EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 4, 2011, 370-423
ISSN 1307-5543 – www.ejpam.com

Euler and Divergent Series

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Abstract. Euler's reputation is tarnished because of his views on divergent series. He believed that all series should have a value, not necessarily a limit as for convergent series, and that the value should remain invariant irrespective of the method of evaluation. Via the key concept of regularisation, which results in the removal of the infinity in the remainder of a divergent series, regularised values can be evaluated for elementary series outside their circles of absolute convergence such as the geometric series and for more complicated asymptotic series called terminants. Two different techniques for evaluating the regularised values are presented: the first being the standard technique of Borel summation and the second being the relatively novel, but more powerful, Mellin-Barnes regularisation. General forms for the regularised values of the two types of terminants, which vary as the truncation parameter is altered, are presented using both techniques over the entire complex plane. Then an extremely accurate and extensive numerical study is carried out for different values of the magnitude and argument of the main variable and the truncation parameter. In all cases it is found that the MB-regularised forms yield identical values to the Borel-summed forms, thereby vindicating Euler's views and restoring his status as perhaps the greatest of all mathematicians.

2000 Mathematics Subject Classifications: 00A30, 01A45, 01A50, 01A65, 01A70, 03A05, 03B30, 30B10, 30B30, 30D20, 30E15, 30E20, 34E05, 34E10, 40A05, 40G10, 40G99

Key Words and Phrases: Absolute Convergence, Asymptotic Form, Asymptotics, Asymptotic Series, Borel Summation, Cauchy Integral, Cauchy Principal Value, Complete Asymptotic Expansion, Conditional Convergence, Divergent Series, Domain of Convergence, Dominant Series, Equivalence, Euler's Constant, Gamma Function, Geometric Series, Grandi's Series, Harmonic Series, Jump Discontinuity, Logarithmic Divergence, Mellin-Barnes (MB) Regularisation, Mellin Transform, Poincaré Prescription, Recurring Series, Regularisation, Regularised Value, Renormalisation, Stokes Line, Stokes Phenomenon, Stokes Sector, Subdominant Terms, Terminant, Truncation, Truncation Parameter

1. Introduction

Despite being regarded as one of the four greatest mathematicians of all time, to this day Euler's reputation is tarnished because of the views he held on divergent series. First, he believed that every series, both convergent and divergent, should be assigned a certain value,

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but because of the fallacies and paradoxes surrounding the latter type of series, he felt that such a value should not be denoted by the name sum [6]. Second, he believed that the value should be independent of the actual method or technique used to determine it. Later, when the foundations of analysis were laid down, initially by Abel and Cauchy, and then by Weierstrass ("the father of modern analysis") and Dedekind, divergent series were virtually banished from the mathematical lexicon. Consequently, Euler's reputation suffered. In fact, the extremely gifted Abel, who died at the tragically young age of 26, described divergent series as "the invention of the devil" and that it was "totally shameless to base any demonstration on them whatsoever". As recently as 2007, in an article celebrating the tercentenary of Euler's birth Varadarajan [32] wrote that whilst Euler certainly had some misconceptions regarding the summation of divergent series, his greatness on this topic was not appreciated for a century after his death when mathematicians began to consider the development of a general theory of divergent series [12]. Later in the same article he states that although in his opinion Euler had taken the first steps towards creating a true theory of divergent series, which is still lacking today, the situation is much more subtle than Euler could ever have anticipated. Unfortunately, he does not elaborate on exactly what he means by "more subtle".

Over the past few centuries mathematicians have, for the most part, tended to steer clear from divergent series, but unfortunately, there is one discipline or field where series of this type abound— asymptotics. In this discipline special methods or techniques, e.g. steepest descent, Laplace's method and the iterative solution to differential equations to name a few, are used to derive solutions in form of the power series expansions whose coefficients eventually diverge quite rapidly. Although it is not clear whether such expansions are always divergent, they are invariably truncated according to the Poincaré prescription or definition as described on p. 151 of Ref. [33]. Generally, this involves truncating an expansion after a few terms. Then one is left with an approximation to a given function, whose accuracy is dependent upon whether the variable in the expansion tends to a limit point, which is often zero or infinity. Hence, depending upon whether the limit point is zero and infinity, we say that a function "goes as" or "is approximately equal to" the truncated expression in the limit as such and such variable goes to zero or infinity. In other instances the Landau symbols of $O()$ and $o()$, or even $+\dots$, are used to signify that the remaining terms dropped from the truncated expression can be bounded. This is fiction of course, because the remainder is only bounded as long as there is an optimal point of truncation [24]. Even more troubling is the fact that the domain over which an optimal point of truncation exists is often unspecified or even unknown. In fact, for most values of the variable there is simply no optimal point of truncation. So, we have the situation today where standard asymptotics represents an inexact, if not crude, mathematical discipline composed of truncated asymptotic expansions that suffer from the drawbacks of vagueness and severe limitation in accuracy and range of applicability as a result of an overly permissive Poincaré prescription. It is no wonder that the discipline is frequently subject to derisory remarks from pure mathematicians in particular, who point out that mathematics is supposed to be an exact science.

During the last twenty years or so, outstanding problems in dendritic crystal growth, the directional solidification of crystals, viscous flows in the presence/absence of surface tension, quantum field theory including tunnelling, ordinary differential equations, optics, number theory, non-local solitary waves, fluid mechanics and a host of other fields [4, 5, 21, 29] have required improved methods aimed at obtaining meaningful corrections that lie beyond all orders of a standard asymptotic expansion. In addition to these applications, analysts have been engaged in developing exponentially improved asymptotics of special functions such as the confluent hypergeometric and gamma functions as described in Ch. 6 of Ref. [27]. For these exceptional and important problems standard asymptotic analysis is simply inadequate. Therefore, the sub-discipline or field known as exponential asymptotics or asymptotics beyond all orders, also occasionally referred to as hyperasymptotics, has evolved. Whilst this field seeks to derive the terms in an asymptotic expansion that are neglected by the application of the Poincaré prescription, as explained at the beginning to Sec. 3 here, it still suffers from the same problem in standard asymptotics, which is: how does one obtain meaningful values to divergent series? This is because frequently these subdominant terms are themselves divergent series. Worse still, they are usually masked by a divergent dominant series. Hence, in order to determine both contributions to the overall solution, we again require a theory of divergent series for only then will it be possible to determine the exact values of the original function, which is the ultimate goal of asymptotics. If such a methodology could be formulated, then asymptotics would be elevated to a true mathematical discipline eliciting precise answers. This would not only have a profound effect on mathematics, but also on physics and engineering.

2. Divergent Series

When one wishes to discuss Euler's "unorthodox" views on divergent series, one is inevitably drawn into a study of the geometric series for it is this series that was used as the basis for his views. We shall do likewise, although it should be pointed out that the series has a fascinating history of its own going way back to Archimedes, who used it to calculate the area under a parabola intersected by a line. This became the precursor to integral calculus.

Before the geometric series can be introduced, however, we first need to understand what is meant by a divergent series. In actual fact, there is no formal or rigorous definition of a divergent series. Instead, we must examine what a convergent series is. Then by a process of elimination, anything that is not a convergent series is regarded as being divergent. Copson's definition [7] begins with the symbol of $a_0 + a_1 + a_2 + \ldots + a_k + \ldots$, which involves the sum of an infinite number of complex numbers. To assign a meaning to this symbol, he then considers the partial sums, s_0, s_1,s_2,..., where each partial sum is given by

$$s_k = a_0 + a_1 + a_2 + \ldots + a_k \ . \tag{1}$$

If this sequence tends to a finite limit s, then the infinite series is convergent with the value of the limit equal to s. That is, $s = \sum_{k=0}^{\infty} a_k$.

On the other hand, if the sequence of partial sums does not converge, the series is said to be divergent. This is certainly a strange definition for it not only includes series yielding an obvious infinity such as $1 + 1 + 1 + 1 + \ldots$ and $1 + 2 + 4 + \ldots$, but also examples, where the series possess indeterminate limits such as $1 - 1 + 1 - 1 + 1 - 1 + \ldots$. All these examples can be regarded as special cases of the geometric series, although the last example is now known as Grandi's series since he was the first to provide a simplistic account of it in 1703*. In particular, he noticed that bracketing the series as $(1-1)+(1-1)+\ldots$ appears to yield a limit of zero, while bracketing it as $1+(-1+1)+(-1+1)+\ldots$ appears to yield a limit of unity. Therefore, we see that there are two possible limits for the series. However, because one can bound the series, one does not get the impression that it is divergent. Nevertheless, according to Copson [7], the series is divergent. In fact, Grandi himself did not think that it summed to either value, but to 1/2 for various reasons, none of which would be considered a mathematical proof today. Leibniz went further by introducing a "law of justice", which amounted to averaging the two possible limits. Consequently, the series is also known as Leibniz's series.

It was Euler who gave what could be regarded as the first proper mathematical treatment of Grandi's series. To do so, we express the series in Eq. (1) as the geometric series by replacing a_k by x^k, where x can be any value. When the magnitude of x is less than unity, i.e. for $|x|<1$, the limit s equals $1/(1-x)$. The series is said to be absolutely convergent for these values of x. E.g., for $x=1/4$, the series becomes $1+1/4+1/16+1/64+1/256+\ldots=1/(1-1/4)=4/3$. Thus, Archimedes was able to show that the area enclosed by a parabola and straight line is 4/3 times the area of the triangle inscribed within this area. If we replace x by the complex variable $z(=x+iy)$, then $|z|<1$ represents the unit disk centred in the complex plane. Furthermore, if we put x equal to -1, then we find that $s=1/2$, but the series is no longer absolutely convergent, which means that it is invalid to use the limit value of $1/(1-x)$. Instead, Euler wrote the series in terms of $-x$ as

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^k x^k+\frac{(-x)^{k+1}}{1+x}\ . \tag{2}$$

Then the main objection to the use of $1/(1+x)$ when $x=1$ is that the final term or remainder cannot be disregarded as k goes to infinity. His idea was that since an infinite series has no last term, it could be neglected. Later, he used finite differences to attack the problem, but in reality his explanation would not be considered valid today. As a consequence, in time his belief that every series should be assigned a certain value came under attack and his reputation began to wane as indicated earlier.

According to Varadarajan [32], Euler had several different methods for summing divergent series, but most of all he used what is now known as Abel summation. This amounts to extending the limit inside the unit disk of absolute convergence to a domain with $z=1$. Unfortunately, for more intricate examples of divergent series, e.g. a_k equal to $(-1)^k\Gamma(k+1)$, where $\Gamma(k+1)=k!=k(k-1)\ldots2\cdot1$, this method breaks down completely, which is why Euler referred to such series as divergent series par excellence. Unlike the geometric series, which we have already stated possesses a radius of absolute convergence equal to unity, the latter type of series possesses zero radius of absolute convergence. We shall return to these series later in this article.

Despite the tone of his papers, Euler expressed doubt in private correspondence over his methods for handling divergent series, but he never found a counterexample to Grandi's series

*History of Grandi's Series., `http://en.wikipedia.org/wiki/History_of_Grandi's_Series` equalling 1/2. Then in 1771 Daniel Bernoulli, who had accepted the result, noticed that by inserting zeros into the series, one could obtain any value between 0 and 1. For example, he found that $1+0-1+1+0-1+\ldots=2/3$ [12]. This is really counter-intuitive and a theory of divergent series would need to account for how the introduction of an infinite number of zeros can yield a different limit. It is precisely this type of result that Abel was referring to when criticising divergent series for producing fallacies and paradoxes. Worse still, if the zeros and minus ones are re-ordered so that the series becomes $1-1+0+1-1+0+1-1+\ldots$, then we would obtain a different limit. In this case the limit would be 1/3.

In 1799 more than a decade after Euler's death, the situation became even worse when Callet pointed out to Lagrange that

$$\frac{1+z}{1+z+z^2}=1-z^2+z^3-z^5+z^6+\ldots\ , \tag{3}$$

gives Grandi's series for $z=1$, but now the limit is 2/3 instead of 1/2. Lagrange defended Euler by stating that the rhs of Eq. (3) is not a true power series since many powers are missing. When these are included by writing the series as

$$1+0\cdot z-z^2+z^3+0\cdot z^4-z^5+z^6+0\cdot z^7-z^8+\ldots\ , \tag{4}$$

the series reduces to $1+0-1+1+0-1+\ldots$ for $z=1$, which as stated previously, was found by Bernoulli to yield a limit of 2/3. Whilst this may have silenced Callet, it is particularly alarming for applied mathematicians who derive divergent series in the form of asymptotic series for often such series are in the form where z is a power of another variable. For example, the asymptotic series for the error function, which appears in Sec. 9, is actually in powers of $-1/z^2$. The above would imply that those missing powers, namely the odd powers of $1/z$, would have to be included in the analysis to obtain the limit.

