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Subordination Results For Spirallike Functions

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Abstract. In this paper, we introduce a new class of functions which is defined by Dziok-Srivastava operator and obtain the subordination results for this class of functions. Some known and new results, which follow as special cases of our results, have also been mentioned.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.2}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \qquad (1.3) \\ (l \leq m+1;\ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U) \end{aligned}$$

where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} 1, & n = 0 \\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & n \in N. \end{cases} \tag{1.4}$$

Let $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : A \to A$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(f)](z) \quad &:= \quad z\ {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z) \\ &= \quad z + \sum_{n=2}^{\infty} \Gamma_n\, a_n z^n \qquad (1.5) \end{aligned}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(n-1)!(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}}\ . \tag{1.6}$$

For notational simplicity, we can use a shorter notation
$H_m^l[\alpha_1, \beta_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel. The linear operator $H_m^l[\alpha_1, \beta_1]$ is called Dziok-Srivastava operator (see [3]), includes (as its special cases) various other

linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [2], Libera [4], Livingston [6], Ruscheweyh [7] and Srivastava-Owa [11].

For $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $\frac{-\pi}{2} < \eta < \frac{\pi}{2}$, we let $\mathbb{R}_m^l(\eta, \gamma, \lambda)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{e^{i\eta}\frac{z(H_m^l[\alpha_1,\beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'}\right\} > \gamma\cos\eta, \quad z \in U, \tag{1.7}$$

where $H_m^l[\alpha_1, \beta_1]f(z)$ is given by (1.5).

Several known and new subclasses can be obtained from the class $\mathbb{R}_m^l(\eta, \gamma, \lambda)$, by suitably specializing the values of l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, λ, γ and η. We present below some of these subclasses of $\mathbb{R}_m^l(\eta, \gamma, \lambda)$ consisting of functions of the form (1.1). We observe that

Example 1.1. *If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ then*

$$\begin{aligned}&\mathbb{R}_1^2(\eta,\gamma,\lambda) \equiv \mathbb{S}(\eta,\gamma,\lambda)\\ &:= \left\{f \in A : Re\ \left\{e^{i\eta}\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\} > \gamma\cos\eta,\ |\eta| < \frac{\pi}{2},\ 0 \leq \gamma < 1,\ z \in U\right\}.\end{aligned}$$

Also $\mathbb{R}_1^2(\eta, \gamma, 0) \equiv \mathbb{S}(\eta, \gamma)$ denotes the $\eta-$spirallike functions of order γ studied by Libera [5]. Further $\mathbb{R}_1^2(\eta, 0, 0) \equiv \mathbb{S}(\eta)$, $|\eta| < \frac{\pi}{2}$. Spacek [10] proved that the members of $\mathbb{S}(\eta)$, known as $\eta-$spirallike functions, are univalent in U.

Example 1.2. *If $l = 2$ and $m = 1$ with $\alpha_1 = \delta + 1\,(\delta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, then*

$$\begin{aligned}&\mathbb{R}_1^2(\eta,\gamma,\lambda) \equiv \mathbb{D}_\delta(\eta,\gamma,\lambda)\\ &:= \left\{f \in A : Re\ \left\{e^{i\eta}\frac{z(D^\delta f(z))'}{(1-\lambda)D^\delta f(z)+\lambda z(D^\delta f(z))'}\right\} > \gamma\cos\eta,\right.\\ &\left.|\eta| < \frac{\pi}{2},\ 0 \leq \gamma < 1,\ z \in U\right\},\end{aligned}$$

where $D^\delta f(z)$ is called Ruscheweyh derivative operator [7] defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv H_1^2(\delta+1, 1; 1)f(z).$$

