EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 4, 2011, 340-360
ISSN 1307-5543 – www.ejpam.com

An Iterative Method for $(AGDDVIP)$ in Hilbert Space and the Homology Theory to Study the $(GDDCP_n)$ in Riemannian n-manifolds in the Presence of Fixed Point Inclusion

Prasanta Kumar Das

School of Applied Science (Mathematics), KIIT University, Bhubaneswar, Orissa, 721024, India

Abstract. The main purpose of this paper is to study the convergence of variable step iterative methods for the defined problem absolutely generalized dominated differential variational inequality problems $(AGDDVIP)$ in Hilbert spaces. The iterative process considered in the paper admit the presence of variable iteration parameters, which can be useful in numerical implementation to find solution of the problem $(AGDDVIP)$. Finally, we study the existence theorems of the problems $(GDDVIP_n)$ and $(GDDCP_n)$ in Riemannian n-manifolds modelled on the Hilbert space in the presence of coincidence index, fixed point theorem of Homology theory and one-point compactification of Topology theory.

2000 Mathematics Subject Classifications: 65K10; 90C33; 47J30

Key Words and Phrases: Quasidomonotone and potential operator, weakly η-invex set, T-η-invex function, Hilbert spaces, Banach space, iterative sequence, Lipschitz function, generalized dominated differential variational inequality problems, absolutely generalized dominated differential variational inequality problems, Maximal fixed point open set, one-point compactification, n-manifold, Riemannian n-manifolds, η-closed, η-invex set, weakly η-invex set, η-invex cone, complete w.r.t. η, tangent bundle, cotangent bundle, coincidence index set, and fixed-point index.

1. Introduction

In the recent decades, there has been a great deal of development in the theory of optimization techniques. The study of variational inequalities is a part of development in the theory of optimization theory because optimization problems can often be reduced to the solution of variational inequalities. Variational inequality theory has emerged as a powerful tool for wide class of unrelated problems arising in various branches of physical, engineering, pure and applied sciences in a unified and general frame work (see for example [14], [15]). In this development, computer science has played a vital role for making it possible to implement such techniques for everyday use as well as stimulating new effort for finding solutions of much more complicated problems. Several authors have proved many fascinating results on

Email address: `dasprasantkumar@yahoo.co.in`, (P. Das)

variational inequality problems. We list some of them, which are used frequently in this paper. The existence of the solution to the problem is studied by many authors such as, J.L. Lions and G. Stampacchia [21], R.W. Cottle, F. Giannessi and J.L. Lions [24] to name only a few. The most general and popular forms of inequality with very reasonable conditions are due to F. E. Browder [7], D. Kinderlehrer and G. Stampacchia [20], C. Bardaro and R. Ceppitelli [2], M. Chipot [9], A. Behera and G. K. Panda [4, 6].

In 1994, A. Hassouni and A. Moudafi [17] introduced and studied a class of variational inclusions and developed an iterative algorithm for the variational inclusions. S. Adly [1], N. J. Haung [18], X. P. Ding [12, 13] and K. R. Kazmi [19], have obtained some important extensions of the result [17] in Hilbert spaces.

The *Variational Inequality Problem*(VIP) is defined as follows.

Let X be a reflexive real Banach space with its dual X^*. Let K be a nonempty subset of X. Let $T : K \to X^*$ be a nonlinear mapping. Let $\langle f, x\rangle$ denote the value of $f \in X^*$ at $x \in K$. Then, the variational inequality problem is to:
(VIP) Find $x_0 \in K$ such that

$$\langle T(x_0), x - x_0\rangle \geq 0 \quad \forall x \in K.$$

The generalized variational inequality problem $(GVIP)$ and generalized complementarity problem (GCP) are defined as follows.

Let X be a reflexive real Banach space with its dual X^* and K be any nonempty subset of X. Let $\eta : K \times K \to X$ be a vector valued continuous mapping. Let $T : K \to X^*$ be a nonlinear mapping. Then, the generalized variational inequality problem is defined by:
$(GVIP)$ Find $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle \geq 0 \quad \forall x \in K,$$

and generalized complementarity problem (GCP) is to:
(GCP) Find $x_0 \in K$ such that

$$\langle T(x_0), \eta(x, x_0)\rangle = 0 \quad \forall x \in K, \tag{1}$$

For our need, we recall some known definitions and results.

Definition 1 ([9]). *A mapping $T : K \to \mathbb{R}$ is said to be monotone if*

$$\langle T(u) - T(v), u - v\rangle \geq 0 \quad \forall u, v \in K,$$

and T is strictly monotone if equality holds for $u = v$.

Definition 2 ([10]). *A mapping $F : K \to \mathbb{R}$ is said to be Lipschitz near each point of K with rank $M > 0$ if*

$$|F(v) - F(x)| \leq M|v - x| \quad \forall v, x \in K.$$

Theorem 1 ([9, Theorem 1.4, p.3]). *Let K be a compact convex subset of a finite dimensional Banach space X with dual X^* and T a continuous mapping of K into X^*. Then there exists $x_0 \in K$ such that for all $y \in K$,*

$$\langle T(x_0), y - x_0 \rangle \geq 0.$$

Theorem 2 ([5, Theorem 5.1, p.900]). *Let K be a closed, convex and bounded subset of a reflexive real Banach space X and X^* be the dual of X. Let $T : K \to X^*$ and $\eta : K \times K \to X$ be two maps such that*

(i) $\langle T(y), \eta(y, y) \rangle = 0$ *for all* $y \in K$.

(ii) the map $x \mapsto \langle T(x), \eta(y, x) \rangle$ *of K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), for each $y \in K$,*

(iii) the map $y \mapsto \langle T(x), \eta(y, x) \rangle$ *of K into $\mathbb{R}$ is convex for each $x \in K$,*

(iv) $\langle T(x), \eta(y, x) \rangle + \langle T(y), \eta(x, y) \rangle \leq 0$ *for all* $x, y \in K$.

Then there exists $x_0 \in K$ such that for all $y \in K$,

$$\langle T(x_0), \eta(y, x_0) \rangle \geq 0.$$

In 1981, M. A. Hanson [16] introduced the invex function which is the generalized concept of convex function and concave function. The concept of invexity of a function brought a new edge to generalize the variational inequality problem, that is, in particular case, the generalization of optimization problems, complementarity problems and fixed point problems. In 2006, A. Behera and P.K. Das [3] generalized the concept of invexity of any function to T-η-invexity of the function in ordered topological vector spaces. For our need we recall the following definitions.

Definition 3 ([16]). *The set K is said to be η-invex set where $\eta : K \times K \to X$ is a vector valued continuous mapping, if for all $x, u \in K$, and for all $t \in (0,\ 1)$ such that*

$$u + t\eta(x, u) \in K.$$

Definition 4 ([3, Condition C_0]). *A vector function $\eta : K \times K \to X$ is said to satisfy condition C_0 if the following hold :*

(a) $\eta(x' + \eta(x, x'), x') + \eta(x', x' + \eta(x, x')) = 0,$

(b) $\eta(x' + t\eta(x, x'), x') + t\eta(x, x') = 0,$ *for all* $x, x' \in K$ *and for all* $t \in (0, 1)$.

For our need, we define the following definitions.

Definition 5. *The set $K \subset X$ is said to be weakly η-invex set where $\eta : K \times K \to X$ is a vector valued continuous mapping, if for all $x, u \in K$, there exists a $t \in (0,\ 1)$ such that*

$$z + t\eta(x, u) \in K \text{ where } z \in \{u, x\}.$$

Definition 6. *The set $K \subset X$ is said to be complete w.r.t. η, if there exists a vector valued continuous function $\eta : K \times K \to X$ such that for each two points $x, u \in K$, we have*

$$z + t\eta(x,u) \in K \text{ where } z \in \{u, x\},\ t \in [0,\ 1].$$

Remark 1. *In particular, if $\eta(x,u) = u - x$ when $z = x$ and $\eta(x,u) = x - u$ when $z = u$, then K is complete. In this case, if we get the vector $\overrightarrow{AB}$ for $z = x$, then for $z = u$, we get the vector $\overrightarrow{BA}$ contained in K.*

Remark 2. *If K is complete w.r.t. η then K is both weakly η-invex set and η-invex set but not conversely.*

In section 2, we proposed the problem of absolutely generalized differential dominated variational inequality problem $(AGDDVIP)$ and find the iterative process of it in the Hilbert space.

