimes d)\Delta(a), \forall a \in A. \tag{23}$$

Hence, using the fact that Δ_L and Δ_R preserve the commutation relations (14-16) and consistency of the exterior differential operator d with the noncommutative relations (10), we obtain the commutation coefficients in terms of p and q as follows

$$x dx = dx x, \quad x dy = p dy x, \quad x dz = q dz x \tag{24}$$

$$y dx = p^{-1} dx y, \quad y dy = dy y, \quad y dz = p^{-n} q^m dz y \tag{25}$$

$$z dx = q^{-1} dx z, \quad z dy = p^n q^{-m} dy z, \quad z dz = dz z. \tag{26}$$

Applying the exterior differential operator to the noncommutative relations in (24-26), one has

$$dx \wedge dx = 0, \quad dy \wedge dy = 0, \quad dz \wedge dz = 0 \tag{27}$$

and

$$dx \wedge dy = -pdy \wedge dx \tag{28}$$

$$dx \wedge dz = -qdz \wedge dx \tag{29}$$

$$dy \wedge dz = -p^{-n}q^m dz \wedge dy. \tag{30}$$

Finally, we have

$$\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2 \oplus \Omega_3 \oplus 0 \oplus 0 \dots \tag{31}$$

and the following coproduct yields a graded Hopf algebra over Ω

$$\hat{\Delta} = \Delta_L + \Delta_R, \tag{32}$$

which implies

$$\hat{\Delta}(dx) = dx \otimes x + x \otimes dx \tag{33}$$

$$\hat{\Delta}(dy) = mdxx^{m-1} \otimes y + dy \otimes x^m + x^m \otimes dy + y \otimes mdxx^{m-1} \tag{34}$$

$$\hat{\Delta}(dz) = ndxx^{n-1} \otimes z + dz \otimes x^n + x^n \otimes dz + z \otimes ndxx^{n-1}. \tag{35}$$

Note that the multiplication of two elements in $\Omega \otimes \Omega$ is given by the graded tensor product as follows

$$(X \otimes Y)(Z \otimes T) = (-1)^{\widehat{Z}} XZ \otimes YT, \tag{36}$$

where \widehat{w} , the parity of a differential n - form w in Ω , is given by $\widehat{w} = n$.

Partial derivative operators corresponding to the differantial calculus (24-26) act on A as follows

$$\begin{aligned} \partial_x(x^i y^j z^k) &= ix^{i-1} y^j z^k \\ \partial_y(x^i y^j z^k) &= jp^i x^i y^{j-1} z^k \\ \partial_z(x^i y^j z^k) &= kp^{-nj} q^{mj+i} x^i y^j z^{k-1}. \end{aligned} \tag{37}$$

To show the actions (37), let $f \in A$. From (24-26) and the Leibniz rule there exists the unique $f_a \in A$, $a \in \{x, y, z\}$ such that

$$d(f) = dx f_x + dy f_y + dz f_z. \tag{38}$$

We, therefore, could assume that there exists a linear operator $\partial_a : A \rightarrow A$ such that $\partial_a(f) = f_a$. Thus, the differential operator d could be then given by

$$d = dx \partial_x + dy \partial_y + dz \partial_z.$$

Hence, the action of the derivative operator ∂_a on the monomial $x^i y^j z^k$ is deduced by applying the Leibniz rule inductively to $x^i y^j z^k$ and substituting differential calculus (24-26) as follows

$$\begin{aligned} d(x^i y^j z^k) &= d(x^i y^j) z^k + x^i y^j d(z^k) \\ &= dx(ix^{i-1} y^j z^k) + dy(jp^i x^i y^{j-1} z^k) \end{aligned}$$

$$+ dz(kp^{-nj}q^{mj+i}x^i y^j z^{k-1}).$$

This results in (37), and we extend the action of ∂_a on the monomial to f by its linearity. When $p, q \rightarrow 1$, the algebra A becomes the usual commutative algebra and these operators reduce to the usual partial derivative operators.

To obtain Weyl algebra corresponding to these operators, we need commutation relations between x, y, z and the corresponding operators. Let $f \in A$. From the Leibniz rule, we have

$$d(xf) = dx f + x(dx\partial_x + dy\partial_y + dz\partial_z)(f). \tag{39}$$

Substituting (24) to (39) implies

$$(dx\partial_x + dy\partial_y + z + \partial_z)(xf) = [dx(1 + x\partial_x) + pdyx\partial_y + qdzx\partial_z](f). \tag{40}$$

This last equation results in

$$\partial_x x = 1 + x\partial_x, \partial_y x = px\partial_y, \partial_z x = qx\partial_z. \tag{41}$$

In similar way, the following relations could be obtained

$$\partial_x y = p^{-1}y\partial_x, \partial_y y = 1 + y\partial_y, \partial_z y = p^{-n}q^m y\partial_z \tag{42}$$

$$\partial_x z = q^{-1}x\partial_x, \partial_y z = p^n q^{-m} z\partial_y, \partial_z z = 1 + z\partial_z. \tag{43}$$