Example 1.3. *If $l = 2$ and $m = 1$ with $\alpha_1 = \mu + 1 (\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu + 2$, then*

$$\begin{aligned}&\mathbb{R}_1^2(\eta, \gamma, \lambda) \equiv B_\mu(\eta, \gamma, \lambda)\\ &:= \left\{ f \in A : Re\left\{ e^{i\eta} \frac{z(J_\mu f(z))'}{(1-\lambda)J_\mu f(z) + \lambda z(J_\mu f(z))'} \right\} > \gamma \cos\eta, \ |\eta| < \frac{\pi}{2}, \right.\\ &\left. 0 \leq \gamma < 1, \ z \in U \right\},\end{aligned}$$

where J_μ is a Bernardi operator [1] defined by

$$J_\mu f(z) := \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \equiv H_1^2(\mu+1, 1; \mu+2) f(z).$$

Note that the operator J_1 was studied earlier by Libera [4] and Livingston [6].

Example 1.4. *If $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c > 0)$, then*

$$\begin{aligned}&\mathbb{R}_1^2(\eta, \gamma, \lambda) \equiv L_c^a(\eta, \gamma, \lambda)\\ &:= \left\{ f \in A : Re\left\{ e^{i\eta} \frac{z(L(a,c)f(z))'}{(1-\lambda)L(a,c)f(z) + \lambda z(L(a,c)f(z))'} \right\} > \gamma \cos\eta, \ |\eta| < \frac{\pi}{2}, \right.\\ &\left. 0 \leq \gamma < 1, \ z \in U \right\},\end{aligned}$$

where $L(a,c)$ is a well-known Carlson-Shaffer linear operator [2] defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) * f(z) \equiv H_1^2(a, 1; c) f(z).$$

The object of the present paper is to investigate the coefficient estimates and subordination properties for the class of functions $\mathbb{R}_m^l(\eta, \gamma, \lambda)$. Some interesting consequences of the results are also pointed out.

2. Main Results

To prove our results we need the following definitions and lemmas.

Definition 2.1. *For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.*

Definition 2.2. *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in U. \tag{2.1}$$

In 1961, Wilf [12] proved the following subordinating factor sequence.

Lemma 2.1. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re \left\{ 1 + 2\sum_{n=1}^{\infty} b_n z^n \right\} > 0, \quad z \in U. \tag{2.2}$$

Next we obtain the coefficient inequality theorem for the class $\mathbb{R}_m^l(\eta, \gamma, \lambda)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathbb{R}_m^l(\eta, \gamma, \lambda)$ if*

$$\sum_{n=2}^{\infty} [(1-\lambda)(n-1)\sec\eta + (1-\gamma)(1+n\lambda-\lambda)]\Gamma_n \, |a_n| \le 1-\gamma, \tag{2.3}$$

where $|\eta| < \frac{\pi}{2}$, $0 \le \lambda < 1$, $0 \le \gamma < 1$ and Γ_n is given by (1.6).

Proof. Suppose the inequality (2.3) holds true. Then we get,

$$\begin{aligned}
&\left| z(H_m^l[\alpha_1, \beta_1]f(z))' - [(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'] \right| \\
&\quad - (1-\gamma)\cos\eta \left| [(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'] \right| \\
&\le \left| \sum_{n=2}^{\infty} [(n-1)(1-\lambda)a_n \Gamma_n z^n] \right| - (1-\gamma)\cos\eta \left| z + \sum_{n=2}^{\infty} (1+n\lambda-\lambda)a_n \Gamma_n z^n] \right| \\
&\le \sum_{n=2}^{\infty} (n-1)(1-\lambda)|a_n|\Gamma_n|z|^n - (1-\gamma)\cos\eta|z| + \sum_{n=2}^{\infty} (1-\gamma)\cos\eta(1+n\lambda-\lambda)|a_n|\Gamma_n|z|^n
\end{aligned}$$

By taking $z \to 1$ on the real axis we obtain

$$\leq \sum_{n=2}^{\infty}[(n-1)(1-\lambda)+(1-\gamma)\cos\eta(1+n\lambda-\lambda)]|a_n|\Gamma_n-(1-\gamma)\cos\eta$$
$$\leq 0.$$

This completes the proof of the Theorem 2.1.

In the view of Examples 1.1 to 1.4, we state the following corollaries.