In section 3, we introduce the maximal fixed open set and defined the generalized differential dominated variational inequality problems $(GDDVIP_n)$ and generalized differential dominated complementarity problem $(GDDCP_n)$ in Riemannian n-manifolds modelled on the Hilbert spaces. Further, we study the existence theorems of the problems $(GDDVIP_n)$ and $(GDDCP_n)$ in the presence of coincidence index, fixed point inclusion of Homology theory and one-point compactification of Topology theory.

2. The Iterative Method for $(AGDDVIP)$ in Hilbert Space

The notion of η-invex function was introduced by Hanson [16] as a generalization of convex function. In 2006, A. Behera and P.K. Das [3] generalized the concept of invexity of any function to T-η-invexity of the function in ordered topological vector spaces.

Let $F : M \to \mathbb{R}$ be a differentiable function where $\nabla F(u)$ is the differential of F at $u \in M$. Then, T-η-invex function is defined as follows.

Definition 7 ([3]). *Let $F : M \to \mathbb{R}$ be any function. Then,*

(a) F is T-η-invex on M if

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \geq 0 \quad \forall x, u \in M, \tag{2}$$

(b) F is T-η-invex at point $u \in M$ if

$$F(x) - F(u) - \langle T(u), \eta(x,u)\rangle \geq 0 \quad \forall x \in M. \tag{3}$$

In this section, we proposed the *generalized dominated differential variational inequality problems* $(GDDVIP)$ and *generalized dominated differential complementarity problems* $(GDDCP)$ in reflexive real Banach spaces. We prove the existence of the solution of the absolutely generalized dominated differential variational inequality problems $(AGDDVIP)$ using the iterative process in Hilbert spaces.

Let X be a reflexive real Banach space and K be any nonempty η-invex subset of X. Let $T : K \to X^*$ be a nonlinear mapping and $F : K \to \mathbb{R}$ be a differentiable map where ∇F is the derivative of F. Let $\langle f, x\rangle$ denote the value of $f \in X^*$ at point $x \in X$. The problem $(GDDVIP)$ is defined as follows.
$(GDDVIP)$ Find $x_0 \in K$ such that

$$\langle(\nabla F - T)(x_0), \eta(x, x_0)\rangle \geq 0 \quad \forall x \in K. \tag{4}$$

and the $(GDDCP)$ is defined as follows.
$(GDDCP)$ Find $x_0 \in K$ such that

$$\langle(\nabla F - T)(x_0), \eta(x, x_0)\rangle = 0 \quad \forall x \in K. \tag{5}$$

In this section, everywhere V is considered as an Hilbert space space with the inner product $\langle\cdot\rangle$ satisfies the Euclidean norm $|\cdot|$ by the rule $|v| = \sqrt{\langle v, v\rangle}$ and M is a nonempty subset of V. Let V^* be the dual of V. Let $\eta : M \times M \to V$ be a vector valued function.

Definition 8. *Let $M \subset V$. An operator $A : V \to V$ is said to be quasi-pseudomonotone (in short; quasidomonotone) with respect to η on M if for all $t \in (0, 1)$, there exists a vector function $\eta : M \times M \to V$ such that*

$$\langle A(tu), \eta(u, u)\rangle - \langle A(tv), \eta(v, v)\rangle = \langle A(v + t\eta(v, u)), \eta(v, u)\rangle \quad \forall\, u, v \in M.$$

Definition 9. *Let $M \subset V$. An operator $A : V \to V$ is said to be quasidomonotone and potential with respect to η on M if for all $t \in (0, 1)$, there exists a vector function $\eta : M \times M \to V$ such that*

$$\int_0^1 \langle A(tu), \eta(u, u)\rangle\, dt - \int_0^1 \langle A(tv), \eta(v, v)\rangle\, dt = \int_0^1 \langle A(v + t\eta(v, u)), \eta(v, u)\rangle\, dt \quad \forall\, u, v \in M.$$

$(AGDDVIP)$ and the Iterative Process

Let $F : M \to \mathbb{R}$ be a differentiable map where ∇F is the derivative of F and $T : M \to V^*$ be a nonlinear map. The *absolutely generalized dominated differential variational inequality problems* $(AGDDVIP)$ is defined as follows:
$(AGDDVIP)$ Find $u^* \in M$ such that

$$\langle(\nabla F - T)(u^*), \eta(v, u^*)\rangle \geq 0 \quad \forall v \in M, \tag{6}$$

if there exists a finite subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that

$$u_{n_k} \xrightarrow{w} u^* \text{ as } k \to \infty,$$

and satisfying

$$\limsup_{k\to\infty} \langle T(u_{n_k}), \eta(u^*, u_{n_k})\rangle \leq \limsup_{k\to\infty} \langle \nabla F(u_{n_k}), \eta(u^*, u_{n_k})\rangle.$$

Let the following properties satisfies.

(P1) (a) ∇F and T are quasidomonotone and potential with respect to η on M,

(b) $|T(v+t\eta(v,u))| \leq |T(u+t\eta(v,u))| \quad \forall u,v \in M$,

(c) $\langle T(u),\eta(v,u)\rangle \leq \langle T(u),\eta(v,z)\rangle + \langle T(u),\eta(u,z)\rangle \ \forall u,v \in M$, fixed $z \in M$,

(d) $|\nabla F(u) - T(u)| \leq \alpha \ \forall \ u \in M$.

(P2) Let η be an absolutely bounded function satisfying the condition

$$|\eta(v,u)| \leq \frac{2}{\alpha}\delta(\epsilon) \quad \forall u,v \in M, \tag{7}$$

where $\alpha > 0$ and for each $\epsilon > 0$, $\delta(\epsilon)$ be a bounded continuous function satisfying $\delta(\epsilon) < \frac{\alpha}{2}$.

(P3) ∇F and T are Lipschitz continuous with ranks $L_1 > 0$ and $L_2 > 0$ respectively.

We introduce a functional $\Gamma : V \to \mathbb{R}$ by the relation

$$\Gamma(u) = A_\eta(u) - B_\eta(u), \tag{8}$$

$$\text{where } A_\eta(u) = \int_0^1 \langle \nabla F(tu), \eta(u,u)\rangle dt \tag{9}$$

$$\text{and } B_\eta(u) = \int_0^1 \langle T(tu), \eta(u,u)\rangle \ dt. \tag{10}$$

By the quasi-pseudomonotone and potential property (i.e., $P1(a)$) of ∇F and T, we have

$$\begin{aligned} A_\eta(u) - A_\eta(v) &= \int_0^1 \langle \nabla F(tu), \eta(u,u)\rangle dt - \int_0^1 \langle \nabla F(tv), \eta(v,v)\rangle \ dt \\ &= \int_0^1 \langle \nabla F(v+t\eta(v,u)), \eta(v,u)\rangle \ dt. \end{aligned} \tag{11}$$

and

$$\begin{aligned} B_\eta(u) - B_\eta(v) &= \int_0^1 \langle T(tu), \eta(u,u)\rangle dt - \int_0^1 \langle T(tv), \eta(v,v)\rangle \ dt \\ &= \int_0^1 \langle T(v+t\eta(v,u)), \eta(v,u)\rangle \ dt. \end{aligned} \tag{12}$$

Hence, we get

$$\begin{aligned}\Gamma(u)-\Gamma(v) =& A_\eta(u)-B_\eta(u)-A_\eta(v)+B_\eta(v),\\ =& \left(A_\eta(u)-A_\eta(v)\right)-\left(B_\eta(u)-B_\eta(v)\right),\\ =& \int_0^1 \langle \nabla F(v+t\eta(v,u)),\eta(v,u)\rangle dt\\ &- \int_0^1 \langle T(v+t\eta(v,u)),\eta(v,u)\rangle\ dt. \end{aligned} \tag{13}$$

To solve the problem $(AGDDVIP)$, we consider the following iterative process.