Since Lagrange, many mathematicians have introduced various methods for summing divergent series. Most of these sum Grandi's series to 1/2. Others motivated by Bernoulli's treatment sum the series to another value, while a small minority take the safe option of not bothering to sum it at all. Therefore, the issue has become whether all the inconsistencies or apparent paradoxes that have been raised here can be resolved.

3. Regularisation

In 1993 my colleague T. Taucher and I carried out a numerical study into the complete asymptotic expansion of a particular case of a number theoretic exponential series, which we called the generalised Euler-Jacobi series [21]. Specifically, our investigation concentrated on the series, $S_3(a)=\sum_{k=0}^{\infty}\exp(-ak^3)$, which represented the $p/q=3$ case. This series was found to possess unimportant constant terms, which were removed so that remaining terms or the tail denoted by $T_3(a)$ yielded an asymptotic expansion, which was composed of two separate divergent series. One of these series denoted by $T_3^K(a)$ was subdominant to the other, which was denoted by $T_3^L(a)$. Specifically, we found that

$$T_3(a)=T_3^L(a)+T_3^K(a)\ , \tag{5}$$

where

$$T_3^L(a) = 2\sum_{k=0}^{\infty} \frac{(-1)^{k+1} a^{2k+1}}{(2\pi)^{6k+4}} \frac{\Gamma(6k+4)}{\Gamma(2k+2)}\,\zeta(6k+4)\ , \tag{6}$$

whilst the subdominant series was given by

$$T_3^K(a) = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \sum_{n=1}^{\infty} \frac{e^{-\sqrt{2z}}}{(6\pi n a)^{1/4}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/6)}{(4\sqrt{z})^k} \times \frac{\Gamma(k+5/6)}{\Gamma(k+1/2)} \cos\Big(\sqrt{2z} - \frac{\pi}{8} - \frac{3k\pi}{4}\Big)\ . \tag{7}$$

In these results $z = (2n\pi/3)^3 a^{-1}$, while $\zeta(s)$ represents the Riemann zeta function.

Subdominance in an asymptotic expansion means that one of the component series possesses an exponential factor that causes the entire series to vanish as the main variable tends to the limit point, which in the above example refers to either $a \to 0$ or $z \to \infty$. That is, in this limit the exponential factor of $\exp(-\sqrt{2z}\,)$ appearing in Eq. (7) becomes vanishingly small in comparison with the dominant series given in Eq. (6). It should also be noted that subdominant terms can become the dominant terms and vice-versa as the main variable or a in the above example undergoes changes in its argument or phase. However, at the time we were only interested in real values of a. As described in the introduction, subdominant terms such as those in Eq. (7) are said to lie beyond all orders of the dominant part of the expansion [4, 5, 29] and are generally neglected by practitioners of standard asymptotics. Nevertheless, we found that they were necessary for obtaining exact values of the series regardless of the size of a. For example, when the first fifteen terms of the dominant series and the first twenty-one terms of the subdominant series are subtracted from $T_3(a)$ with a equal to 0.2, one obtains a value of

$$T_3(0.2) - T_3^L(0.2, 15) - T_3^K(0.2, 21) = -8.458\,470\,156\,185\,480 \cdots \times 10^{-7}\ . \tag{8}$$

On the lhs of the above equation, we have introduced the truncation parameter N into the series given by Eqs. (6) and (7) to indicate that the sums over k have been evaluated partially by setting $N = 15$ in the first series and $N = 21$ in the second series. The value on the rhs now represents the combined remainder of two divergent series. By using our newly-discovered mathematical technique, we were able to evaluate the remainder of $T_3^L(a)$, which when subtracted from the right hand side (rhs) of the above equation yielded a value of $-1.588955334 \cdots \times 10^{-17}$. Then by applying the same technique to the expression for the remainder of the subdominant series $T_3^K(a)$, we obtained the same value. The analysis was repeated for numerous values of a ranging from 0.01 to 10. On each occasion we obtained the exact numerical values of the remainder for the subdominant series. Therefore, for the first time in the history of mathematics we had shown that a complete asymptotic expansion could be used to generate the values of the original function it represented. All the results from this spectacular study were eventually documented and discussed in Chs. 7 and 8 of Ref. [21].

The mathematical technique mentioned in the previous paragraph is known today as Mellin-Barnes regularisation. At its heart lies the key concept of regularisation, which is defined as the removal of the infinity in the remainder of a divergent series so as to make the series summable. It is the absence of this concept that has resulted in the fallacies and paradoxes occurring in divergent series as described in the previous section. So, let us examine how regularisation applies to the geometric series since it represents a generalisation of Grandi's series. To do so, we write the geometric series as

$$\sum_{k=0}^{k} z^k = \sum_{k=0}^{\infty} \Gamma(k+1)\,\frac{z^k}{k!} = \lim_{p\to\infty} \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_0^p dt\; e^{-t}\, t^k \ . \tag{9}$$

In obtaining the above equation we have multiplied the summand z^k by $k!/k!$, substituted $k!$ by its more general form in terms of the gamma function and then introduced the integral representation for the latter. That is, $\Gamma(k+1)$ has been replaced by its integral representation of $\int_0^\infty dt\, t^k \exp(-t)$.

Although the integral in Eq. (9) actually extends from zero to infinity, the upper limit has been replaced by the finite value p, which we let go to infinity later. Since the resulting integral in the above equation is now technically finite, we can interchange the order of the summation and integration. In reality, an impropriety is occurring here, which will be explained shortly. Nevertheless, if we persevere with interchanging the order of the summation and integration, then we find that the summation is not only absolutely convergent, but it also represents the Taylor series expansion for $\exp(zt)$. Therefore, replacing the series by this limit, we find that Eq. (9) becomes

$$\sum_{k=0}^{\infty} z^k = \lim_{p\to\infty} \int_0^p dt\; e^{-t(1-z)} = \lim_{p\to\infty} \left[-\frac{e^{-p(1-z)}}{1-z} + \frac{1}{1-z} \right] \ . \tag{10}$$

When the real part of z is less than unity, i.e. $\Re\, z < 1$, the first term in the last member of Eq. (10) vanishes and the series yields the finite value of $1/(1-z)$. Hence, we see that the same value is obtained for the series when $\Re\, z < 1$ as for when lies in the unit disk of absolute convergence.

According to the definition on p. 18 of Ref. [33], this means that the series is conditionally convergent for $\Re\, z < 1$ and $|z| > 1$. That is, it is not divergent, but it is also not absolutely convergent either. For $\Re\, z > 1$, however, the first term in the last member of Eq. (10) yields infinity. Since we have defined regularisation as the process of removing the infinity so that the series becomes summable, we remove or neglect the first term of the last member of Eq. (10). Then we are left with a finite result that once again equals $1/(1-z)$. We shall call this result the regularised value of the series when it is divergent. Hence, for all complex values of z except for $\Re\, z = 1$, we arrive at

$$\sum_{k=0}^{\infty} z^k \begin{cases} \equiv 1(1-z), & \Re\, z > 1, \\ = 1/(1-z), & \Re\, z < 1. \end{cases} \tag{11}$$

Frequently, it is not known for which values of the variable, e.g. z in the above example, an asymptotic series is convergent and for which it is divergent. In these cases we replace the equals sign by the less stringent equivalence symbol on the understanding that we may be dealing with a series that is absolutely convergent for some values of the variable. As a result, we adopt the shorthand notation of

$$\sum_{k=N}^{\infty} z^k = z^N \sum_{k=0}^{\infty} z^k \equiv \frac{z^N}{1-z} \ . \tag{12}$$

Obviously, such mathematical statements are no longer equations for it is simply invalid to refer to the above as an equation because the left hand side (lhs) is infinite when $\Re z > 1$, while the right hand side (rhs) remains finite for these values of z. Instead, we shall refer to such results as equivalence statements or simply equivalences, for short. It should also be noted that the above notation is only applicable when the result for the regularised value of a divergent series is identical to the limiting value of the convergent series. This is not always the case as can be seen from the final example in Ch. 4 of Ref. [17].

An important property of the above result is that it is one-to-one or bijective for each value of z in the principal branch of the complex plane. This is critical for developing a theory of divergent series since it means that each value of z will yield a unique regularised value, which is beginning to accord with Euler's belief that each series has a specific value. Thus, there is now a possibility that the fallacies and paradoxes that led to the banishment of divergent series from the mathematical lexicon can start to disappear.

At the barrier of $\Re z = 1$, the situation appears to be unclear. For $z = 1$ the last member of Equivalence (10) vanishes, which is consistent with removing the infinity due to $1/(1-z)$. For other values of $\Re z = 1$, the last member of Eq. (10) is clearly undefined, which is expected because this line forms the border or boundary between the domains of convergence and divergence for the series. Because the finite value is the same to the right and to the left of the barrier or line at $\Re z = 1$ and in keeping with the fact that regularisation is effectively the removal of the first term in the last member or rhs of Eq. (10), we take $1/(1-z)$ to be the finite or regularised value when $\Re z = 1$. Hence, Equivalence (11) becomes

$$\sum_{k=0}^{\infty} z^k \begin{cases} \equiv 1/(1-z), & \Re z \geq 1, \\ = 1/(1-z), & \Re z < 1. \end{cases} \tag{13}$$

Since the equals sign is less stringent than the equivalence symbol, we can replace the former symbol by the latter in the above result. Then we find that Equivalence (12) is valid for all values of z.

The standard rules of differentiation and integration apply to an equivalence statement just as they would to an equation. That is, the regularised value has to be either differentiable or integrable in order to operate on the series. For example, differentiating the preceding result j times yields

$$\sum_{k=j}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-j+1)} \, z^{k-j} \begin{cases} \equiv (-1)^j \Gamma(j+1)(1-z)^{j+1}, & \Re z \geq 1, \\ = (-1)^j \Gamma(j+1)(1-z)^{j+1}, & \Re z < 1, \end{cases} \tag{14}$$

while if we replace z by $-z$ and integrate from 0 to z, then we obtain

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{k+1}}{k+1} \begin{cases} \equiv \log(1+z), & \Re z \leq -1, \\ = \log(1+z), & \Re z > -1. \end{cases} \tag{15}$$

The series in the above result is often used as a textbook example of conditional series, e.g. see p. 18 of Ref. [33]. If one puts $z=1$, then one obtains

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots = \log 2 \ . \tag{16}$$

On the other hand, by putting $z = -1$ in Equivalence (15), one obtains the logarithmically divergent and quite famous harmonic series of $1 + 1/2 + 1/3 + 1/4 + \ldots$. This series, which, as described in Sec. 7, represents a very different prospect to regularise from the geometric series, was studied in great detail by Euler [13]. Unbeknownst to him at the time, in writing down the equation for the constant that now bears his name from the harmonic series, viz.

$$\gamma = \lim_{k\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} - \log k \right) \ , \tag{17}$$

he was actually displaying for the first time ever a formula for regularising a divergent series.

4. Divergent Integrals

As discussed in Refs. [14, 17], the regularised value of a divergent series is analogous to the Hadamard finite part that arises in the regularisation of divergent integrals in the theory of generalised functions [10, 23]. As a typical example, let us consider the general integral representation of the gamma function that was used to derive Eq. (9). This is

$$\Gamma(\alpha) = \int_0^{\infty} dx \; x^{\alpha-1} e^{-x} \ . \tag{18}$$

The above integral is convergent for $\Re\, \alpha > 0$, but is divergent for all other values of α.

The divergence in the above integral is associated with the lower limit. If the lower limit is replaced by ϵ and the limit of $\epsilon \to 0$ is taken, then the above integral can be evaluated by integrating by parts continuously so that after k integrations one finds that $\Re(k+\alpha) > 0$. Then one obtains $\Gamma(k+\alpha)/\alpha(\alpha+1)\ldots(\alpha+k-1) = \Gamma(\alpha)$ plus a whole lot of contributions such as $-\epsilon^{\alpha} \exp(-\epsilon)/\alpha$, $-\epsilon^{\alpha+1} \exp(-\epsilon)/\alpha$ and so on. In accordance with regularisation, these infinities are omitted or removed, leaving only $\Gamma(\alpha)$, which is the same result for $\Re\, \alpha > 0$. According to p. 32 of Ref. [23], the remaining term was called the "finite part" by Hadamard, who showed that it obeys many of the ordinary rules of integration.