Corollary 2.1. *A function $f(z)$ of the form (1.1) is in $\mathbb{S}(\eta,\gamma,\lambda)$ if*

$$\sum_{n=2}^{\infty}[(1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda)]\;|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\lambda<1$ and $0\leq\gamma<1$.

Remark 2.1. *We observe that Corollary 2.1, yields the result of Silverman [8] for the special values of η, λ and γ.*

Corollary 2.2. *A function $f(z)$ of the form (1.1) is in $\mathbb{D}_\delta(\eta,\gamma,\lambda)$ if*

$$\sum_{n=2}^{\infty}[(1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda)]\;\frac{(\delta+1)\ldots(\delta+n-1)}{(n-1)!}\;|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\lambda<1$, $0\leq\gamma<1$ and $\delta>-1$.

Corollary 2.3. *A function $f(z)$ of the form (1.1) is in $B_\mu(\eta,\gamma,\lambda)$ if*

$$\sum_{n=2}^{\infty}[(1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda)]\;\left(\frac{\mu+1}{\mu+n}\right)\;|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\lambda<1$, $0\leq\gamma<1$ and $\mu>-1$.

Corollary 2.4. *A function $f(z)$ of the form (1.1) is in $L_c^a(\eta,\gamma,\lambda)$ if*

$$\sum_{n=2}^{\infty}[(1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda)]\;\frac{(a)_{n-1}}{(c)_{n-1}}\;|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\lambda<1$, $0\leq\gamma<1$ and $a>0$, $c>0$.

Next we obtain the subordination result for the class $\mathbb{R}^l_m(\eta,\gamma,\lambda)$.

Theorem 2.2. *Let $f \in \mathbb{R}^l_m(\eta,\gamma,\lambda)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}(f*g)(z) \prec g(z) \tag{2.4}$$

where $|\eta| < \frac{\pi}{2}$, $0 \le \gamma < 1$; $0 \le \lambda < 1$, *with*

$$\Gamma_2 = \frac{\alpha_1 \ldots \alpha_l}{\beta_1 \ldots \beta_m} \tag{2.5}$$

and

$$Re\ \{f(z)\} > -\frac{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2},\ \ z \in U. \tag{2.6}$$

The constant factor $\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}$ *in (2.4) cannot be replaced by a larger number.*

Proof. Let $f \in \mathbb{R}^l_m(\eta,\gamma,\lambda)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in C$. Then

$$\begin{aligned}
&\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}(f*g)(z)\\
&\quad= \frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}\left(z+\sum_{n=2}^{\infty} c_n a_n z^n\right).
\end{aligned} \tag{2.7}$$

Thus, by Definition 2.2, the subordination result holds true if

$$\left\{\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 2.1, this is equivalent to the following inequality

$$\text{Re}\ \left\{1+\sum_{n=1}^{\infty}\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}a_n z^n\right\} > 0,\ \ z \in U. \tag{2.8}$$

By noting the fact that $\frac{((1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda))\Gamma_n}{(1-\gamma)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{(1-\gamma)} \leq \frac{((1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda))\Gamma_n}{(1-\gamma)},$$
$$n \geq 2,\ |\eta| < \frac{\pi}{2},$$

therefore, for $|z| = r < 1$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}\sum_{n=1}^{\infty}a_n z^n\right\}\\
&=\mathrm{Re}\left\{1+\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}z+\right.\\
&\left.\frac{\sum\limits_{n=2}^{\infty}((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2 a_n z^n}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}\right\}\\
&\geq 1-\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}r\\
&\quad-\frac{1}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}\times\\
&\sum_{n=2}^{\infty}((1-\lambda)(n-1)\sec\eta+(1-\gamma)(1+n\lambda-\lambda))\Gamma_n|a_n|r^n\\
&\geq 1-\frac{((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}r-\\
&\frac{1-\gamma}{[1-\gamma+((1-\lambda)\sec\eta+(1-\gamma)(1+\lambda))\Gamma_2]}r\\
&>0,\quad |z|=r<1,
\end{aligned}$$