Let u_0 be an arbitrary element of M. For $n=0,1,2,\ldots$, we define $u_{n+1}\in M$ as the solution of the variational inequality problem

$$\langle \eta(u_n,u_{n+1}),\eta(v,u_{n+1})\rangle+\rho_n\langle \nabla F(u_n),\eta(v,u_n)\rangle \geq 0 \quad \forall v\in M, \tag{14}$$

where the sequence $\{\rho_n\}_{n=0}^\infty$ of the iteration parameters satisfies the conditions

$$0\leq \rho_*\leq \rho_n\leq \rho^*\leq \frac{2}{L_1+L_2}. \tag{15}$$

For the sequence $\{\epsilon_n\}_{n=1}^\infty$, we assume that

$$\sum_{n=1}^\infty \delta(\epsilon_n)=\sigma<\infty. \tag{16}$$

Since $|\eta(v,u)|\leq \frac{2}{\alpha}\delta(\epsilon)$ for all $u,v\in M$ and $\delta(\epsilon)<\frac{2}{\alpha}$, we have

$$|\eta(u_n,u_{n+1})|<1 \Rightarrow \lim_{n\to\infty}|\eta(u_n,u_{n+1})|=0.$$

Hence, the series $\sum\limits_{n=0}^\infty |\eta(u_n,u_{n+1})|^2$ is convergent. Let the limit point of the series $\sum\limits_{*n=0}^\infty |\eta(u_n,u_{n+1})|^2$ be $\frac{2\sigma}{L_1+L_2}$.

To analyze the convergence of the iterative process, we need the following assertion.

Lemma 1 ([26, p.93]). *Let $\{a_k\}_{k=0}^\infty$ be a numerical sequence such that*

$$a_{k+1}\leq a_k+\delta_k \textit{ where } \delta_k\geq 0,\ k=0,1,2,\ldots,\sum_{k=0}^\infty \delta_k<\infty,$$

then, there exists a limit

$$\lim_{k\to\infty} a_k<\infty.$$

In addition, the sequence $\{a_k\}_{k=0}^\infty$ is bounded below then the limit is finite.

Theorem 3. *Let $M \subset V$ be complete w.r.t. η (or at least weakly η-invex set) where $\eta : M \times M \to V$ is a continuous function. Let the condition given in (15) be satisfied. Then, the iterative sequence $\{u_n\}_{n=0}^{\infty}$ given by (14) is bounded in V, and all its weak limit points are solutions of the problem $(AGDDVIP)$.*

Proof. Let us assume $M \subset V$ as weakly η-invex set. Let $S(u_0) \subset M$ be a bounded subset of M defined by

$$S(u_0) = \{u \in M : \Gamma(u) \leq \Gamma(u_0) + 3\sigma\}.$$

Then, $S(u_0)$ is nonempty because by definition of $S(u_0)$, we have $u_0 \in S(u_0)$. Next, we show that the iterative sequence defined by

$$\{u_n\}_{n=0}^{\infty} \subset S(u_0) \tag{17}$$

is bounded, i.e, to show if $u_n \in S(u_0)$ then

$$u_{n+1} \in S(u_0).$$

Since ∇F and T are Lipschitz continuous with rank L_1 and L_2 respectively, so that, we get

$$|\nabla F(v) - \nabla F(x)| \leq L_1 |v - x| \tag{18}$$

and

$$|T(v) - T(x)| \leq L_2 |v - x| \tag{19}$$

for all $x, u \in M$. Since M is weakly η-invex set, for each $x, u \in M$, there exists a $t \in (0,1)$ such that $z + t\eta(x,u) \in M$ where $z \in \{x, u\}$, implies that, $x + t\eta(x,u) \in M$ for all $x, u \in M$. From property $P3$ (i.e. Lipschitz continuity of T), we get

$$|\nabla F(x + t\eta(x,u)) - \nabla F(x)| \leq L_1 t |\eta(x,u)| \tag{20}$$

and

$$|T(x + t\eta(x,u)) - T(x)| \leq L_2 t |\eta(x,u)| \tag{21}$$

for all $x, u \in M$ and for each $t \in (0,1)$. Replacing x by u_n and u by u_{n+1} in (20) and (21), we get

$$|\nabla F(u_n + t\eta(u_n, u_{n+1})) - \nabla F(u_n)| \leq L_1 t |\eta(u_n, u_{n+1})| \tag{22}$$

and

$$|T(u_n + t\eta(u_n, u_{n+1})) - T(u_n)| \leq L_2 t |\eta(u_n, u_{n+1})|. \tag{23}$$

Hence from (22), we have

$$\begin{aligned} |\langle \nabla F(u_n + t\eta(u_n, u_{n+1})) &- \nabla F(u_n), \eta(u_n, u_{n+1}) \rangle| \\ &\leq |\nabla F(u_n + t\eta(u_n, u_{n+1})) - \nabla F(u_n)| \, |\eta(u_n, u_{n+1})| \\ &= L_1 t |\eta(u_n, u_{n+1})|^2 \end{aligned} \tag{24}$$

for each $t \in (0,1)$ and from (23), we have

$$\begin{aligned}
|\langle T(u_n + t\eta(u_n,u_{n+1}))-T(u_n), \eta(u_n,u_{n+1})\rangle| \\
&\leq |T(u_n + t\eta(u_n,u_{n+1})) - T(u_n)|\ |\eta(u_n,u_{n+1})| \\
&= L_2 t|\eta(u_n,u_{n+1})|^2 \qquad (25)
\end{aligned}$$

for each $t \in (0,1)$. Therefore, substituting $v = u_n$ and $u = u_{n+1}$ in (13) and using the equations from (22) to (25), we have

$$\begin{aligned}
\Gamma(u_{n+1}) - \Gamma(u_n) &= \int_0^1 \langle \nabla F(u_n + t\eta(u_n,u_{n+1})), \eta(u_n,u_{n+1})\rangle dt \\
&\quad - \int_0^1 \langle T(u_n + t\eta(u_n,u_{n+1})), \eta(u_n,u_{n+1})\rangle dt \\
&= \int_0^1 \left[\langle \nabla F(u_n + t\eta(u_n,u_{n+1})) - \nabla F(u_n)\right], \eta(u_n,u_{n+1})\rangle dt \\
&\quad - \int_0^1 \left[\langle T(u_n + t\eta(u_n,u_{n+1})) - T(u_n), \eta(u_n,u_{n+1})\rangle\right] dt \\
&\quad + \int_0^1 \langle (\nabla F - T)(u_n), \eta(u_n,u_{n+1})\rangle dt \\
&\leq \int_0^1 |\nabla F(u_n + t\eta(u_n,u_{n+1})) - \nabla F(u_n)|\ \eta(u_n,u_{n+1})| dt \\
&\quad + \int_0^1 |\ T(u_n + t\eta(u_n,u_{n+1})) - T(u_n)||\eta(u_n,u_{n+1})| dt \\
&\quad + \int_0^1 |(\nabla F - T)(u_n)|\ |\eta(u_n,u_{n+1}| dt \\
&\leq \int_0^1 L_1 t|\eta(u_n,u_{n+1})|^2 dt + \int_0^1 L_2 t|\eta(u_n,u_{n+1})|^2 dt
\end{aligned}$$

$$+\int_0^1 \alpha\cdot\frac{2}{\alpha}\delta(\epsilon)dt \text{ by P1(d), (24) and (25))}$$