To demonstrate the relationship between the finite part of a divergent integral and the regularised value of a divergent series, consider the following integral:

$$I = \int_0^{\infty} dx \; e^{ax} = \lim_{p\to\infty} \int_0^{p} dx \, e^{ax} = \lim_{p\to\infty} \left[\frac{e^{ap}-1}{a} \right] \ . \tag{19}$$

For $\Re\, a < 0$, the integral in Eq. (19) is convergent, yielding a value of $-1/a$. On the other hand, for $\Re\, a > 0$, it is divergent, but removing the infinity or first term in the last member yields, once again, a finite part of $-1/a$. Now to connect the above result with regularisation of a divergent series, we write the integral in terms of an arbitrary positive real parameter, say b, as

$$I = \int_0^\infty dx\, e^{-bx} e^{(a+b)x} = \int_0^\infty dx\, e^{-bx} \sum_{k=0}^\infty \frac{(a+b)^k x^k}{k!} \ . \tag{20}$$

In obtaining this result we have employed the asymptotic method of expanding most of the exponential as described on p. 113 of Ref. [8]. Next we interchange the order of the summation and integration. Because most of the exponential has been expanded, an impropriety has occurred. Evaluating the resulting integral yields a divergent series, depending, of course, on the values of a and b. As a consequence, we have to replace the equals sign by an equivalence symbol in the final result. Hence, the integral I becomes

$$I \equiv \sum_{k=0}^\infty \frac{(a+b)^k}{k!} \int_0^\infty dx\, e^{-bx}\, x^k = \frac{1}{a+b} \sum_{k=1}^\infty (1+a/b)^k \ . \tag{21}$$

The series in the final member of the above result is merely the geometric series with the variable equal to $(1+a/b)$. If we introduce the regularised value of the series, viz. $-(1+a/b)/(a/b)$, into the last member of the above equivalence, then we obtain the finite value of $-1/a$ as we did when we evaluated the integral in Eq. (19). That is, by regularising the series in Eq. (21), we have found that $I \equiv -1/a$, which is identical to the direct evaluation of the divergent integral and removal of the infinity or the first term in the last member of Eq. (19). Hence, regularisation of a divergent series is equivalent to evaluating the finite part of a divergent integral.

In Ref. [9] Farassat discusses the issue of whether the appearance of divergent integrals in applications constitutes a breakdown in physics or mathematics. He concludes that divergent integrals arise as a result of incorrect mathematics because an ordinary derivative has been wrongly evaluated inside an improper integral. Therefore, he finds regularisation of a divergent integral or taking the finite part as a necessary corrective measure. In view of the equivalence between divergent integrals and divergent series, the same can be said of divergent series in asymptotic expansions. By itself, a divergent series yields infinity, but when regularised, one obtains a finite value or part. Yet the original function from which an asymptotic series is derived is finite. Therefore, when an asymptotic method is used to obtain a power series expansion, an impropriety or flaw associated with the method has been invoked. The derivation of asymptotic expansions from integral representations, e.g. by Laplace's method or the method of steepest descent, invariably involves integrating over a range that is outside the circle or disk of absolute convergence of the expanded function. The derivation of an asymptotic series by applying the iterative method to differential equations also involves introducing an infinity as is explained in Ch. 2 of Ref. [17]. Therefore, it is to be expected that series, which are either divergent or conditionally convergent outside the radius of absolute convergence will possess vastly different properties than within the radius of absolute convergence. Moreover, whilst regularisation has been presented as a mathematical abstraction for obtaining the finite value of a divergent series so far, it is required in asymptotics for correcting the impropriety of the method used to obtain the expansion from the original function.

With regard to the issue of whether the appearance of divergent integrals and series in applications constitutes a breakdown in physics or mathematics, it is more than likely to be a combination of both when dealing with the very small such as the Planck scale in particle physics and physical cosmology. However, before the physical issues can be tackled, the mathematics needs to be corrected first. At the moment it appears that the wrong sort of mathematics is being employed, which has resulted in the rather bizarre predictions being made by eminent cosmologists and particle theorists today. We shall consider the physicist's approach to renormalisation and compare it with the mathematical concept of regularisation in a later section.

5. Grandi's Series Revisited

Let us now return to Grandi's series, which is obtained by putting z equal to -1 in the geometric series. From our study of the geometric series in the previous section we know that Grandi's series is conditionally convergent, and not divergent according to Copson's definition. Hence, the limit of the series is simply 1/(1-(-1)) or 1/2, a result which is entirely consistent with Leibniz's law of justice. If it is not divergent, then how can the introduction of an infinite number of zeros affect the limit as Bernoulli found?

Before we can consider this question, we need to examine what happens when an infinite number of zeros is introduced into an absolutely convergent series. Therefore, we write the geometric series as

$$S(z) = 1 + 0 + z + 0 + z^2 + 0 + z^3 + 0 + z^4 + 0 + \dots . \tag{22}$$

Since every second element is zero, we can express $S(z)$ alternatively as

$$S(z) = \sum_{k=1}^{\infty} \Big(\frac{1-(-1)^k}{2} \Big) z^{(k-1)/2} + \sum_{k=2}^{\infty} \Big(\frac{1+(-1)^k)}{2} \Big) 0^k \ . \tag{23}$$

The above result represents the sum of four separate series. Hence, separating each series we obtain

$$S(z) = \frac{1}{2\sqrt{z}} \sum_{k=1}^{\infty} z^{k/2} - \frac{1}{2\sqrt{z}} \sum_{k=1}^{\infty} (-1)^k z^{k/2} + \frac{1}{2} \sum_{k=2}^{\infty} 0^k - \frac{1}{2} \sum_{k=2}^{\infty} (-0)^k \ . \tag{24}$$

The last two terms in Eq. (24) vanish according to Equivalence (13), which now becomes an equation since $z = 0$. That is, the equivalence symbol can be replaced by an equals sign. Furthermore, for $|z| < 1$ the first two series can be evaluated with the equation form of Equivalence (13). Then we find that

$$S(z) = \frac{1}{2\sqrt{z}} \, \frac{\sqrt{z}}{1-\sqrt{z}} - \frac{1}{2\sqrt{z}} \, \frac{(-\sqrt{z}\,)}{1+\sqrt{z}} = \frac{1}{1-z} \ . \tag{25}$$

Hence, we see that the introduction of an infinite number of zeros into the geometric series has no effect on its limit when $|z|<1$.

Let us now consider the introduction of an infinite number of zeros into Grandi's series as Bernoulli did. Then we can write the modified version of Grandi's series as

$$\begin{aligned} S_3(1,0,-1) &= 1+0-1+1+0-1+\ldots \\ &= 1+0+0+1+0+0+1+\ldots \\ &\quad +0+0-1+0+0-1+0+\ldots \ . \end{aligned} \tag{26}$$

That is, Grandi's series represents the sum of two separate series, $S_3(1,0,0)$ and $S_3(0,0,-1)$. The first series in Eq. (26) can be written alternatively as

$$S_3(1,0,0) = \frac{1}{3}\sum_{k=0}^{\infty}\Big(1^k + e^{2\pi ik/3} + e^{4\pi ik/3}\Big) \ , \tag{27}$$

while the other series can be expressed as

$$S_3(0,0,-1) = -\frac{1}{3}\sum_{k=0}^{\infty}\Big(1^k + e^{2\pi i(k-2)/3} + e^{4\pi i(k-2)/3}\Big) \ . \tag{28}$$

Therefore, we see that both $S_3(1,0,0)$ and $S_3(0,0,-1)$ are themselves the sums of three specific geometric series. In each case the first series yields a regularised value of infinity, but as we are interested in the sum of $S_3(1,0,0)$ and $S_3(0,0,-1)$, these infinities cancel.

In order to evaluate the other series in Eqs. (27) and (28), we introduce the rhs of Equivalence (13). As z is equal to either $\exp(2\pi i/3)$ or $\exp(4\pi i/3)$ in these series, we have $\Re z<1$. Hence, the equivalence symbol can be replaced by an equals sign, resulting in an equation. Combining the regularised values of all the series gives

$$S_3(1,0,-1) = \frac{1}{3}\Big(\frac{1-e^{-4\pi i/3}}{1-e^{2\pi i/3}} + \frac{1-e^{-2\pi i/3}}{1-e^{4\pi i/3}}\Big) = \frac{2}{3} \ , \tag{29}$$

which is the identical result obtained by Bernoulli. In addition, if the series had been given by $1-1+0+1-1+0+\ldots$, then the series would have become $S_3(1,-1,0)$, which is the sum of $S_3(1,0,0)$ and $S_3(0,-1,0)$. The limit of the latter series is given by

$$S_3(0,-1,0) = -\frac{1}{3}\sum_{k=0}^{\infty}\Big(1^k + e^{2\pi i(k-1)/3} + e^{4\pi i(k-1)/3}\Big) \ . \tag{30}$$

By combining this result with the limit for $S_3(1,0,0)$, we arrive at

$$S_3(1,-1,0) = \frac{1}{3}\Big(\frac{1-e^{-2\pi i/3}}{1-e^{2\pi i/3}} + \frac{1-e^{-4\pi i/3}}{1-e^{4\pi i/3}}\Big) \ . \tag{31}$$

After a little algebra we find that $S_3(1,-1,0)$ equals $1/3$, once again demonstrating that the position of the infinite number of zeros in a conditionally convergent series affects the limit value.

6. Recurring Series

As a result of the foregoing analysis, we can consider any periodically recurring series of the form where

$$S_k(a_1,a_2,a_3,\ldots,a_k)=a_1+a_2+\ldots+a_k+a_1+a_2+\ldots+a_k+\ldots \quad . \tag{32}$$

From the previous section we know that $S_k(a_1,a_2,a_3,\ldots,a_k)$ can be expressed as a finite sum of series involving zeros and one. That is, the above can be written as

$$\begin{aligned}S_k(a_1,a_2,a_3,\ldots,a_k)&=a_1S_k(1,0,0,\ldots,0)+a_2S_k(0,1,0,\ldots,0)+\ldots\\ &+\ a_kS_k(0,0,0,\ldots,1)\ .\end{aligned} \tag{33}$$

Each series on the rhs of Eq. (33) possesses an infinite sum over unity, just as in Eqs. (27) and (28). In the preceding cases we found that they eventually cancelled each other when evaluating the limits for $S_3(1,0,-1)$ and $S_3(1,-1,0)$. In the above result the sums over unity will cancel each other if and only if $\sum_{j=1}^{k} a_j=0$. Otherwise, one obtains infinity. Hence, we need to make this assumption or condition to obtain a finite limit for $S_k(a_1,a_2,a_3,\ldots,a_k)$. In addition, if we denote $S_k(0,\ldots,i_j,\ldots,0)$ as the series composed of zeros and ones, where the ones only appear at the j-th position of each cycle of k terms, e.g. $S_3(0,1_2,0)$ has unity appearing at the second position of every triple (0,1,0), then the above equation can be represented as

$$\begin{aligned}S_k(a_1,a_2,a_3,\ldots,a_k)&=\sum_{j=1}^{k}a_jS_k(0,0,\ldots,1_j,\ldots,0)\\ &=\ \frac{1}{k}\sum_{j=1}^{k}a_j\sum_{l=1}^{k-1}\sum_{n=0}^{\infty}e^{2(n-j+1)l\pi i/k}\quad .\end{aligned} \tag{34}$$

The infinite sum over n can be removed by introducing the rhs of Equivalence (13) except that because $\Re\,\exp(2l\pi i/k)<1$ for all values of l in the above result, the equivalence symbol can be replaced by an equals sign. Then Eq. (34) reduces to

$$S_k(a_1,a_2,a_3,\ldots,a_k)=\frac{i}{2k}\sum_{l=1}^{k-1}\frac{1}{\sin(l\pi/k)}\sum_{j=1}^{k}a_j\,e^{-(2j-1)l\pi i/k}\quad . \tag{35}$$

Therefore, we arrive at a finite double sum that is very much dependent upon the values of the a_j.

It should be noted that in Eq. (35) the a_j need not necessarily be real. That is, they can be complex provided that $\sum_{j=1}^{k} a_j=0$. On the other hand, if they are purely real, then Eq. (32) can be simplified even further because the real part of the final sum, viz. the sum over j, must vanish. Therefore, in this case Eq. (35) yields

$$S_k(a_1,a_2,a_3,\ldots,a_k)=\frac{a_1}{2}\Big(1-\frac{1}{k}\Big)+\frac{i}{2k}\sum_{l=1}^{k-1}\frac{1}{\sin(l\pi/k)}$$

$$\times \sum_{j=2}^{k} a_j \Big(\frac{\sin((2j-1)l\pi/k)}{\sin(l\pi/k)} \Big) \ . \tag{36}$$

If we let $a_1=1$, $a_2=0$, $a_3=-1$, and $k=3$ in the above equation, then, as expected, we find that $S_3(1,0,-1)=2/3$, while for $a_1=1$, $a_2=-1$, $a_3=0$, and $k=3$, we find that $S_3(1,-1,0)=1/3$. For a more complicated series such as $S_4(3,2,-4,-1)$, we obtain a limit value of $9/4$.