where we have also made use of the assertion (2.3) of Theorem 2.1. This evidently proves the inequality (2.8) and hence also the subordination result (2.4) asserted by Theorem 2.2. The inequality (2.6) follows from (2.4) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2} z^2$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$, $0 \leq \lambda < 1$ and Γ_2 is given by (2.5). Clearly $F \in \mathbb{R}^l_m(\eta, \gamma, \lambda)$. For this function (2.4)becomes

$$\frac{((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma + ((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\text{Re}\ \left(\frac{((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma + ((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2]} F(z)\right)\right\} = -\frac{1}{2}, \quad z \in U.$$

This shows that the constant $\frac{((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2}{2[1-\gamma + ((1-\lambda)\sec\eta + (1-\gamma)(1+\lambda))\Gamma_2]}$ cannot be replaced by any larger one.

By taking different choices of l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, λ, γ and η in the above theorem and in view of the Examples 1 to 4 in Section 1, we state the following corollaries for the subclasses defined in those examples.

Corollary 2.5. *If $f \in \mathbb{S}(\eta, \gamma, \lambda)$, then*

$$\frac{(1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)}{2[1-\gamma + (1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)]} (f * g)(z) \prec g(z) \tag{2.9}$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$; $0 \leq \lambda < 1$, $g \in C$ and

$$Re\ \{f(z)\} > -\frac{[1-\gamma + (1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)]}{(1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)}, \quad z \in U.$$

The constant factor $\frac{(1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)}{2[1-\gamma + (1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)]}$ in (2.9) cannot be replaced by a larger one.

Corollary 2.6. *If $f \in \mathbb{D}_\delta(\eta, \gamma, \lambda)$, then*

$$\frac{(\delta+1)[(1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)]}{2[1-\gamma + (\delta+1)\{(1-\lambda)\sec\eta + (1-\gamma)(1+\lambda)\}]} (f * g)(z) \prec g(z), \tag{2.10}$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$; $0 \leq \lambda < 1$, $\delta > -1$, $g \in C$ *and*

$$Re\ \{f(z)\} > -\frac{[1-\gamma+(\delta+1)\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}{(\delta+1)[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}, \quad z \in U.$$

The constant factor $\frac{(\delta+1)[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}{2[1-\gamma+(\delta+1)\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}$ *in (2.10) cannot be replaced by a larger one.*

Corollary 2.7. *If* $f \in B^*_\mu(\eta,\gamma,\lambda)$, *then*

$$\frac{(\mu+1)[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}{2[(\mu+2)(1-\gamma)+(\mu+1)\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}(f*g)(z) \prec g(z), \quad (2.11)$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$; $0 \leq \lambda < 1$, $\mu > -1$, $g \in C$ *and*

$$Re\ \{f(z)\} > -\frac{[(\mu+2)(1-\gamma)+(\mu+1)\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}{(\mu+1)[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}, \quad z \in U.$$

The constant factor $\frac{(\mu+1)[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}{2[(\mu+2)(1-\gamma)+(\mu+1)\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}$ *in (2.11) cannot be replaced by a larger one.*

Corollary 2.8. *If* $f \in L^{*a}_c(\eta,\gamma,\lambda)$, *then*

$$\frac{a[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}{2[c(1-\gamma)+a\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}(f*g)(z) \prec g(z), \quad (2.12)$$

where $|\eta| < \frac{\pi}{2}$, $0 \leq \gamma < 1$; $0 \leq \lambda < 1$, $a > 0$, $c > 0$, $g \in C$ *and*

$$Re\{f(z)\} > -\frac{[c(1-\gamma)+a\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]}{a[(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)]}, \quad z \in U.$$

The constant factor

$$2[c(1-\gamma)+a\{(1-\lambda)\sec\eta+(1-\gamma)(1+\lambda)\}]$$

in (2.12) cannot be replaced by a larger one.

Remark 2.2. *We observe that Corollary 2.5, yields the results obtained by Singh [9] for the special values of* λ,γ *and* η.

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