$$=(L_1+L_2)|\eta(u_n,u_{n+1})|^2\int_0^1 t\,dt+2\delta(\epsilon)\int_0^1 dt$$

$$=\frac{L_1+L_2}{2}|\eta(u_n,u_{n+1})|^2+2\delta(\epsilon)$$

Thus, we obtained the relation

$$\Gamma(u_{n+1})\le\Gamma(u_n)+\frac{L_1+L_2}{2}|\eta(u_n,u_{n+1})|^2+2\delta(\epsilon) \tag{26}$$

which is valid for all $n=0,1,2,\ldots$. Putting $n=0,1,2,\ldots,N$ in (26), we get

$$\begin{aligned}\Gamma(u_{N+1})&\le\Gamma(u_0)+\left(\frac{L_1+L_2}{2}\right)\sum_{n=0}^{N}|\eta(u_n,u_{n+1})|^2+2\sum_{n=0}^{N}\delta(\epsilon_n)\\&\le\Gamma(u_0)+\left(\frac{L_1+L_2}{2}\right)\left(\frac{2\sigma}{L_1+L_2}\right)+2\sigma\\&\le\Gamma(u_0)+3\sigma,\end{aligned} \tag{27}$$

Thus, $u_{N+1}\in S(u_0)=\{u\in M:\Gamma(u)\le\Gamma(u_0)+3\sigma\}$. Since N is arbitrary, replacing N by n, we get $u_{n+1}\in S(u_0)$ and hence, $\{u_n\}_{n=0}^{\infty}\subset S(u_0)$. Now by (16), the assumptions of Lemma 1 are valid the sequence

$$\{\Gamma(u_n)\}_{n=1}^{\infty}.$$

Next to show, the sequence $\{\Gamma(u_n)\}_{n=1}^{\infty}$ is bounded above and has a finite limit. Taking limit $N\to\infty$ in (27), and using (16) we get

$$\lim_{N\to\infty}\Gamma(u_{N+1})\le\Gamma(u_0)+\left(\frac{L_1+L_2}{2}\right)\sum_{n=0}^{\infty}|\eta(u_n,u_{n+1})|^2+2\sum_{n=0}^{\infty}\delta(\epsilon_n)\le\Gamma(u_0)+3\sigma,$$

equivalently, we have

$$\lim_{n\to\infty}\Gamma(u_{n+1})\le\Gamma(u_0)+3\sigma \tag{28}$$

Hence, the sequence $\{\Gamma(u_n)\}_{n=1}^{\infty}$ is bounded above. Again, by the property $P1(c)$, for fixed $z\in M$, we have

$$\langle T(u),\eta(v,u)\rangle\le\langle T(u),\eta(v,z)\rangle+\langle T(u),\eta(u,z)\rangle\quad\forall u,v\in M.$$

Taking $u=u_n$, $z=u_{n+1}$ in the above inequality, we get

$$\langle T(u_n),\eta(v,u_n)\rangle\le\langle T(u_n),\eta(v,u_{n+1})\rangle+\langle T(u_n),\eta(u_n,u_{n+1})\rangle$$

$$
\begin{aligned}
&\leq \langle T(u_n), \eta(u_n, u_{n+1})\rangle + \langle T(u_n), \eta(v, u_{n+1})\rangle + \langle \nabla F(u_n), \eta(v, u_n)\rangle \\
&\quad + \frac{1}{\rho_n}\langle \eta(u_n, u_{n+1}), \eta(v, u_{n+1})\rangle \quad \forall v \in M \text{ (from equation (14))} \\
&\leq |T(u_n)|\; |\eta(v, u_{n+1})| + |T(u_n)|\; |\eta(u_n, u_{n+1})| + \langle \nabla F(u_n), \eta(v, u_n)\rangle \\
&\quad + \frac{1}{\rho_n}|\eta(u_n, u_{n+1})|\; |\eta(v, u_{n+1})| \\
&= (|T(u_n)| + \frac{1}{\rho_n}|\eta(v, u_{n+1})|)\; |\eta(u_n, u_{n+1})| + |T(u_n)|\; |\eta(v, u_{n+1})| \\
&\quad + \langle \nabla F(u_n), \eta(v, u_n)\rangle \\
&\leq C_v|\eta(u_n, u_{n+1})| + S_v|T(u_n)| \; + \langle \nabla F(u_n), \eta(v, u_n)\rangle
\end{aligned}
\tag{29}
$$

for all $v \in M$, where

$$
C_v = |T(u_n)| + \frac{1}{\rho_n}|\eta(v, u_{n+1})| \text{ and } S_v = |\eta(v, u_{n+1})| \tag{30}
$$

are the nonnegative constants limits to 0 as $n \to \infty$ depending on $v \in M$. Since the iterative sequence is bounded, it has a subsequence which is of finite limit. We claim that, there exists a finite subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that

$$
u_{n_k} \xrightarrow{w} u^* \text{ as } k \to \infty.
$$

and satisfying the inequality

$$
\lim_{k\to\infty} \sup \langle T(u_{n_k}), \eta(u^*, u_{n_k})\rangle \leq \lim_{k\to\infty} \sup \langle \nabla F(u_{n_k}), \eta(v, u_{n_k})\rangle.
$$

Taking $v = u^*$ in the (29) and using (30), we have

$$
\begin{aligned}
\lim_{k\to\infty} \sup \langle T(u_{n_k}), \eta(u^*, u_{n_k})\rangle &\leq \lim_{k\to\infty} \sup \; C_{u^*}|\eta(u_{n_k}, u_{n_k+1})| + \lim_{k\to\infty} \sup \; S_{u^*}|T(u_{n_k})| \\
&\quad + \lim_{k\to\infty} \sup \langle \nabla F(u_{n_k}), \eta(u^*, u_{n_k})\rangle \\
&\leq \lim_{k\to\infty} \sup \langle \nabla F(u_{n_k}), \eta(u^*, u_{n_k})\rangle.
\end{aligned}
$$

Next, we show that u^* solves the problem $(AGDDVIP)$. From (29), we have

$$
C_v|\eta(u_n, u_{n+1})| + S_v|T(u_n)| \geq \langle T(u_n), \eta(v, u_n)\rangle - \langle \nabla F(u_n), \eta(v, u_n)\rangle,
$$

i.e.,

$$
C_v|\eta(u_{n_k}, u_{n_k+1})| + S_v|T(u_{n_k})| \geq \langle T(u_{n_k}), \eta(v, u_{n_k})\rangle - \langle \nabla F(u_{n_k}), \eta(v, u_{n_k})\rangle
$$

for all $v \in M$. Thus

$$
\begin{aligned}
&\lim_{k\to\infty} \inf \Big[\; C_v|\eta(u_{n_k}, u_{n_k+1})| + S_v|T(u_{n_k})|\Big] \\
&\geq \lim_{k\to\infty} \inf \Big[\langle T(u_{n_k}), \eta(v, u_{n_k}\rangle - \langle \nabla F(u_{n_k}), \eta(v, u_{n_k}\rangle)\Big],
\end{aligned}
$$

i.e.,

$$0 \geq \liminf_{k\to\infty} \langle T(u_{n_k}), \eta(v, u_{n_k})\rangle + \liminf_{k\to\infty} \left[-\langle \nabla F(u_{n_k}), \eta(v, u_{n_k})\rangle\right]$$

for all $v \in M$. Thus

$$0 \geq \langle T(u^*), \eta(v, u^*)\rangle - \langle \nabla F(u^*), \eta(v, u^*)\rangle,$$

i.e.,

$$\langle \nabla F(u^*), \eta(v, u^*)\rangle - \langle T(u^*), \eta(v, u^*)\rangle \geq 0$$

for all $v \in M$. Thus

$$\langle (\nabla F - T)(u^*), \eta(v, u^*)\rangle \geq 0$$

for all $v \in M$. Hence u^* solves $(AGDDVIP)$. This is a **proof**.