Since we have seen that Grandi's series is conditionally convergent rather than divergent, we now turn to the question of who is correct: Callet or Lagrange? In actual fact, both are correct, but for different reasons. First, we note that Grandi's series admits an infinite number of encodings. To see this more clearly, consider the following series:

$$\lim_{z\to 1} S(z) = \lim_{z\to 1} (1-z^p) \left(1 + z^q + z^{2q} + z^{3q} + \ldots\right) = \lim_{z\to 1} (1-z^p) \sum_{k=0}^{\infty} z^{qk}, \tag{37}$$

where both $\Re\, p$ and $\Re\, q$ are greater than zero. It is obvious that if we put $z=1$ in the above result, then we will obtain Grandi's series. Introducing the regularised value of the geometric series, viz. Equivalence (13), into the above result, we arrive at

$$\lim_{z\to 1} S(z) = \lim_{z\to 1} \left(\frac{1-z^p}{1-z^q} \right) = \frac{p}{q} \ . \tag{38}$$

Eq. (38) has been obtained by applying l'Hospital's rule [30]. Note that there is no equivalence symbol in Eq. (38) because the infinity in the series is cancelled by the factor of $(1-z^p)$ preceding it. For $p=1$ and $q=2$, we find that

$$S(z) = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 + \ldots \ , \tag{39}$$

while if $p=2$ and $q=3$, then $S(z)$ becomes

$$S(z) = 1 - 0\cdot z - z^2 + z^3 + 0\cdot z^4 - z^5 + z^6 + \ldots \ . \tag{40}$$

Therefore, in the first instance we recover the geometric series with z replaced by $-z$, while in the second case we recover the Callet/Lagrange example. For $p=4$ and $q=5$, however, we find that

$$S(z) = 1 - z^4 + z^5 - z^9 + z^{10} - z^{14} + \ldots \ . \tag{41}$$

In the three preceding examples putting $z=1$ always yields Grandi's series. In fact, Eq. (38) admits an infinite number of representations for Grandi's series. Therefore, the problem is that Grandi's series does not possess a unique representation. To obtain a specific representation, we need to impose conditions so that one specific representation can be isolated for $S(z)$. This is essentially what Lagrange did by stipulating that $S(z)$ had to be a "true" power series. Consequently, the powers of z in $S(z)$ had to be positive integers, which automatically excludes p and q from being anything other than positive integers. It also implies that all coefficients of $S(z)$ have to be non-zero. Then one finds that $p=1$ and $q=2$, which yields the value for the limit of the series obtained by Grandi, Euler and Leibniz, not to mention Lagrange, of course.

The problem concerning uniqueness does not arise in asymptotics because an asymptotic expansion is generally determined over a range of values for the variable. Hence, the asymptotic expansion is valid for an infinite number of values of the variable, which guarantees its uniqueness. However, Grandi's series represents an infinite series for one value of the variable, viz. $z=1$. Consequently, a multitude of valid representations exist for such a series as we have witnessed above. This situation resembles the application of boundary conditions in order to derive a specific solution from the general solution to a differential equation.

7. Logarithmic Divergence

It should be emphasised again that the regularisation of series which diverge logarithmically such as the harmonic series presented earlier is a much different proposition from that for the geometric series. To see this more clearly, if we put $z=-1$ in Equivalence (15), then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{k+1} - \log(\infty) \equiv 0 \ . \tag{42}$$

where $-\log(0)$ has been replaced by $\log(\infty)$. The problem with this result is that it has been obtained by integrating the singularity in the geometric series, bearing in mind that the singularity is now situated at $z=-1$ rather than at $z=1$ due to fact that z has been replaced by $-z$ in the derivation of Equivalence (15). As yet, a theory of integrating singularities does not exist and it could well be that there may be a missing term like a constant of integration. This means that more rigorous mathematics is required to establish whether the above equivalence is correct.

As indicated earlier, the quantity on the left hand side of the above result was made famous, again by Euler, who found that it yielded a constant. In fact, he was effectively regularising the series. Today, the constant that remains in this regularisation process is known as Euler's constant [13]. Sometimes it is called the Euler-Mascheroni constant because the latter calculated it to 32 decimal places. Not long afterwards, a controversy arose, where it was found that the last 12 decimal places Mascheroni had calculated were incorrect. Specifically, Euler found that

$$\sum_{k=0}^{\infty} \frac{1}{k+1} - \log(\infty) = \gamma = 0.577\,215\,664\,901\ldots \ . \tag{43}$$

Because of this result, one cannot simply subtract a logarithmic infinity from an integrated divergent series at its most singular point as in Equivalence (42). Previously, we were successful in regularising the geometric series by introducing the gamma function into the analysis. Let us do the same here by multiplying the summand of $1/(k+1)$ by k!/k! and introducing the integral representation for the gamma function in the numerator. Then the harmonic series can be written as

$$\sum_{k=1}^{\infty}\frac{1}{k}=\int_0^{\infty} dt\; e^{-t}\sum_{k=1}^{\infty}\frac{t^{k-1}}{k!}=\int_0^{\infty} dt\;\left(1-e^{-t}\right)/t\ . \tag{44}$$

The integral in the above result is not singular at the lower limit, but is logarithmically divergent at the upper limit. Since $\log t$ is obtained by integrating $1/t$ between 0 and t, we subtract the integral $\int_0^1 dt\; t^{-1}$ from the rhs of the above result in order to regularise it. Thus, the rhs becomes

$$I=\int_0^1 dt\;\left(\frac{1}{\log t}+\frac{1}{1-t}\right)\ , \tag{45}$$

which, according to No. 8.367(6) of Ref. [11], is the integral representation for Euler's constant. That is, the above result is finite, not equal to zero as implied by Equivalence (42).

Euler's regularisation formula provides us with a method or scheme for regularising logarithmically divergent series with far more complicated summands than that in the harmonic series. In such cases, all we need to do is subtract the entire harmonic series and add Euler's constant. E.g., consider the following series

$$S(z)=\sum_{k=0}^{\infty}\frac{1}{k+z}\ . \tag{46}$$

By introducing Euler's regularisation formula into the above result, we find that $S(z)-\sum_{k=0}^{\infty}1/(k+1)+\gamma$ is now finite. In fact, multiplying this result by -1, we see that according to No. 8.362(1) of Ref. [11], it is the representation for the digamma function, which is defined as $\psi(z)=d\log\Gamma(z)/dz$.

The reason why regularisation of a logarithmically divergent series is a much different proposition than regularisation of a divergent series with an algebraic infinity such as the geometric series is because we can truncate the series at N in the sum, replace the logarithmic infinity by $\log N$ and still come up with an accurate approximation to γ. That is, Eq. (43) can be written as an approximation given by

$$\sum_{k=0}^{N}\frac{1}{k+1}-\log N\approx\gamma\ . \tag{47}$$

As N increases, the lhs becomes more and more accurate as an approximation to Euler's constant. In fact, this is the standard approach for determining numerical values of γ. In our study of the geometric series we could not expect to obtain an accurate approximation to the limit of $1/(1-z)$ for z outside the disk of absolute convergence by truncating the series at ever increasing values of N and then subtracting values of N instead of infinity.

Euler's formula for γ is a unique example of a mathematical quantity that has been first evaluated by employing the concept of regularisation, albeit of a logarithmically diverging series. Only recently has a rapidly converging formula for γ been discovered in terms of an infinite set of relatively novel numbers known as the reciprocal logarithm numbers or A_k given in Ref. [18]. According to p. 137 of Ref. [2], the magnitudes of these numbers have been referred to in the past as either the Gregory or the Cauchy numbers, but important properties for them have only appeared for the first time in Ref. [18]. There, the new result for γ, which is known as Hurst's formula, is given as

$$\gamma = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} A_k \ , \tag{48}$$

where $A_0 = 1, A_1 = 1/2, A_2 = -1/12$, and

$$A_k = \frac{(-1)^k}{k!} \int_0^1 dt \, \frac{\Gamma(k+t-1)}{\Gamma(t-1)} \ . \tag{49}$$

Furthermore, by using the properties of Volterra functions and the orthogonality of Laguerre polynomials, Apelblat obtains on p. 156 of Ref. [2] an alternative result for the A_k, which for $k \geq 1$ is given by

$$A_k = (-1)^k \int_0^{\infty} dt \, \frac{1}{(t+1)^k \, (\pi^2 + \log^2 t)} \ . \tag{50}$$

If this result is introduced into Hurst's formula and the order of the integration and summation are interchanged, then with the aid of the lower result in Equivalence (15), i.e. the equation form, we arrive at a new integral representation for Euler's constant, which is

$$\gamma = -\int_0^{\infty} \frac{dt}{\pi^2 + \log^2 t} \, \log\Big(\frac{t}{t+1}\Big) \ . \tag{51}$$

8. Regularisation versus Renormalisation

The regularisation formula discovered by Euler can also be used to see whether it is consistent with the physicist's concept of renormalisation. The longitudinal dielectric response function of an electron-positron plasma in an external magnetic field denoted by $\epsilon(\mathbf{q}, \omega, B)$ arises in the response theory of particle-anti-particle plasmas [22]. This quantity can be separated into particle and vacuum parts denoted by the subscripts p and v respectively. Because it is derived via quantum mechanics, it needs to be renormalised in accordance with standard quantum electrodynamic theory (QED). Hence, the divergent vacuum polarisation term must be removed, which means evaluating

$$\Re \, \bar{\epsilon}(\mathbf{q}, \omega, B) = \Re \, \epsilon_p(\mathbf{q}, \omega, B) + \Re \, \epsilon_V(\mathbf{q}, \omega, B) - \Re \, \epsilon_V(0,0,0) \ . \tag{52}$$

In this equation the field-free vacuum term is given by the following divergent integral:

$$\Re \, \epsilon_V(0,0,0) = \frac{e^2}{4\pi^2} \int d^3p \left(\frac{m^2 + 2p^2/3}{(p^2+m^2)^{5/2}} \right) \ . \tag{53}$$

Because the last two terms of Eq. (52) diverge, they are both renormalised by introducing $-\Re\,\epsilon_V(0,0,B)$ after the second term on the rhs and $+\Re\,\epsilon_V(0,0,B)$ before the final term. This results in the emergence of a quantity known as the longitudinal static uniform polarisability, which is defined as

$$\alpha_{\|}(B) = \Re\,\epsilon_V(0,0,B) - \Re\,\epsilon_V(0,0,0) = \frac{e^3 B}{4\pi}\sum_{n=0}^{\infty} a_n \times \int_{-\infty}^{\infty} dp_z \left(\frac{m^2+2neB}{(p_z^2+m^2+2neB)^{5/2}}\right) - \frac{e^2}{4\pi^2}\int d^3p \left(\frac{m^2+2p^2/3}{(p^2+m^2)^{5/2}}\right) \quad . \tag{54}$$

In Eq. (54), $a_n = 2$ for $n > 0$, while for $n = 0$, $a_n = 1$.

For those with a physical bent, e and m represent the charge and mass of an electron, while the momentum p is expressed in components p_x, p_y and p_z. Hence, in the vacuum term $d^3p = dp_x dp_y dp_z$, where each component of the momentum ranges from $-\infty$ to ∞. The summation over n arises from summing over the Landau levels that result from the solutions for the Dirac equation. The introduction of a magnetic field into response theory has the effect of suppressing the x and y components of the momentum. That is, $2neB$ takes on the role of $p_x^2 + p_y^2 = p_\perp^2$.

The first term on the rhs of Eq. (54) is logarithmically divergent, which can be observed by evaluating the integral over p_z. With the aid of No. 2.271(6) in Ref. [11], we find that

$$\frac{e^3 B}{4\pi}\sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} dp_z \left(\frac{m^2+2neB}{(p_z^2+m^2+2neB)^{5/2}}\right) = \frac{e^3 B}{3\pi}\left(\frac{1}{m^2} + \frac{1}{eB}\sum_{n=1}^{\infty}\frac{1}{n+m^2/2eB}\right) . \tag{55}$$

The series in Eq. (55) can be regularised by introducing Euler's regularisation formula. Denoting the finite part by $P(b)$, where $b = 2eB/m^2$, we obtain

$$P(b) = \sum_{n=1}^{\infty}\frac{1}{n+1/b} - \sum_{n=0}^{\infty}\frac{1}{n+1} + \gamma \quad . \tag{56}$$

From No. 8.362(1) of Ref. [11], which states that the digamma function is given by

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n+1}\right) \quad . \tag{57}$$

we arrive at

$$P(b) = -\psi(1/b) - b \quad . \tag{58}$$

Then by introducing this result into Eq. (53) we obtain

$$\frac{e^3 B}{4\pi}\sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} dp_z \left(\frac{m^2+2neB}{(p_z^2+m^2+2neB)^{5/2}}\right) \equiv -\frac{e^2}{3\pi}\Big[b/2 + \psi(1/b)\Big] \quad . \tag{59}$$

Now let us examine when the static uniform polarisability is renormalised according to the physicist's approach. First, it is found that the free-field vacuum term or the second term on the rhs of Eq. (54) is logarithmically divergent, but does not match the $B \to 0$ limit of the preceding term, which is also logarithmically divergent. Consequently, the second term on the rhs is replaced by the $B \to 0$ limit of the first term. Then with the aid of Eq. (55) the longitudinal static uniform polarisability becomes

$$\alpha_{||}(B) = \frac{e^2 b}{3\pi} \sum_{n=0}^{\infty} \left(\frac{1}{1+nb} \right) - \frac{e^2 b}{6\pi} - \lim_{b \to 0} \frac{e^2 b}{3\pi} \sum_{n=0}^{\infty} \left(\frac{1}{1+nb} \right) . \tag{60}$$

Converting the last term to an integral yields a logarithmically divergent integral, but the resulting integral is difficult to match with the first term when the latter is also converted into an integral. Instead, the process of renormalisation involves:

1. converting the first term into an integral by replacing $1/(1+nb)$ with $\int_0^{\infty} dt \, \exp(-(1+nb)t)$,
2. interchanging the order of the summation and integration,
3. subtracting the $B \to 0$ limit of the resulting integral.