Moreover, using the nilpotency rule $d^2 = 0$ and (27-30) implies relations:

$$\partial_x \partial_y = p\partial_y \partial_x, \partial_x \partial_z = q\partial_z \partial_x, \partial_y \partial_z = p^n q^{-m} \partial_z \partial_y, \tag{44}$$

which are compatible with the actions (37); for example,

$$\begin{aligned} (\partial_x \partial_y)(x^i y^j z^k) &= \partial_x(\partial_y(x^i y^j z^k)) \\ &= \partial_x(j p^i x^i y^{j-1} z^k) \\ &= j i p^i x^{i-1} y^{j-1} z^k \\ &= p i p^{i-1} j x^{i-1} y^{j-1} z^k \\ &= p \partial_y(i x^{i-1} y^j z^k) \\ &= p \partial_y(\partial_x(x^i y^j z^k)) \\ &= (p \partial_y \partial_x)(x^i y^j z^k). \end{aligned}$$

Finally, one could easily see Weyl algebra corresponding to A as $\mathbb{C} \langle x, y, z, \partial_x, \partial_y, \partial_z \rangle$ modulo the commutation relations (10) and (41-44). We also get the usual Weyl algebra in three commutative variables when $p, q \rightarrow 1$.

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References

- [1] N Aizawa and R Chakrabarti. Noncommutative geometry of super-Jordanian $OSph(2/1)$ covariant quantum space, *J. Math. Phys.*, 45:1623-1638, 2004.
- [2] E Baz, Gauge theory on a four-dimensional quantum space *Modern Phys. Letters A*, 21(30): 2323-2330, 2006
- [3] W Behr and A Sykora, Construction of gauge theories on curved noncommutative space-time, *Nuclear Physics B*, 3:473-502, 2004.
- [4] Z Bentalha and M Tahiri, A new approach in bicovariant differential calculus on $SU_q(2)$, *Int. J. Geo.M. .Mod.Phys.*, 4: 1087-1097, 2007.
- [5] P Bouwknegt, J McCarthy and P Nieuwenhuizen. Fusing the coordinates of quantum superspace, *Phys.Lett.B*, 394 : 82-86, 1997.
- [6] T Brzezinski, Quantum group related to the space of differential operators on the quantum hyperplane, *J. Phys.A.Math Gen.*, 26 : 905-911, 1993.
- [7] S Celik and E Yasar, Differential Geometry of the quantum 3-dimensional space, *Czech. J.Phys.*, 56:229-236, 2006
- [8] A Connes, Noncommutative differential geometry *Institut des Hautes Etudes Scientifiques.Extrait des Publications Mathematiques*, 1986
- [9] R Coquereaux, A Garcia and R Trinchero. Differential calculus and connections on a quantum plane at a cubic root of unity, *Reviews in Mathematical Phys.*, 12: 227-285, 2000.
- [10] V Drinfeld, Quantum groups in Proceedings of the International congress of mathematicians Berkeley, CA, 1986.
- [11] A El-Hassouni Y Hassouni and E Tahri, Differential calculi on the quantum superplane, *Int J. Th. Phys.*, 35:2517-2525, 1996.
- [12] E El-Rifai, A Hegazi and E Ahmed, *Int. J.Theori.Phys.*, 37:2757,1998.
- [13] L Fadeev, N Reshetikin and L Takhtajan, Quantization of Lie groups and Lie algebras, *Algebraic Analysis*, 129-139 ,1989.
- [14] E Falaki and E Tahri, Quantum supergroup structure of (1+1)-dimensional quantum superplane, its dual and its differential calculus ,*J.Phys.A;Math.Gen.*,34: 3403, 2001
- [15] J Frohlich, Statistic of fields, the Yang Baxter equation, and the theory of knots and links. Zurich Preprint, 1987.
- [16] M Jimbo, *Lett. Math. Phys.*, 10(1):63-69,1985.

- [17] M Jimbo, *Lett. Math. Phys.*, 11(3):247-252, 1986
- [18] A Klimyk and K Schmüdgen, *Quantum Groups and their Representations*, Springer-Verlag, Heidelberg, 1997
- [19] T Kobayashi and T Uematsu, Differential calculus on the quantum superspace and deformation of phase space, *Z. Phys.*, C 56:193-200, 1992.
- [20] S Majid, *Fondation of Quantum Group Theory*. Cambridge. Cambridge University Press, 1995.
- [21] Y Manin, Multiparametric quantum deformation of the general Linear supergroup, *Comm. Math. Phys.*, 123(1):163-175, 1989.
- [22] Y Manin, Quantum groups and noncommutative geometry *Preprint Montreal Uni, CRM*, 1988.
- [23] A Sudbery, Canonical differential calculus on quantum general linear groups and supergroups, *Physics Lett. B*, 284: 61-65, 1992.
- [24] R Ubricco, Noncommutative differential calculus and q-analysis. *J. Phys. A; Math. Gen.*, 25(1): 169-174, 1992.
- [25] J Wess and B Zumino, Covariant differential calculus on the quantum hyperplane, *Nucl. Phys.*, 18(2): 302-312, 1990.
- [26] S Woronowicz, Differential calculus on compact matrix pseudogroups, *Commun. Math. Phys.*, 122(1):125-170, 1989.
- [27] S Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.*, 111(4):613-665, 1987.
- [28] S Woronowicz, Pseudospaces, Pseudogroups and Pontriagin duality, *Lect. Notes in Phys.*, 116: 407-412, 1980.
- [29] C Yang, *Braid group, knots theory and statistical mechanics I-II*. Singapore: World Scientific, 1994.
- [30] R Zhang, M Gould, and A Bracken. University of Queensland, Preprint, 1989.