3. $GDDVIP$ in Riemannian n-Manifolds

In order to make the paper self-contained, we recall the necessary terminologies of the coincidence index, differential of any function on a differentiable manifold, and the Riemannian metric.

If $f,\ g : M_1 \to M_2$ are maps between closed oriented n-manifolds, a coincidence of f and g is a point $x \in M_1$ such that $f(x) = g(x)$. Geometrically, if $G(f)$ and $G(g)$ are the graphs of the respective functions in $M_1 \times M_2$, their points of intersection correspond to the coincidences [27].

$$\begin{array}{ccc} H_m(M_1;\mathbb{Q}) & \xrightarrow{f_*} & H_m(M_1;\mathbb{Q}) \\ \cong\uparrow \mu & & \cong\uparrow \nu \\ H^{n-m}(M_1;\mathbb{Q}) & \xleftarrow{g^*} & H^{n-m}(M_1;\mathbb{Q}) \end{array}$$

where the vertical homomorphisms are Poincarė duality isomorphisms. The homomorphism

$$\Theta_m : H_m(M_1;\mathbb{Q}) \to H_m(M_1;\mathbb{Q})$$

is defined by $\Theta_m = \mu g^* \nu^{-1} f_*$. Then the coincidence number of f and g is given by

$$L(f,\ g) = \sum_{k=0}^{n} (-1)^k\ tr(\Theta_m),$$

where $L(f, g)$ is the intersection number of $G(f)$ and $G(g)$; hence if $L(f,\ g) \neq 0$, then f and g have a coincidence [27].

Let M_1 and M_2 be closed, connected, oriented n-manifolds with fundamental classes $z_i \in H_n^*(M_i)$ and corresponding Thom classes

$$U_i \in H_n^*(M_i \times M_i,\ M_i \times M_i - \triangle(M_i)),\ i = 1, 2.$$

Suppose that W is an open set in M_1 and $f, g : W \to M_2$ are the maps for which the coincidence set $C = \{x \in W : f(x) = g(x)\}$ is a compact subset of W. By normality of M_1 there exists an open set V in M_1 with $C \subseteq V \subseteq \bar{V} \subseteq W$. The coincidence index of the pair (f, g) on W is defined to be the integer $I^W_{f,\,g}$ given by the image of the fundamental class of z_1 under the composition

$$H_n(M_1) \to H_n(M_1,\ M_1 - V) \xrightarrow[\text{excision}]{\cong} H_n(W,\ W - V)$$
$$\xrightarrow{(f,g)_*} H_n(M_2 \times M_2,\ M_2 \times M_2 - \triangle(M_2)) \cong \mathbb{Z},$$

where the map $(f, g) : W \to M_2 \times M_2$ is given by

$$(f, g)(x) = (f(x), g(x)),$$

and the identification

$$H_n(M_2 \times M_2,\ M_2 \times M_2 - \triangle(M_2)) \cong \mathbb{Z}$$

is given by sending a class α into the integer $\langle U_2,\ \alpha\rangle$. If $M_1 = M_2$ (denoted by M) and g = identity on the open set W, the coincidence index $I^W_{f,id}$ is denoted by I^W_f, called the fixed-point index of f on W [27] and fixed-point index of f on M is denoted by I_f.

Theorem 4 ([27, Lemma 6.7, p.180])**.** *Let X be a closed, convex and oriented Riemannian n-manifold. Let W be an open set in X. If $I^W_f \neq 0$, then f has a fixed point on W.*

Let X be a differentiable manifold with tangent bundle τX where $\tau(X, u)$ is the tangent bundle at $u \in K$. Let K be a closed convex cone in the manifold X. Mititelu [22] introduced the differential application of a differentiable vector function in the differentiable manifold for a development of the η-invex function. This enhanced to develop the the scope of $(GVVIP)$ in differentiable manifold.

Definition 10 ([22])**.** *Let X be a differentiable manifold with tangent bundle τX. Let $\phi : X \to \mathbb{R}^n$ be a differentiable vector function. The application $d\phi_u : \tau(X, u) \to \tau(\mathbb{R}^n, \phi(u)) = \mathbb{R}^n$ is said to be differential of ϕ at $u \in K$, if $d\phi_u(v) = d\phi(u)(v)$ for all $v \in \tau(X, u)$.*

Now, if X is modelled in the Hilbert space $\mathbb{H}$, then $\tau X = X$. In this section, we obtain the Generalized Vector Variational Inequality Problem and Generalized Vector Complementarity Problems in Riemannian n-manifolds. Let X be a closed, convex and oriented Riemannian n-manifold, modeled on the Hilbert space $\mathbb{H}$ with Riemannian metric g. It is well known that the tangent bundle $\tau(X)$ can be identified with the cotangent bundle $\tau^*(X)$ by the Riemannian metric, because $\mathbb{H}^*$, the dual of $\mathbb{H}$ can be identified with $\mathbb{H}$ [25]. If $v, w \in \tau(X, x)$, then we write

$$g_x(v, w) = \langle v, w\rangle_x.$$

Definition 11 ([11])**.** *Let X be a Riemannian n-manifold. Let $\eta : X \times X \to \tau X$ defined by, for each $u \in X$, $\eta(x, u) \in \tau(X, u)$. Then X is said to be η-closed if for every $p \in X$, there is an unique $x \in X$ closest to p with respect to η, that is,*

$$\langle x, \eta(z, x)\rangle_x \geq \langle p, \eta(z, x)\rangle_x \textit{ for all } z \in X.$$

For our need, we define the differential application as follows. Let X and Y be two differentiable manifolds with tangent bundles τX and τY respectively. Let K be a closed convex cone in the manifold X and P be a closed convex ordered cone in Y with $intP \neq \emptyset$. Let $F : K \to Y$ be the differentiable vector function. Denote the differential of F at $u \in K$ as $dF_u : \tau(X,u) \to \tau(Y,F(u))$ where

$$dF_u(v) = dF(u)(v) = \langle \nabla F(u), v\rangle_u.$$

Behera and Das [3] introduced the variational inequality problem and complementarity problem in the Riemannian n-manifold. For our purpose, we call these problems as *Differential Inequality Problem*(DIP_n) and *Differential Complementarity problem*(DCP_n) in Riemannian n-manifold respectively. We recall the known results for our need.

The Differential Inequality Problem in Riemannian n-manifold (DIP_n) is defined as follows:
(DIP_n) Find $y_0 \in X$ such that

$$\nabla F(y_0) \in \tau^*(X) \text{ and } g_{y_0}(\nabla F(y_0), z - y_0) = \langle \nabla F(y_0), z - y_0\rangle_{y_0} \geq 0 \text{ for all} z \in X.$$

and the Differential Complementarity problem in Riemannian n-manifold (DCP_n) is defined as:
(DCP_n) Find $y_0 \in X$ such that

$$\nabla F(y_0) \in \tau^*(X) \text{ and } g_{y_0}(\nabla F(y_0), y_0) = \langle \nabla F(y_0), y_0\rangle_{y_0} = 0.$$

Theorem 5 ([3, Theorem-6.1]). *Let X be a closed, convex and oriented Riemannian n-manifold, modelled on the Hilbert space $\mathbb{H}$ with Riemannian metric g and $f : X \to X$ with Lipschitz number L(f). Let $F : X \to \mathbb{H}$ be an operator. Then there exists a unique $y_0 \in X$ such that*

$$\nabla F(y_0) \in \tau^*(X) \textit{ and } g_{y_0}(\nabla F(y_0), y_0) = \langle \nabla F(y_0), y_0\rangle_{y_0} = 0.$$

In this section, we extend the problems (DIP_n) and (DCP_n) as the *generalized differential dominated variational inequality problem in Riemannian n-manifold* ($GDDVIP_n$) and the *generalized differential dominated complementarity problem in Riemannian n-manifold* ($GDDCP_n$) respectively. We also establish the existence of their solutions in the presence of fixed point index set and fixed point theorem. We use the following notations for our need.