Carrying out these steps yields

$$\alpha_{||}(B) = \frac{e^2}{3\pi} \left[\int_0^{\infty} dt \left(\frac{be^{-t}}{1-e^{-bt}} - \frac{e^{-t}}{t} \right) - \frac{b}{2} \right] . \tag{61}$$

By introducing some of the integral identities that appear in Secs. 8.361 and 8.367 of Ref. [11], one eventually arrives at

$$\alpha_{||}(B) = -(e^2/3\pi) \left[b/2 + \log(b) + \psi(1/b) \right] . \tag{62}$$

This result was first obtained by Bakshi, Cover and Kalman in Ref. [3].

By comparing the rhs of Equivalence (62) with the rhs of Eq. (59), we see that there is a discrepancy of $\log(b)$ in the bracketed terms. Hence, we have seen that the mathematical approach to regularising a divergent series can yield a different result from the physicist's approach of renormalisation. That is, regularising a divergent mathematical quantity arising out of a physical theory may not necessarily yield the correct physical result. This vindicates the statements made earlier concerning whether the appearance of divergent series and integrals in theoretical physics constitutes a breakdown in the mathematics or the physical theory. It is likely to be a combination of both with the creation of a physical theory out of new mathematics.

9. Terminants

All the divergent series that have been considered so far have been relatively elementary, but in order to develop a theory of divergent series, more complicated examples will need to be analysed. Although this is well and truly beyond the scope of the present work, we can at least discuss the issue of regularising those series, which Euler referred to as divergent par excellence. Previously, it was remarked that such series had rapidly diverging coefficients a_k, which were equal to $(-1)^k k!$. In fact, these series can be generalised by replacing the $k!$ factor in the coefficients by the gamma function $\Gamma(k+\alpha)$. Then they become what are known today as terminants. This terminology was introduced by Dingle [8] after he noticed that the late terms in many asymptotic expansions for the special functions of mathematical physics could be approximated by them. Specifically, there are two types of such series: the first type is defined as

$$T_I(N,\alpha,z) = \sum_{k=N}^{\infty} \Gamma(k+\alpha)(-z)^k \ , \tag{63}$$

while the second type is defined as

$$T_{II}(N,\alpha,z) = \sum_{k=N}^{\infty} \Gamma(k+\alpha) z^k \ . \tag{64}$$

In these results N is referred to as the truncation parameter.

Since the limit point is zero in the above series, both types of terminants represent small z asymptotic series. Had they been expressed in terms of powers of $1/z$, which is how Dingle defined them originally in Ref. [8], then they would have represented large z asymptotic series. By asymptotic, we mean here according to the standard Poincaré prescription discussed on p. 151 of Ref. [33]. As mentioned in the introduction, by adopting this prescription for sufficiently small values of z, namely $|z| \ll 1$, one can truncate the series after only a few terms and still produce an accurate approximation to the actual value of the original function from which the series has been derived. Furthermore, the point at which the approximation begins to break down, known as the optimal point of truncation and denoted by N_T in this work, increases or diverges to infinity as $z \to 0$. For those seeking an understanding of the important concept of optimal truncation, they should consult Sec. 4.6 of Ref. [24]. As long as $N < N_T$ or even for $N \approx N_T$, one can still obtain an accurate approximation to the original function. However, the accuracy of the approximation wanes dramatically as $N_T \to 0$, so that truncation of the series is no longer a valid option for those values of z in either the intermediate region, typically given by $0.1 < |z| < 2$, or for "large values" of $|z|$ greater than 2. As was also discussed in the introduction, since asymptotic series possess limited ranges of applicability and suffer from deficiencies in accuracy, asymptotics as a mathematical discipline has often been ridiculed by pure mathematicians, who point out that mathematics is supposed to be an exact science, not composed of vague concepts and quantities. In reality, the cause for this state of affairs is the adoption of the Poincaré prescription, particularly truncating asymptotic series.

Because of the rapid divergence in the coefficients of both types of terminants, which results in a zero radius of absolute convergence, they too represent a different proposition to regularise compared with the geometric series studied earlier. Thus, the question becomes: how do we regularise them? When discussing regularisation of the geometric series above, we introduced the integral representation for the gamma function in the numerator, interchanged the order of the summation and integration and finally evaluated the sum. This approach to obtaining limits to divergent series is known more commonly as Borel summation. So let us do the same to the first type of terminant. Then we find that

$$T_I(N,\alpha,z) = \int_0^\infty dt\; t^{\alpha-1}\, e^{-t} \sum_{k=N}^{\infty} (-zt)^k \;\; . \tag{65}$$

Now we see that the first type of terminant has been expressed in terms of the geometric series. Therefore, if we introduce the regularised value of the latter series into the above result, then we obtain the regularised value of the first type of terminant. In addition, according to our analysis of the geometric series, it is conditionally convergent for $\Re(-zt) < 1$. As t ranges from 0 to infinity, this means that the terminant is conditionally convergent for $\Re z > 0$ and divergent for all other values of z. As a consequence, we observe that an asymptotic series need not necessarily be divergent. That is, an asymptotic expansion is not always divergent; it can also be conditionally convergent.

The introduction of the regularised value of the geometric series into Eq. (65) yields

$$T_I(N,\alpha,z) \equiv (-z)^N \int_0^\infty dt\; \frac{t^{N+\alpha-1}\, e^{-t}}{1+zt} \;\; . \tag{66}$$

Since we have already stated that the geometric series is bijective within the principal branch of the complex plane, i.e. for $|\arg z| < \pi$, the regularised value of the first type of terminant given by the above Cauchy integral is also bijective. That is, there is a definite value for each value of z within the principal branch of the complex plane, which means, in turn, that we are moving closer to Euler's unorthodox view of there being a definite value connected with each divergent series. On the other hand, if in Equivalence (66) we replace z with $z\exp(-2il\pi)$, where l is an arbitrary integer, then we find that

$$T_I(N,\alpha,z\exp(-2il\pi)) \equiv (-z)^N \int_0^\infty dt\; \frac{t^{N+\alpha-1}\, e^{-t}}{1+z\exp(-2il\pi)t} \;\; . \tag{67}$$

We can express the above result as a contour integral in terms of the complex variable s and C, the line contour along the positive real axis. Then Equivalence (67) becomes

$$T_I(N,\alpha,z\exp(-2il\pi)) \equiv (-1)^N z^{N-1} \int_C ds\; \frac{s^{N+\alpha-1}\, e^{-s}}{s-(z^{-1}\exp((2l-1)i\pi))} \;\; , \tag{68}$$

where $-\pi < \arg(z\exp(-2il\pi)) < \pi$ or$(2l-1)\pi < \arg z < (2l+1)\pi$. The problem with Equivalence (68) is that it appears to yield the same regularised value for any value of l. That is, the regularised value appears to be the same for all branches of the complex plane when we might expect it to be different. This is because when asymptotic series including terminants are derived, they are usually expressed in powers of z such as z^β, where $\beta > 1$. For example, the asymptotic expansion for the complementary error function, which is given as No. 7.1.23 in Ref. [1], is

$$\operatorname{erfc}(z) \equiv \frac{e^{-z^2}}{\pi z} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{(-z^2)^k} \ , \quad |\arg z| < 3\pi/4 \, . \tag{69}$$

Note that it has been necessary to introduce the equivalence symbol into this result as a consequence of our previous discussion on the properties of an asymptotic series.

According to Equivalence (68) the value of $\operatorname{erfc}(\exp(-3\pi i/8))$ is expected to be equal to $\operatorname{erfc}(\exp(5\pi i/8))$. Yet, the former yields a value of $0.663\,282\ldots$, while the latter equals $7.117\,400\ldots \times 10^{-24}$. Therefore, Equivalence (68) must be restricted to one branch given by either $2l\pi < \arg(-z^2) < (2l+2)\pi$ or $(l-1/2)\pi < \arg z < (l+1/2)\pi$. Then we are left with the problem of deciding whether the equivalence is valid for either $-3\pi/2 < \arg z < -\pi/2$, $|\arg z| < \pi/2$, or for $\pi/2 < \arg z < 3\pi/2$ within the principal branch of the complex plane.

Another problem with Equivalences (66) and (67) is: what do we do when $\arg z = \pm\pi$? For these values of z the Cauchy integral is singular. Whilst this might not be a serious problem with Equivalence (66) where nearly all the principal branch of the complex plane has been covered anyway, for asymptotic expansions written in terms of z^{β}, where $\beta > 1$, it will mean that the regularised value will be singular well within inside the principal branch. For example, in the case of the complementary error function mentioned above, the Cauchy integral is singular along the positive and negative imaginary axes.

These issues can be resolved as a result of a remarkable discovery made in 1857 by Stokes [31] of what is known today as the Stokes phenomenon. Stokes found that as one moved across specific sectors of the complex plane, called Stokes sectors, asymptotic expansions suddenly acquired extra or jump discontinuous terms. These terms appear at specific rays in the complex plane known as Stokes lines. Along these lines the regularised value as indicated by the Cauchy integral in Equivalence (65) is singular. Moreover, Ch. 1 of Ref. [8] states that the Stokes lines occur at those values of $\arg z$, where all the terms in the terminants are of the same sign and homogeneous in phase. For the first type of terminant this means they occur whenever $\arg z = (2k+1)\pi$, where k is an arbitrary integer. Hence, the regularised value given by Equivalence (67) develops extra terms as $\arg z$ moves across these lines. In addition, on the lines the Cauchy integral will need to be modified.

It should be emphasised that Stokes sectors and lines are fictitious with regard to the original function from which an asymptotic expansion is derived. That is, although the asymptotic expansion develops jump discontinuities as the argument of variable in the expansion changes in the complex plane, it does not necessarily mean that the original function is discontinuous. In fact, it is more often than not continuous across the Stokes lines of discontinuity.

Now the only problem that remains is determining the value of l for which Equivalence (67) is valid. In actual fact, this value is arbitrary, but once it is fixed, the regularised value will change on reaching the Stokes lines at its boundaries and then from each Stokes line to the adjacent Stokes sectors. Because of the arbitrariness in the choice of a primary Stokes sector, one can no longer only provide a regularised value to a series expansion to represent a function. Accompanying the regularised value must also be the values of $\arg z$ for which it is valid. This applies to both Stokes sectors and lines. We shall refer to the combination of the series expansion, regularised value and the Stokes sector over which the latter is valid as an asymptotic form.

Frequently, asymptotic expansions are derived when the argument of the variable is real, positive and situated in the principal branch of the complex plane. If this is the case, then we let the primary Stokes sector for Equivalence (67) be given by the $l=0$ value or in other words, by $|\arg z|<\pi$. Hence, Equivalence (66) as an asymptotic form becomes

$$T_I(N,\alpha,z) \equiv (-z)^N \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{1+zt} \quad , \quad |\arg z| < \pi \ . \tag{70}$$

Let us now turn our attention to the second type of terminant given by Eq. (64). Borel summation of this result yields

$$T_{II}(N,\alpha,z) \equiv z^N \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{1-zt} \ . \tag{71}$$

If z is replaced by $z\exp(-2li\pi)$, where l is an arbitrary integer integer, then Equivalence (71) becomes

$$T_{II}(N,\alpha,z\exp(-2il\pi)) \equiv z^N \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{1-z\exp(-2il\pi)t} \ . \tag{72}$$

Whilst the integral in the above equivalence is defined for complex values of z, it is singular for positive real values of z. This is a problem since it has already been stated that whenever a Type II asymptotic expansion is derived, it is usually for these values of z.