Definition 12. *Let X be a Riemannian n-manifold modelled on the Hilbert space $\mathbb{H}$ with the Riemannian metric g. Let $H : X \to L(X,\mathbb{H}) \equiv \mathbb{H}$. Then, the the positive orthant and the negative orthant of X are defined as follows.*

(1) For each $x \in X$, the positive orthant of X denoted by $X^{\oplus}_{\eta}$ where

$$X^{\oplus}_{\eta} = \{H(x) \in L(X,\mathbb{H}) \equiv \mathbb{H} : \langle H(x), \eta(z,x)\rangle_x \geq 0 \textit{ for all } z \in X\},$$

(2) For each $x \in X$, the negative orthant of X denoted by $X^{\ominus}_{\eta}$ where

$$X^{\ominus}_{\eta} = \{H(x) \in L(X,\mathbb{H}) \equiv \mathbb{H} : \langle H(x), \eta(z,x)\rangle_x \leq 0 \textit{ for all } z \in X\}.$$

Everywhere, in this section $B(x,r)$ is assumed to be the open ball of radius r and center x. We recall the following known definitions.

Definition 13 (Accumulation point, [8]). *Let $W \subset X$. An element $x \in X$ is called accumulation point(or limit point) of W if for every $r > 0$, there is some y in $B(x,r) \cap W$ with $y \neq x$.*

Definition 14 (Closure, [8]). *Let $W \subset X$. The closure of W is the set of all accumulation points of W and is denoted by $\overline{W}$ where*

$$\overline{W} = \{x \in X : B(x,r) \cap W \neq \varnothing \textit{ for every } r > 0\}.$$

Fixed Point Inclusion set

Let $\Phi : X \to X$ be an open map. Let $x^* \in X$ be the fixed point of Φ, then $\Phi(x^*) = x^*$, that is $(\Phi - 1_X)(x^*) \in 0_X$, implies that, $x^* \in (\Phi - 1_X)^{-1}(0_X) \subset X$ holds because Φ is an open map, implies that, $(\Phi - 1_X)^{-1}$ is continuous in X. Hence, the concept, fixed point inclusion set of the map Φ states that Φ has a fixed point in X, that is, if the restricted map $((\Phi - 1_X)\backslash_W)^{-1}$ is continuous on $(\Phi - 1_X)(X)$, then Φ has a fixed point on W, i.e., $I^W_\Phi \neq 0$.

For our purpose, we define the following definitions.

Definition 15 (Accumulated fixed point w.r.t. Φ). *Let $W \subset X$. Let $\Phi : X \to X$ be any map. An element $x \in X$ lies outside W is called accumulated fixed point of W with respect to Φ if $(\Phi - 1_X)^{-1}$ is continuous on X, $(\Phi - 1_X)^{-1}(0_X) \subset \overline{W}$, the closure of W and for every $r > 0$, there is some y in $B(x,r) \cap W$ for $y \neq x$.*

Definition 16 (Fixed point closure w.r.t. Φ). *Let $W \subset X$. Let $\Phi : X \to X$ be any map such that $(\Phi - 1_X)^{-1}(0_X) \subset \overline{W}$. The fixed point closure of W, denoted by $\widehat{W}$ which is the set of all fixed accumulated points of W w.r.t. Φ and is defined by*

$$\widehat{W} = \{x \in X : (\Phi - 1_X)^{-1}(0_X) \bigcup \{B(x,r) \cap W \neq \varnothing \textit{ for every } r > 0\}\}.$$

Remark 3. *The definition of fixed point closure w.r.t. Φ, that is, Definition 16 coincides with the definition of closure, that is, Definition 14 if Φ has no fixed point. By the definition, we have if W is fixed point closure w.r.t. Φ, then W is fixed point closure but not conversely.*

Definition 17 (Fixed point dense w.r.t. Φ). *Let $W \subset X$. Let $\Phi : X \to X$ be any map. Then W is fixed point dense in X with respect to Φ if $\overline{\widehat{W}} = X$, which means, $\overline{W} = X$ and $(\Phi - 1_X)^{-1}$ is continuous on X, that is, if $\mathscr{F}$ is the collection of all fixed points of Φ which are also the cluster points or accumulated points of Φ, then $X = W \cup \mathscr{F}$ is the extended set of W.*

Definition 18 (Maximal open set w.r.t. Φ). *Let X be any set then the open set $W \subset X$ is said to be maximal open set if W is dense in X w.r.t. Φ, that is, $\overline{W} = X$ and there exist no open set $U \subset X$ such that $W \subset U$ and $\overline{U} = X$.*

Definition 19 (Maximal fixed point open set w.r.t. Φ). *Let X be any set. Let $\Phi : X \to X$ be any map. Then, the open set $W \subset X$ is said to be maximal fixed point open set w.r.t. Φ if W is fixed point dense in X with respect to Φ, that is, $\overline{\widehat{W}} = X$.*

Definition 20 ([23]). *Let X be a locally compact Housdorff space. Then $Y = X \cup \{\infty\}$ is one-point compactification of X where the symbol ∞ is some object lies outside X.*

Theorem 6 ([23, Theorem 8.1]). *Let X be a locally compact Housdorff space which is not compact; let Y be the one-point compactification of X. Then Y is compact Housdorff space; X is a subspace of Y; the set $Y - X$ consists of singleton point; and $\overline{X} = Y$.*

Example 1. *Since the one point compactification of the real line $\mathbb{R}$ is isomorphic with the circle. So the real line $W = \mathbb{R}$ is a maximal fixed point open set of $X = \mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$, the extended real line where Φ is the required isomorphism.*

Example 2. *Since the one point compactification of the plane $\mathbb{R}^2$ is homeomorphic with the Riemannian sphere $(S)^2$ which is also known as the extended complex plane $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$, so $W = \mathbb{R}^2$ is maximal fixed point open set of $X = (S)^2$ or $\mathbb{C}^\infty$ where Φ is the stereographic projection.*

Remark 4. *Let W be locally compact Housdorff space and X be the one-point compactification of W. Then W is maximal fixed point open set of X, but not conversely because of the existence of more than one fixed point. A möbius transformation has two fixed points in $\mathbb{C}$.*

The Main Problems and Their Existence Theorems

Let X be a closed, convex and oriented Riemannian n-manifold, modelled on the Hilbert space $\mathbb{H}$ with Riemannian metric g. The tangent bundle $\tau(X)$ is identified with the cotangent bundle $\tau^*(X)$ by the Riemannian metric g. Let $T : X \to L(\tau^*(X), \mathbb{H}) \equiv \mathbb{H}$ be any application. Let $F : K \to \mathbb{H}$ be the differentiable vector function. Let
$\nabla F(u) = dF_u : \tau(X, u) \to \tau(\mathbb{H}, F(u)) \equiv \mathbb{H}$ be the differential of F at $u \in K$. Let
$\eta : X \times X \to \tau(X, u) \equiv X$ be an vector application.

The Generalized Differential Dominated Variational Inequality problem in Riemannian n-manifold is defined as follows:
($GDDVIP_n$) Find $y_0 \in X$ such that

$$(\nabla F - T)(y_0) \in \big(\tau^*(X)\big)^{\oplus}_{\eta}$$

and

$$g_{y_0}\big((\nabla F - T)(y_0), \eta(z, y_0)\big) = \langle(\nabla F - T)(y_0), \eta(z, y_0)\rangle_{y_0} \geq 0 \text{ for all } z \in X.$$

The Generalized Differential Dominated Complementarity problem in Riemannian n-manifold is defined as follows:
($GDDCP_n$) Find $y_0 \in X$ such that

$$(\nabla F - T)(y_0) \in \big(\tau^*(X)\big)^{\oplus}_{\eta}$$

and

$$g_{y_0}\big((\nabla F - T)(y_0), \eta(z, y_0)\big) = \langle(\nabla F - T)(y_0), \eta(z, y_0)\rangle_{y_0} = 0 \text{ for all } z \in X.$$

Theorem 7. *Let X be a η-closed, η-invex and oriented Riemannian n-manifold, modelled on the Hilbert space $\mathbb{H}$ with Riemannian metric g. Let $f : X \to \mathbb{H}$, $\eta : X \times X \to \tau(X,u) \equiv X$ are two continuous maps. Let $T : X \to L(\tau^*(X), \mathbb{H}) \equiv \mathbb{H}$ be an operator. Let $F : X \to \mathbb{H}$ be an operator such that the differential operator $\nabla F : X \to \tau(\mathbb{H}, F(u)) \equiv \mathbb{H}$. Let W be a maximal fixed point open set in X with respect to $\nabla F - T$. Then, there exists a unique y_0 such that $y_0 \in (\nabla F - T - 1_X)^{-1}(0_X)$ and y_0 solves the problem $(GDDVIP_n)$, that is,*

$$(\nabla F - T)(y_0) \in (\tau^*(X))^{\oplus}_{\eta}$$

and

$$g_{y_0}\left((\nabla F - T)(y_0), \eta(z, y_0)\right) = \langle(\nabla F - T)(y_0), \eta(z, y_0)\rangle_{y_0} \geq 0 \textit{ for all } z \in X.$$

Proof. Since X is an n-manifold, it is a Housdorff space with a countable basis such that each point x of X has a neighborhood that is homeomorphic with an open set of $\mathbb{R}^n$. Again, since W be a maximal fixed point open set in X, by the definition, we have $\overline{W} = X$. Now, to apply Theorem 6, we show that X is one-point compactification of W by proving f has only one fixed point in X which lies outside W.

Let I_f be the fixed point index of f. Then at first we show, f has a fixed point by proving $I_f \neq 0$. Since X is η-closed endowed with the Riemannian metric g, $\overline{W}$ is η-closed endowed with the Riemannian metric g. Thus, for every $y \in \overline{W}$, there is an unique $x \in W$ which is closest to $y - \nabla F(y) + T(y)$ with respect to η.

Let the mapping $f : \overline{W} \to X$ defined by the rule

$$f(y) = y - \nabla F(y) + T(y) + x$$

for every $y \in \overline{W}$ where x is the unique element corresponding to y. Now for every $y \in \overline{W}$,

$$(1_{\overline{W}} - f)(y) = 1_{\overline{W}}(y) - f(y) = \nabla F(y) - T(y) - x = (\nabla F - T)(y) - x.$$

Let $A : \overline{W} \to \mathbb{H}^* = \mathbb{H}$ be a mapping defined by the rule

$$A(y) = (\nabla F - T)(y) \text{ for all } y \in \overline{W}.$$

Then from the above expression, we have

$$(1_{\overline{W}} - f)(y) = A(y) - x$$

and at $y = x$, we have

$$(1_{\overline{W}} - f)(x) = A(x) - x = (A - 1_{\overline{W}})(x),$$

that is, $1_{\overline{W}} - f = A - 1_{\overline{W}}$ at the unique $x \in \overline{W}$, that is, $1_{\overline{W}} - f = A - 1_{\overline{W}}$ at the unique $x \in \overline{W}$.

Define $G : \overline{W} \times I \to \overline{W}$ by the rule

$$G(y,t) = \begin{cases} (1_{\overline{W}} - f)(2tx + (1-2t)y) & \text{if } 0 \leq t \leq \frac{1}{2}; \\ (A - 1_{\overline{W}})(2(1-t)x + (2t-1)y) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $G(y,0) = \left(1_{\overline{W}} - f\right)(y)$, $G(y,1) = \left(A - 1_{\overline{W}}\right)(y)$ for each $y \in \overline{W}$ and at $t = \frac{1}{2}$,

$$G(y,1/2) = \left(1_{\overline{W}} - f\right)(x) = \left(A - 1_{\overline{W}}\right)(x).$$

Thus G is continuous by Pasting Lemma and $G : (1_{\overline{W}} - f) \simeq (A - 1_{\overline{W}})$ where $\simeq$ denotes "homotopically equivalent to" . Hence the coincidence index set of f is given by

$$I_f^{\overline{W}} = \left(1_{\overline{W}} - f\right) * 0_{\overline{W}} = \left(A - 1_{\overline{W}}\right) * 0_{\overline{W}}. \tag{31}$$

By (31), we have $I_f^{\overline{W}} \neq 0$. Since invex set is the generalization of convex set, applying Theorem 4, we have f has a fixed point on $\overline{W}$. Again, since W is maximal fixed point open set in X with respect to A, by the definition, we get $\widehat{\overline{W}} = X$, implies that, $\overline{W} = X$ and $(A - 1_{\overline{W}})^{-1}(0_{\overline{W}}) \subset \overline{W}$, that is, $(A - 1_X)^{-1}(0_X) \subset X$, implies that, $(1_X - f)^{-1}(0_X) \subset X$. Hence, f has a fixed point in X. Let the fixed point be y_0 in X, that is, $f(y_0) = y_0$. Let x_0 be the unique element that corresponds y_0.

By η-closedness of X, we have, for every $y \in X$, there is an unique $x \in X$ which is closest to $y - A(y)$ with respect to η. That is,

$$\langle x, \eta(z,x)\rangle_x \geq \langle y - A(y), \eta(z,x)\rangle_x \text{ for all } z \in X.$$

At $x = x_0$, we have

$$\langle x_0, \eta(z,x_0)\rangle_{x_0} \geq \langle y_0 - A(y_0), \eta(z,x_0)\rangle_{x_0},$$

i.e.,

$$\langle x_0, \eta(z,x_0)\rangle_{x_0} \geq \langle f(y_0) - x_0, \eta(z,x_0)\rangle_{x_0}$$

for all $z \in X$. Thus

$$\langle x_0, z - x_0\rangle_{x_0} \geq \langle y_0 - x_0, \eta(z,x_0)\rangle_{x_0},$$

i.e.,

$$\langle 2x_0 - y_0, \eta(z,x_0)\rangle_{x_0} \geq 0 \tag{32}$$

for all $z \in X$. Again at $y = y_0$, we get $f(y_0) = y_0 - A(y_0) + x_0$, that is, $x_0 = A(y_0)$. Since $\overline{W} = X$, we have $I_f^{\overline{W}} = I_f^X = I_f$. Hence, by definition of coincidence index set, we have

$$I_f = (1_X - f) * 0_X = (A - 1_X) * 0_X = I_A,$$

which means f and A has same fixed point in X, that is, $f(y_0) = y_0 = A(y_0)$. Therefore, $(A - 1_X)^{-1}(0_X)$ contains y_0 only, that is, $y_0 \in (\nabla F - T - 1_X)^{-1}(0_X)$.

Now, since $x_0 = A(y_0)$, implies that, $x_0 = y_0$. Substituting $x_0 = y_0$ in (32), we get

$$\langle A(y_0), \eta(z,y_0)\rangle_{y_0} \geq 0,$$

i.e.,

$$\langle(\nabla F - T)(y_0), \eta(z,y_0)\rangle_{y_0} \geq 0$$

for all $z \in X$. Thus $(\nabla F - T)(y_0) \in (\tau^*(X))_\eta^\oplus$ and $y_0 \in X$ solves $(GDDVIP_n)$. This is a **proof**.