The situation can be resolved by noting that the Stokes lines for this type of terminant occur at $\arg z=2k\pi$, where k is an arbitrary integer. Consequently, instead of nominating a primary Stokes sector, we must now nominate a primary Stokes line. Again, this is arbitrary, but we shall take it to be the $k=0$ line. Furthermore, in accordance with the rules for the Stokes phenomenon given in Ch. 1 of Ref. [8], as soon as $\arg z$ moves off this line in either direction, the regularised value must acquire jump discontinuous terms. This produces two more problems:

1. Because of the singularity occurring at $t=1/z$, how do we interpret Equivalence (72) along the primary Stokes line?

2. What are the jump discontinuous terms when $\arg z \neq 0$?

Note that these problems apply to the first type of terminant when $\arg z$ reaches the boundary of the principal branch of the complex plane, i.e. when $\arg z=\pm\pi$. We shall be able to consider this situation when the Type II situation has been resolved.

Both the problems mentioned in the previous paragraph are addressed and actually resolved by Dingle in Ref. [8]. For the first problem he points out that since the variable and terms in the series are all positive and real along the primary Stokes line, which we have taken to be $\arg z = 0$, an initially real function cannot acquire an imaginary part. This concept is based on the pioneering work of Zwaan [35] and is referred to as the Zwaan-Dingle principle in Ref. [17]. Basically, it means that the regularised value must be real when $\arg z$ is situated on a Stokes line initially. Furthermore, the singularity at $t = 1/z$ results in a complex term according to Cauchy's residue theorem. So, in order to guarantee that the regularised value is real along the primary Stokes line, we need to evaluate the Cauchy principal value of the integral in Equivalence (66). Hence, the asymptotic form for the second type of terminant becomes

$$T_{II}(N, \alpha, z) \equiv z^N \int_0^\infty dt \, \frac{t^{N+\alpha-1} \, e^{-t}}{1 - zt} \quad , \quad \arg z = 0 \ . \tag{73}$$

In regard to the second question, by using remarkable insight, Dingle points out on p. 411 of Ref. [8] that the discontinuous terms in the Stokes phenomenon arise from the pole in the singular integral of Equivalence (72). In fact, the jump discontinuous terms in the regularised value when $\arg z$ moves off the primary Stokes line in either direction are related to the residue at $t = 1/z$. This is found to be

$$i\gamma \operatorname{Res} I_{II}(z, \alpha) = -iz^{N-1} \int_0^\gamma d\theta \, (1/z)^{N+\alpha-1} e^{-1/z} \ . \tag{74}$$

The above result has been derived by converting the integral in Equivalence (72) to a complex integral along the positive real axis where the variable t has been replaced by the complex variable s. Eq. (74) follows once s is set equal to $1/z + \epsilon \exp(i\theta)$ in the vicinity of the singularity and the limit $\epsilon \to 0$ is taken. Moreover, the question of whether an anti-clockwise rotation or clockwise rotation around the residue should be taken has been left open for the time being with the introduction of γ in the upper limit of the integral. If $\arg z$ is situated just above the positive real axis, then the semi-circular contour around $t = 1/z$ is taken in a clockwise direction in order to be consistent with attempting to avoid its contribution as we did when evaluating the Cauchy principal value. Hence, in this case $\gamma = -\pi$. Conversely, if $\arg z$ is situated just below the positive real axis, then the semi-circular contour is taken in an anti-clockwise direction, i.e. $\gamma = \pi$. In both cases because the semi-residue contribution is removed completely in the process of evaluating the Cauchy principal value in Equivalence (73), we must remove the semi-residue contributions from the integral in Equivalence (72). Therefore, the regularised value of the second type of terminant becomes

$$T_{II}(N, \alpha, z) \equiv \begin{cases} z^N \int_0^\infty dt \, \frac{t^{N+\alpha-1} \, e^{-t}}{1-zt} - i\pi z^{-\alpha} e^{-1/z}, & \arg z = 0 \\ z^N P \int_0^\infty dt \, \frac{t^{N+\alpha-1} \, e^{-t}}{1-zt}, & \arg z = 0, \\ z^N \int_0^\infty dt \, \frac{t^{N+\alpha-1} \, e^{-t}}{1-zt} + i\pi z^{-\alpha} e^{-1/z}, & -2\pi < \arg z < 0. \end{cases} \tag{75}$$

Since we have seen that it is the residues of the Cauchy integrals which are responsible for the jump discontinuities in the Stokes phenomenon, we can now examine the change in the regularised value when z encounters a Stokes line, viz. when $\arg z = \pm\pi$, for the first type of terminant or $T_I(N, \alpha, z)$. In both cases we expect that the Cauchy integral given in Equivalence (68) will form part of the regularised value except that it will have to be modified so that only the principal value is evaluated. Furthermore, the extra terms or jump discontinuous terms to the regularised value along the Stokes lines will be dependent upon the semi-residue contributions, while as arg z moves off the Stokes lines, the extra contributions to the regularised value will become full-residue contributions.

As discussed in Ref. [17], where both types of terminants are generalised by replacing $\Gamma(k+\alpha)$ and z with $\Gamma(pk+q)$ and z^{β} respectively, moving to a higher Stokes sector means that either z or z^{β} undergoes an anti-clockwise rotation of 2π. This means that we need to consider terminants with $z\exp(2i\pi)$ or $l=-1$ in Equivalence (68). According to Sec. 10.1 of Ref. [17], the difference between the regularised value of the first terminant for $z\exp(2i\pi)$ and that for z can be derived via the theory of Mellin transforms [25] and is given by

$$T_I(N,\alpha,z\exp(2i\pi)) - T_I(N,\alpha,z) \equiv 2\pi i \,\mathrm{Res}\left\{I_I(z,\exp(-i\pi),\alpha)\right\} \ , \tag{76}$$

where $I_I(z,\alpha)$ represents the integral on the rhs of Equivalence (68). The residue for this integral is found to be

$$i\gamma \,\mathrm{Res}\left\{I_I(z\exp(-i\pi),\alpha)\right\} = -i\int_0^{\gamma} d\theta\ (1/z)^{\alpha} e^{-i\pi\alpha}\, e^{1/z} \ . \tag{77}$$

By introducing the regularised value given by Equivalence (70) for the second term on the lhs of Equivalence (76) and Eq. (77) into its rhs, one obtains the regularised value of the first series on the lhs. This yields

$$T_I(N,\alpha,z) \equiv (-1)^N z_{-1}^{N-1}\int_C ds\ \frac{s^{N+\alpha-1}\,e^{-s}}{s-(z_{-1}^{-1})} - 2\pi i\, z_{-1}^{-\alpha}\, e^{-i\pi\alpha} e^{1/z_{-1}} \ . \tag{78}$$

In the above result C is, again, the line contour along the positive real axis and $\pi < \arg z < 3\pi$, while $z_{-1} = z\exp(-2i\pi)$.

To obtain the regularised value when $\arg z = \pi$, all we need to do is average the results for the adjacent Stokes sectors, viz. Equivalences (67) and (75), while ensuring that only the Cauchy principal value is evaluated in the resulting integral. Then for $\arg z = \pi$, we find that the regularised value of the first type of terminant is given by

$$T_I(N,\alpha,z) \equiv |z|^N P\int_0^{\infty} dt\ \frac{t^{N+\alpha-1}\,e^{-t}}{1-|z|t} - \pi i\,|z|^{-\alpha}\, e^{-1/|z|} \ . \tag{79}$$

To determine the asymptotic forms for the other or higher Stokes sectors and lines, we continue with more anti-clockwise rotations of 2π. In fact, the generalisation of Equivalence (76) to l rotations yields

$$\begin{aligned} T_I(N,\alpha,z\exp(2li\pi)) &- T_I(N,\alpha,z\exp(2(l-1)i\pi)) \\ &\equiv \ 2\pi i\,\mathrm{Res}\left\{I_I(z\exp(-(2l-1)i\pi),\alpha)\right\} \ . \end{aligned} \tag{80}$$

For $l > 1$, we replace the second term on the lhs by introducing the $l = l-1$ version of Equivalence (80). We continue this process recursively stopping only when we reach the rhs of Equivalence (73). Hence, we see that for the higher Stokes sectors the regularised value is given by the rhs of Equivalence (76) except the second term becomes a sum over all the residues from $k=1$ to l. This means that

$$T_I(N,\alpha,z\exp(2li\pi)) - T_I(N,\alpha,z) \equiv 2\pi i \sum_{k=1}^{l} \text{Res}\left\{ I_I(z\exp(-(2k-1)i\pi),\alpha)\right\} \quad . \tag{81}$$

Replacing $z\exp(2li\pi)$ by z, where$(2l-1)\pi < \arg z < (2l+1)\pi$, and carrying out the finite sum, we find that the regularised value of the first type of terminant reduces to

$$T_I(N,\alpha,z) \equiv (-z_{-l})^N \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{1+z_{-l}\, t} - 2\pi i\, z_{-l}^{-\alpha} e^{-li\pi\alpha} e^{1/z_{-l}}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)}, \tag{82}$$

where $z_{-l} = z\exp(-2li\pi)$. For the Stokes line of $\arg z = (2l+1)\pi$, the regularised value of the first type of terminant is obtained by averaging the regularised values of the abutting Stokes sectors, whilst ensuring that the principal value is evaluated in the resulting contour integral. Thus, we find that

$$T_I(N,\alpha,z) \equiv |z|^{N-1} P \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{t-|z|} - \pi i\, |z|^{-\alpha} e^{-1/|z|} \times \left(2e^{-(l+1)i\pi\alpha}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)} + 1 \right) \quad . \tag{83}$$

To determine the regularised value for $\arg z$ less than zero, we need to consider clockwise rotations of 2π. For example, Equivalence (76) becomes

$$T_I(N,\alpha,z\exp(-2i\pi)) - T_I(N,\alpha,z) \equiv -2\pi i\, \text{Res}\left\{ I_I(z,\exp(i\pi),\alpha)\right\} \quad , \tag{84}$$

which is merely the complex conjugate of Equivalence (76). Consequently, the regularised value of the first type of terminant for negative values of $\arg z$ will be the complex conjugate of the corresponding regularised value for the complex conjugate of z. Hence, the regularised value of the first type terminant for $-(2l+1)\pi < \arg z < -(2l-1)\pi$ is given by

$$T_I(N,\alpha,z) \equiv (-z_l)^N \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{1+z_l t} + 2\pi i\, z_l^{-\alpha} e^{i\pi\alpha} e^{1/z_l}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)} \quad , \tag{85}$$

where $z_l = z\exp(2li\pi)$. Similarly, for the Stokes lines, where $\arg z = -(2l+1)\pi$, we find that

$$T_I(N,\alpha,z) \equiv |z|^{N-1} P \int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{t-|z|} + \pi i\, |z|^{-\alpha} e^{-1/|z|} \times \left(2e^{(l+1)i\pi\alpha}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)} + 1 \right) \quad . \tag{86}$$

Naturally, higher Stokes sectors or branches of the complex plane will also affect the regularised value of the second type of terminant. For this type of terminant the theory of Mellin transforms [25] yields

$$T_{II}(N,\alpha,z\exp(-2li\pi)) - T_{II}(N,\alpha,z\exp(-2(l-1)i\pi)) \equiv 2\pi i\, z^{-\alpha}\, e^{2li\pi\alpha}\, e^{-z^{-\alpha}} \ . \tag{87}$$

The second series on the lhs can be expressed in terms of $T_{II}(N,\alpha,z\exp(-2(l-2)i\pi)$ by replacing l with $l-1$ in the above result. We continue this process stopping at $l=1$. Then with the introduction of the lower form of Equivalence (72) we obtain the regularised value of the second type of terminant for $-2(l+1)\pi<\arg z<-2l\pi$. This is given by

$$T_{II}(N,\alpha,z) \equiv -z^N \int_C ds\, \frac{s^{N+\alpha-1}\, e^{-s}}{1-zs} + \pi i\, |z|^{-\alpha} e^{-1/|z|} \times \left(2e^{-(l+1)i\pi\alpha}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)} + 1 \right) \ . \tag{88}$$

In Equivalence (88) C represents the line contour along the positive real axis as before. For $2l\pi < \arg z < 2(l+1)\pi$, the regularised value is simply the complex conjugate of the above result.

For the Stokes line of $\arg z = -2l\pi$, we can again average the regularised value for each of the abutting Stokes sectors, whilst at the same time ensuring that only the principal value of the resulting integral is evaluated. Then the regularised value of this terminant can be expressed as

$$T_{II}(N,\alpha,z) \equiv |z|^{N-1}\, P\int_0^\infty dt\, \frac{t^{N+\alpha-1}\, e^{-t}}{t-1/|z|} - 2\pi i\, |z|^{-\alpha}\, e^{-1/|z|} \times\ e^{li\pi\alpha}\, \frac{\sin(l\pi\alpha)}{\sin(\pi\alpha)}\, \cos(\pi\alpha) \ . \tag{89}$$

For $\arg z = 2l\pi$, the regularised value of $T_{II}(N,\alpha,z)$ is given by the complex conjugate of the above result.