Theorem 8. *Let X be a η-closed, η-invex and oriented Riemannian n-manifold, modelled on the Hilbert space $\mathbb{H}$ with Riemannian metric g. Let $f : X \to \mathbb{H}$, $\eta : X \times X \to \tau(X,u) \equiv X$ are two continuous maps. Let $T : X \to L(\tau^*(X), \mathbb{H}) \equiv \mathbb{H}$ be an operator. Let $F : X \to \mathbb{H}$ be an operator such that the differential operator $\nabla F : X \to \tau(\mathbb{H}, F(u)) \equiv \mathbb{H}$. Let W be a maximal fixed point open set in X with respect to $\nabla F - T$. Then, there exists an unique y_0 such that $y_0 \in (\nabla F - T - 1_X)^{-1}(0_X)$ and y_0 that solves the problem $(GDDVCP_n)$, that is,*

$$(\nabla F - T)(y_0) \in (\tau^*(X))^{\oplus}_{\eta}$$

and

$$g_{y_0}\left((\nabla F - T)(y_0), \eta(z, y_0)\right) = \langle (\nabla F - T)(y_0), \eta(z, y_0)\rangle_{y_0} = 0 \textit{ for all } z \in X.$$

Proof. By Theorem 7, we get, there exists an unique $y_0 \in X$ such that

$$y_0 \in (\nabla F - T - 1_X)^{-1}(0_X)$$

and y_0 solves the problem $(GDDVIP_n)$, that is,

$$(\nabla F - T)(y_0) \in \left(\tau^*(X)\right)^{\oplus}_{\eta}$$

and

$$g_{y_0}\left((\nabla F - T)(y_0), \eta(z, y_0)\right) = \langle (\nabla F - T)(y_0), \eta(z, y_0)\rangle_{y_0} \geq 0 \tag{33}$$

for all $z \in X$. Since X is η-invex set, for fixed $y_0 \in X$ and $t \in (0,1)$, we have $y_0 + t\eta(z, y_0) \in X$ for all $z \in X$. Replacing z by $y_0 + t\eta(z, y_0)$ in the above inequality, we have

$$\begin{aligned} g_{y_0}\left((\nabla F - T)(y_0), \eta(y_0 + t\eta(z, y_0), y_0)\right) &= \langle (\nabla F - T)(y_0), \eta(y_0 + t\eta(z, y_0), y_0)\rangle_{y_0} \\ &\geq 0 \text{ for all } z \in X. \end{aligned}$$

By condition C_0, we have

$$\langle (\nabla F - T)(y_0),\ \eta(y_0 + t\eta(z, y_0), y_0)\rangle_{y_0} \geq 0 \text{ for all} z \in X,$$

i.e.,

$$\langle (\nabla F - T)(y_0),\ -t\eta(z, y_0)\rangle_{y_0} \geq 0 \text{ for all} z \in X.$$

Thus

$$-t\langle (\nabla F - T)(y_0),\ \eta(z, y_0)\rangle_{y_0} \geq 0 \text{ for all} z \in X,$$

i.e.

$$\langle (\nabla F - T)(y_0),\ \eta(z, y_0)\rangle_{y_0} \leq 0 \tag{34}$$

for all $z \in X$. Hence from (33) and (34), we have

$$\langle (\nabla F - T)(y_0),\ \eta(z, y_0)\rangle_{y_0} = 0 \text{ for all } z \in X.$$

Thus, $y_0 \in X$ solves the problem $(GDDVCP_n)$. This is a **proof**.

4. Conclusion

The convergence of variable step iterative methods of absolutely generalized dominated differential variational inequality problems in Hilbert spaces deals with the parameter that varies from step to step under the given assumptions. We believe that this iterative approach is especially fruitful to solve various minimization problems in mathematical modelling. Furthermore, with the concept of fixed point inclusion set, the application of the existence theorem of (*GDDVIP*) in Riemannian n-manifolds can be useful to study the generalized obstacle problems, generalized elastic plastic torsion problems, and so on.

References

[1] S. Adly. Perturbed algorithms and sensitivity analysis for a general class of variational inclusions. *J. Math. Anal. Appl.*, 201(3):609 – 630, 1996.

[2] C. Bardaro and R. Ceppitelli. Some further generalizations of knaster-kuratowski-mazurkiezicz theorem and minimax inequalities. *Journal of Mathematical Analysis and Applications*, 132(2):484 – 490, 1988.

[3] A. Behera and P.K. Das. Variational inequality problems in h-spaces. *International Journal of Mathematics and Mathematical Sciences, Article ID 78545*, pages 1 – 18, 2006.

[4] A. Behera and G. K. Panda. A generalization of browder's theorem. *Bull. Inst. Math., Acad. Sin.*, 21:183 – 186, 1993.

[5] A. Behera and G. K. Panda. A generalization of minty's lemma. *Indian J. Pure Appl. Math.*, 28(7):897 – 903, 1997.

[6] A. Behera and G. K. Panda. Generalized variational-type inequality in hausdorff topological vector space. *Indian J. Pure Appl. Math.*, 28(3):343 – 349, 1997.

[7] F.E. Browder and J. Mond. Nonlinear monotone operators and convex sets in banach spaces. *Bull. Amer. Math. Soc.*, 17:780 – 785, 1965.

[8] B.V.Limaye. *Functional Analysis.* New Age International (P) Limited, New Delhi, 1997.

[9] M. Chipot. *Variational Inequalities and Flow in Porus Media.* Springer-Verlag, 1984.

[10] F. H. Clarke. *Optimization and Nonsmooth Analysis.* A Wiley-Interscience Publication, New York, 1983.

[11] P. K. Das and S. K. Mohanta. Generalized vector variational inequality problem, generalized vector complementarity problem in hilbert spaces, riemannian n-manifold, $\mathbb{S}^n$ and ordered topological vector spaces: A study using fixed point theorem and homotopy function. *Advances in Nonlinear Variational Inequalities*, 12(2):37 – 47, 2009.

[12] X.P. Ding. Perturbed proximal point algorithm for generalized quasi-variational inclusions. *J. Math. Anal. Appl.*, 210(1):88 – 101, 1997.

[13] X.P. Ding. Perturbed proximal point algorithm for generalized quasi-variational inclusions. *J. Math. Anal. Appl.*, 122:267 – 282, 2001.

[14] G. Duvaut and J. L. Lions. *Inequalities in Mechanics and Physics Translated under the title Neravenstva v mekhanike i fizike, Moscow, 1980.* Berlin, Moscow, 1976.

[15] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* Amsterdam, North-Holland, 1976.

[16] M. A. Hanson. On sufficiency of the kuhn-tucker conditions. *Journal of Mathematical Analysis and Applications*, 142:305 – 310, 1961.

[17] A. Hassouni and A. Moudafi. A perturbed algorithm for variational inequalities. *J. Math. Anal. Appl.*, 185:706 – 712, 1994.

[18] N.J. Haung. Generalized nonlinear variational inclusions with noncompact valued mappings. *Appl. Math. Lett.*, 9(3):25 – 29, 1996.

[19] K. R. Kazmi. Mann and ishikawa type perturbed iterative algorithms for generalized quasivariational inclusions. *J. Math. Anal. Appl.*, 209:572 – 584, 1997.

[20] D. Kinderlehrer and G. Stampacchia. *An introduction to variational Inequalities and their Applications.* Acad. Press, 1980.

[21] J. L. Lions and G. Stampacchia. Variational inequality. *Comm. Pure. Appl. Math.*, 48(1):493 – 519, 1967.

[22] S. Mititelu. Generalized invexity and vector optimization on differentiable manifolds. *Differential Geometry - Dynamical Systems*, 3(1):21 – 31,, 2001.

[23] J. R. Munkres. *Topology.*, volume 13. Prentice Hall of India Pvt. Ltd, New Delhi, 1999.

[24] F. Giannessi R. W. Cottle and J. L. Lions. *Variational inequality and Complementarity Problems-Theory and Application.* John Wiley and Sons, 1980.

[25] L. Serge. *Introduction to Differentiable Manifolds.* Interscience Publishers, John Wiley and Sons, 1962.

[26] F. P. Vasil'ev. *Chislennye metody resheniya ekstremal'nykh zadach (Numerical methods for solving Extremal Problems)*. Moscow, 1988.

[27] J. W. Vick. *Homology Theory, An Introduction to Algebraic Topology.* Academic press, New York, 1973.