As mentioned previously, the preceding analysis is applied in Ref. [17] to generalised versions of both types of terminants. There, expressions for the regularised values of both types of terminant are derived for all values of $\arg z$. These expressions simplify drastically for the cases of p equal to the reciprocal of an integer including unity and $p=2$. In addition, Borel summation is extended in the following chapter by eliminating the need for the gamma function to appear in the coefficients of the asymptotic series. Instead, since it is regularisation of the geometric series, which lies at the heart of Borel summation, all we need to do in order to derive the regularised value is to replace a_k in Eq. (1) by $f(k)z^k$ where $f(k)$ can be expressed as a Mellin transform, viz. $f(k)=\int_0^\infty dx\, x^{k-1}F(x)$. We shall return to this issue at the end of Sec. 11.

10. Mellin-Barnes Regularisation

As stated in Sec. 3, Mellin-Barnes (MB) regularisation was first introduced in Ref. [21] to determine the regularised value of the complete asymptotic expansion for a particular case of the generalised Euler-Jacobi series, viz. $S_3(a)=\sum_{k=0}^{\infty}\exp(-ak^3)$. This fascinating technique for obtaining the regularised value of a divergent series has several advantages over Borel summation. First and foremost, the regularised values, which are expressed in terms of MB integrals, are often more amenable, but not always as we shall see later in this article, to numerical computation than the Cauchy integrals obtained via Borel summation. Second, the technique is not limited to series where the coefficients are expressed in terms of the gamma function. Although it was mentioned at the end of the previous section that Borel summation can be extended to coefficients that can be expressed in terms of Mellin transforms, this is also not necessary for carrying out the MB regularisation of a series. In fact, MB regularisation can be applied to a series with a finite radius of absolute convergence such as the geometric series as is done in Ref. [15] and even to convergent series. Another feature of MB regularisation, which will be seen shortly, is that the MB integrals resulting from this technique are valid over domains of convergence, whose ranges are generally greater than Stokes sectors. This means that not only are Stokes lines non-existent and the resulting MB-regularised values more compact, but that the MB-regularised forms for the regularised value overlap. Hence, in these overlapping sectors or common regions we have two representations for the same regularised value of a series. Consequently, the values obtained from MB-regularisation can be checked against each other in these common regions of the domains of convergence. Such checks cannot be accomplished with Borel-summed forms since we have already seen that the latter forms only apply over specific non-overlapping sectors and lines in the complex plane.

As discussed in Ch. 7 of Ref. [17], we still need to consider both types of asymptotic series studied in the previous section separately when carrying out MB regularisation, but now we can make them more general. Specifically, the first type of general series is represented in terms of the general form below Eq. (1) with $a_k=(-1)^k f(k)z^k$, while in the second type of general series the terms are given by $a_k=f(k)z^k$. Hence, terminants form classes within these general types of series. According to Proposition 3 of the same reference, if there exists a real number c such that the poles of $\Gamma(N-s)$ lie to the right of the line given by $N-1<c=\Re s<N$ and the poles of $f(s)\Gamma(s+1-N)$ to the left of it, then the MB-regularised value of the first type of series is found to be

$$S_I(N,z)=\sum_{k=N}^{\infty} f(k)(-z)^k \equiv \int_{c-i\infty}^{c+i\infty} ds\, \frac{z^s f(s)}{e^{-i\pi s}-e^{i\pi s}} \ . \tag{90}$$

This result is subject to the following conditions:

1. as $L\to\infty$, $|f(s)|=O(\exp(-\epsilon_1 L))$ for $s=c+iL$ and $|f(s)|=O(\exp(-\epsilon_2 L))$ for $s=c-iL$, where ϵ_1, $\epsilon_2>0$,

2. $-\pi<\theta=\arg z<\pi$,

3. $z^s f(s)\Gamma(1+s-N)\Gamma(N-s)$ is single-valued to the right of the line.

In addition, as the offset c continues to increase, the MB integral in the above result will eventually increase exponentially regardless of the magnitude of z. It should also be noted that the situation can be adjusted when the poles of $f(s)\Gamma(s+1-N)$ do not lie to the left of the line $c=\Re s$, but then we must consider the specific form of $f(s)$.

In order to derive Equivalence (90) we need to study the following contour integral:

$$I=(-1)^N \int_{c-i\infty}^{c+i\infty} ds\, z^s f(s)\Gamma(1+s-N)\Gamma(N-s) \ , \tag{91}$$

where $N-1<c=\Re s<N$. The first two conditions given above ensure that the contour integral in Eq. (91) decays exponentially at the endpoints. Since I is defined, it can be closed to the right by introducing a contour integral along the great arc from $c-i\infty$ to $c+i\infty$. The condition on $\arg z$ ensures that the contour integral remains single-valued. Consequently, we can apply the Cauchy residue theorem [7, 30, 33]. The third condition means that when the residues of the contour integral are evaluated, one obtains the series on the lhs of Equivalence (87).

Where regularisation becomes an issue is when we wish to evaluate the contour integral along the great arc. In the case of a convergent series the integral along the great arc vanishes and Equivalence (90) becomes an equation. For $\epsilon_1=\ln|z|$ and $\epsilon_2=\ln|z|$, the integral along the great arc can vanish, but it will depend upon the algebraic part of $f(s)$. However, if $\epsilon_1<\ln|z|$ or $\epsilon_2<\ln|z|$, then the integral along the great arc is infinity. In this situation we neglect the integral, which is equivalent to removing the infinity in accordance with the process of regularisation. For more details on this issue the reader is referred to either p. 85 of Ref. [17] or Ref. [15].

If we introduce the seemingly innocuous factor of 1^k in the form of $\exp(-2\pi ilk)$, where l is an arbitrary integer, into $S_I(N,z)$, then the MB-regularised value of the modified series becomes

$$\begin{aligned} S_I\big(N, z\exp(-2li\pi)\big) &= \sum_{k=N}^{\infty} f(k)\big(-z\exp(-2li\pi)\big)^k \equiv I_l(z) \\ &= \int_{c-i\infty}^{c+i\infty} ds\, \frac{z^s\, e^{-2li\pi s} f(s)}{e^{-i\pi s}-e^{i\pi s}} \ , \end{aligned} \tag{92}$$

where, again, $N-1<c=\Re s<N$. Using the conditions on $f(s)$ below Equivalence (90), one finds that the above integral is convergent when

$$(2l-1)\pi-\epsilon_1<\arg z<(2l+1)\pi+\epsilon_2 \ . \tag{93}$$

If we put $l=l+1$ in Equivalence (90), then the domain of convergence for $I_{l+1}(z)$ becomes $(2l+1)\pi-\epsilon_1<\arg z<(2l+3)\pi+\epsilon_2$. Therefore, there is a common sector which is given by $(2l+1)\pi-\epsilon_1<\arg z<(2l+1)\pi+\epsilon_2$, where the regularised value can be evaluated by either $I_l(z)$ or $I_{l+1}(z)$. Without loss of generality α is assumed to be positive and real since if it is negative, we can separate the finite number of terms up to the first value where $k+\alpha$ becomes positive and re-define α in the remaining infinite series. Furthermore, we can evaluate the difference of the integrals over the common sector. As a result, we find that

$$\Delta I_{l+1,l}(z) = I_{l+1}(z) - I_l(z) = \int_{c-i\infty}^{c+i\infty} ds\, z^s e^{-(2l+1)i\pi s}\, f(s) \ . \tag{94}$$

Hence, the difference between consecutive values of l for $I_l(z)$ yields a standard inverse Mellin transform [25]. In particular, for those series which Euler referred to as divergent par excellence, or more generally standard terminants, $f(s)=\Gamma(s+\alpha)$. Then we obtain

$$\Delta I_{l+1,l}(z) = 2\pi i\, z^{-\alpha}\, e^{(2l+1)i\pi\alpha}\, e^{1/z} \ . \tag{95}$$

Since $I_0(a)$ represents the regularised value of $S_1(N,z)$ for $-\pi < \arg z < \pi$, we see from Eq. (94) that $I_1(z) - \Delta I_{1,0}(z)$ is also the regularised value of the series, but only over the sector of $\pi - \epsilon_1 < \arg z < \pi$. However, we know that $I_1(z)$ is defined over $\pi - \epsilon_1 < \arg z < 3\pi + \epsilon_2$. So, if $\Delta I_{1,0}(z)$ is defined over the same region, which is the case with standard terminants according to Eq. (95), then by analytic continuation $I_1(z) - \Delta I_{1,0}(z)$ becomes the regularised value of over the entire branch given by $\pi - \epsilon_1 < \arg z < 3\pi + \epsilon_2$.

Because of Eq. (94) we can replace $I_1(z)$ by $I_2(z) - \Delta I_{2,1}(z)$ for $3\pi - \epsilon_1 < \arg z < 3\pi + \epsilon_2$. However, the domain of convergence for $I_2(z)$ is $3\pi - \epsilon_1 < \arg z < 5\pi + \epsilon_2$. If $\Delta I_{2,1}(z)$ is defined over the same sector, which is valid for standard terminants, then $I_2(z) - \Delta I_{2,1}(z)$ is the regularised value of $S_I(N,z)$ over $3\pi - \epsilon_1 < \arg z < 5\pi + \epsilon_2$. By continuing this process further, one can obtain the regularised value of $S_I(N,z)$ for all branches of the complex plane. Consequently, we arrive at the result given in Proposition 4 of Ref. [17] for the regularised value of a generalised terminant. For a standard Type I terminant the MB-regularised value simplifies to

$$\begin{aligned} T_I(N,\alpha,z) \equiv & \int_{c-i\infty}^{c+i\infty} ds\, \frac{z^s\, e^{\mp 2Mi\pi s}\Gamma(s+\alpha)}{e^{-i\pi s} - e^{i\pi s}} \mp 2\pi i z^{-\alpha} \\ & \times \ e^{1/z} e^{\pm Mi\pi\alpha}\, \frac{\sin(M\pi\alpha)}{\sin(\pi\alpha)} \ , \end{aligned} \tag{96}$$

where $(\pm 2M - 3/2)\pi < \arg z < (\pm 2M + 3/2)\pi$ since $\epsilon_1 = \epsilon_2 = \pi/2$. In the event that $-\alpha$ is greater than -1, we alter the lower bound on the offset c to $\mathrm{Max}\{N-1, -\alpha\} < c = \Re s < N$. This ensures that the singularity at $s = -\alpha$ remains to the left of the line contour when the truncation parameter equals zero, i.e. for $N = 0$.

Next we turn our attention to the issue of the MB regularisation of the second type of general series, where the terms in Eq. (1) are given by $a_k = f(k) z^k$. As discussed in the previous section, this type of series is often derived where the values of z are situated initially on a Stokes line, e.g. for $\arg z = 0$. As soon as $\arg z$ moves off the initial Stokes line, the regularised value will acquire jump discontinuous terms in either direction. Therefore, if we carry out the MB regularisation of the second type of series, then whilst the resulting regularised value will be valid for the initial Stokes line, it will not be valid for the abutting Stokes sectors despite the fact that we have seen MB regularisation is unaffected by Stokes lines and sectors. Therefore, we have a situation where a Stokes line is critically important initially, but that the other Stokes lines do not affect the regularised value. We refer to this line as the primary Stokes line.

A property of the primary Stokes line is that it is completely arbitrary since it can be set by letting $\arg z = 2k\pi$, where k is an arbitrary integer. Without loss of generality we shall choose the primary Stokes line to be the $k=0$ line or $\arg z = 0$. Choosing another primary Stokes line will only result in a shift in the domains of convergence, as is explained later.

We are now in a position to consider MB regularisation of the second type of general series with the same conditions applying to $f(s)$ as in the derivation of Equivalence (90). Then we arrive at

$$S_{II}(N,z) = \sum_{k=N}^{\infty} f(k) z^k \equiv \int_{c-i\infty}^{c+i\infty} ds \, \frac{(-z)^s f(s)}{e^{-i\pi s} - e^{i\pi s}} \ , \tag{97}$$

where, again, the offset is defined by $N-1 < c = \Re s < N$. The principal difference between this result and Equivalence (90) is the appearance of the multi-valued factor of $(-1)^s$ in the integrand of the above MB integral. Because of this factor the regularised value of the series has become ambiguous since it can be interpreted as being either $\exp(i\pi s)$, $\exp(-i\pi s)$ or even $\exp((2l+1)i\pi s)$, where l is an arbitrary integer. We can drop the last possibility because the primary Stokes line can be shifted to compensate. Nevertheless, the MB integral in Equivalence (94) can be expressed more generally as

$$I_l^*(z) = \int_{c-i\infty}^{c+i\infty} ds \, \frac{z^s \, e^{-(2li+1)\pi s} f(s)}{e^{-i\pi s} - e^{i\pi s}} \ . \tag{98}$$

In addition to introducing ambiguity when the second type of general series is MB-regularised, the multi-valued factor of $(-1)^s$ affects the domain of convergence of the MB integral. If we consider the first interpretation, where $(-1)^s = \exp(i\pi s)$ or $l = -1$ in the above equation, then the domain of convergence for the MB integral in Equivalence (97) is found to be $-\epsilon_1 < \arg z < 2\pi + \epsilon_2$. On the other hand, for $(-1)^s = \exp(-i\pi s)$ or $l = 0$ in the above equation the domain of convergence is given by $-2\pi - \epsilon_1 < \arg z < \epsilon_2$. Therefore, the domains of convergence overlap over the primary Stokes line, which means that either $I_0^*(z)$ or $I_{-1}^*(z)$ is valid in the vicinity of the primary Stokes line. Unfortunately, neither form possesses extra terms to reflect the discontinuity occurring at the primary Stokes line as indicated by the Borel-summed regularised value of the second type of terminant, viz. Equivalence (75). Moreover, both results yield complex values along the primary Stokes line, whereas from the Zwaan-Dingle principle we expect the regularised value to be real.

As stated on p. 103 of Ref. [17] the problem can be resolved by introducing extra terms into the regularised value of the series such that

$$S_{II}(N,z) \equiv \begin{cases} I_0^*(z) + iC(z), & 0 < \arg z < 2\pi + \epsilon_1 \ , \\ I_{-1}^*(z) + iD(z), & -2\pi - \epsilon_1 < \arg z < 0 \ . \end{cases} \tag{99}$$

For z lying on the primary Stokes line we simply average the two regularised values as we did when analysing the first type of series. Then we find that the average of both MB integrals yields a real valued quantity. This means that we are essentially treating $(-z)^s$ in the MB integrals as

$$(-z)^s \equiv \begin{cases} z^s \exp(i\pi s), & \arg z > 0 \ , \\ z^s \cos(\pi s), & \arg z = 0 \ , \\ z^s \exp(-i\pi s), & \arg z < 0 \ . \end{cases} \tag{100}$$

According to p. 12 of Ref. [8], this customary convention for dealing with the multi-valued factor of $(-z)^s$ in asymptotic expansions is an indication that the Stokes phenomenon has occurred. Moreover, because the regularised value is real along the primary Stokes line, $C(z)$ must equal $-D(z)$. We have already seen that MB regularisation is different from Borel summation in that the MB-regularised forms for the regularised value share common regions and that there are no lines of discontinuity, which are fictitious if the original function is continuous. Hence, the results in Equivalence (99) will be equal to another in the common region, which includes the Stokes line of discontinuity. Subtracting both results in Equivalence (99) from each other yields

$$C(z) = \frac{i}{2}\, \Delta I^*_{0,-1}(z) = \frac{1}{2}\left(I^*_0(z) - I^*_{-1}(z) \right) \ . \tag{101}$$

As a consequence, Equivalence (99) can be expressed more precisely as

$$S_{II}(N,z) \equiv \begin{cases} I^*_0(z) - \frac{1}{2}\Delta I^*_{0,-1}(z), & 0 < \arg z < 2\pi + \epsilon_1 \ , \\ \frac{1}{2}\left(I^*_0(z) + I^*_{-1}(z) \right), & \arg z = 0 \ , \\ I^*_{-1}(z) + \frac{1}{2}\Delta I^*_{0,-1}(z), & -2\pi - \epsilon_1 < \arg z < 0 \ . \end{cases} \tag{102}$$

All these results are identical to each other in common region of $-\epsilon_1 < \arg z < \epsilon_2$. So, whilst we have allowed for the jump discontinuity occurring in the Borel-summed forms, there is no jump discontinuity in the final MB-regularised forms for the regularised value of a Type II terminant at the primary Stokes line.

Compared with the first type of series or $S_I(N,z)$ we see that the regularised value of the second type of general series has acquired half the difference between the $l=0$ and $l=-1$ versions of the resulting MB integral derived via MB regularisation. This simply did not occur in the first type of series because its MB-regularised value was derived within a Stokes sector rather than on a singular Stokes line initially. Had we been considering a primary Stokes sector rather than a primary Stokes line, then $C(z)$ would be zero, but the ensuing expressions for the regularised value would still be equal to each other in the common region.

To derive the regularised value for other Riemann sheets in the complex plane, we adopt a similar approach to the analysis of the first type of series. That is, we replace the MB integrals in the first and third results of Equivalence (102) by the MB integrals that overlap with them. In the case of $I^*_0(z)$ it can be replaced by $I^*_1(z) - \Delta I^*_{1,0}(z)$, while $I^*_{-1}(z)$ can be replaced by $I^*_{-2}(z) + \Delta I^*_{-1,-2}(z)$. Of course, the reason we can do these replacements is because of the existence of a common region between the domains of convergence for the MB integrals. Once each replacement has taken place, we can analytically continue the results to the next MB integral provided both $\Delta I^*_{1,0}(z)$ and $\Delta I^*_{-1,-2}(z)$ can be analytically continued. For a standard Type II terminant, where $f(s)=\Gamma(s+\alpha)$, we find for any value of l that

$$\Delta I^*_{l+1,l}(z) = I^*_{l+1}(z) - I^*_{l}(z) = \int_{c-i\infty}^{c+i\infty} ds\, z^s\, e^{-(2l+2)i\pi s}\, \Gamma(s+\alpha) \ . \tag{103}$$

From Ref. [25] we eventually arrive at

$$\Delta I^*_{l+1,l}(z) = 2\pi i\, z^{-\alpha}\, e^{2(l+1)i\pi\alpha}\, e^{-1/z} \ , \tag{104}$$

which can be analytically continued through to all Riemann sheets. Consequently, we can keep replacing MB integrals and analytically continuing the results to the next Riemann sheet over the entire complex plane.

The procedure outlined in the preceding paragraph is the method used in Ref. [17] to derive the regularised value of the generalised Type II terminant given in Proposition 5. For a standard terminant this result reduces to

$$\begin{aligned} T_{II}(N,\alpha,z) \equiv & \int_{c-i\infty}^{c+i\infty} ds\, \frac{z^s\, e^{\mp(2M+1)i\pi s}\,\Gamma(s+\alpha)}{e^{-i\pi s}-e^{i\pi s}} \mp 2\pi i z^{-\alpha} e^{-1/z} \\ & \times \ e^{\pm(M+1)i\pi\alpha}\, \frac{\sin(M\pi\alpha)}{\sin(\pi\alpha)} \mp \pi\, i\, z^{-\alpha}\, e^{-1/z} \ , \end{aligned} \tag{105}$$

where for the upper-signed result, $(2M-1/2)\pi < \arg z < (2M+5/2)\pi$, and for the lower-signed result, $-(2M+5/2)\pi < \arg z < (-2M+1/2)\pi$. Once again, the offset is the same as that given below Equivalence (96), viz. $\mathrm{Max}[N-1,-\alpha] < c = \Re\, s < N$.

To complete this section, we now turn to an example where the second type of terminant series is only valid initially over a sector of complex plane rather than on a line of discontinuity as is the usual case discussed above. In a recent work [31] a series expansion for the trigonometric cosecant function was derived by using the partition method for a power series expansion [16, 18], in which the a_k in Eq. (1) were expressed in terms of special numbers known as the cosecant numbers. Specifically, the following result was derived

$$z\,\csc(z) \equiv \sum_{k=0}^{\infty} c_k z^{2k} \ , \tag{106}$$

where $c_0=1$, $c_1=1/6$, $c_2=7/360$, etc. A more general formulation for the cosecant numbers in terms of the Riemann zeta function is

$$c_k = 2\left(1-2^{1-2k}\right)\, \frac{\zeta(2k)}{\pi^{2k}} \ . \tag{107}$$

The power series expansion given by Equivalence (106) was also found to possess a finite radius of absolute convergence given by $|z|<\pi$.

The simplest method of deriving an asymptotic series with the cosecant numbers in it is to evaluate the Laplace transform of $z\,\csc(az)$. This yields

$$I_{csc}(p,a) = z\int_0^{\infty} dz\, z\, e^{-pz}\,\csc(z) \equiv \frac{1}{ap} + \frac{1}{ap}\sum_{k=1}^{\infty}\Gamma(2k+1)c_k\left(\frac{a}{p}\right)^{2k} \ . \tag{108}$$

If we introduce Eq. (107) into the above result and replace the zeta function by its Dirichlet series form, i.e. by $\zeta(2k) = \sum_{j=1}^{\infty} 1/j^{2k}$, then we obtain

$$I_{csc}(p,a) \equiv \frac{1}{ap} + \frac{2}{ap}\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\Gamma(2k+1)\left(\frac{a}{jp\pi}\right)^{2k} - 2\sum_{k=1}^{\infty}\Gamma(2k+1)\left(\frac{a}{2jp\pi}\right)^{2k}\right) . \tag{109}$$

Therefore, we see that the asymptotic expansion for $I_{csc}(p,a)$ is composed of an infinite series of generalised terminants, which is not so surprising as each terminant corresponds to each singularity lying on the real axis.

There is an interesting anomaly arising out of Equivalences (108) and (109). That is, for either large values of p or small values of a, we can truncate the expansion on the rhs of Equivalence (106), thereby yielding a finite value when both a and p are real. Yet for these values of a and p the original integral on the lhs of Equivalence (108) is singular or undefined. This is a situation where a regularised value can be obtained, but it does not represent the actual function. Since a^2/p^2, not z, represents the variable in the above asymptotic series, the rhs of Equivalence (108) can only apply to the Stokes sectors given by $(j-1)\pi < (a/p) < j\pi$, where j is an arbitrary integer. That is, the real axis represents a line of discontinuity for $I_{csc}(p,a)$ with the expansion on the rhs of Equivalence (108) being applicable to each sector.

Because the Riemann zeta function can be analytically continued into the complex plane, the cosecant numbers can also be continued in the complex plane. That is, $c_k = c(k)$. As a consequence, Equivalence (108) can undergo MB regularisation directly without the need to consider the infinite sum of generalised terminants in Equivalence (109) arising from Borel summation [19]. Therefore, for $N=1$ and $-\pi < \arg(a/p) < 0$, the MB-regularised value of $I_{csc}(p,a)$ is given by

$$I_{csc}(p,a) \equiv \frac{1}{ap}\left(1 + \int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s+1)\, e^{i\pi s/2}}{e^{-i\pi s/2} - e^{i\pi s/2}}\left(\frac{a}{p\pi}\right)^{s}(1-2^{1-s})\zeta(s)\right), \tag{110}$$

while for $0 < \arg(a/p) < \pi$, MB regularisation yields

$$I_{csc}(p,a) \equiv \frac{1}{ap}\left(1 + \int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s+1)\, e^{-i\pi s/2}}{e^{-i\pi s/2} - e^{i\pi s/2}}\left(\frac{a}{p\pi}\right)^{s}(1-2^{1-s})\zeta(s)\right). \tag{111}$$

In both of these results the offset c is given by $0 < c = \Re(s/2) < 1$.

The MB integrals in the above results can be expressed more generally as

$$I_{MB}(j) \equiv \int_{c-i\infty}^{c+i\infty} ds\, \frac{\Gamma(s+1) e^{(j+1/2)i\pi s}}{e^{-i\pi s/2} - e^{i\pi s/2}}\left(\frac{a}{p\pi}\right)^{s}(1-2^{1-s})\zeta(s) . \tag{112}$$

Hence, the MB integral in Equivalence (110) is basically $I_{MB}(0)$, while that in Equivalence (111) is $I_{MB}(-1)$. In addition, the domain of convergence for the integral in Eq. (112) is

$-(j+3/2)\pi < \arg(a/p) < -(j-1/2)\pi$, which means that both $I_{MB}(0)$ and $I_{MB}(-1)$ are valid over the common region or sector given by $-\pi/2 < \arg(a/p) < \pi/2$. On the other hand, the difference between the MB integrals is found from the theory of Mellin transforms [25] to be

$$\Delta I_{MB}^{(-1.0)} = I_{MB}(-1) - I_{MB}(0) = 2\pi i \left(\frac{p\pi}{a}\right) \frac{e^{-p\pi/a}}{(e^{-p\pi/a}+1)^2} \ . \tag{113}$$

Therefore, for $N=1$ and $-\pi/2 < \arg(a/p) < 0$, the regularised value of $I_{csc}(p,a)$ can also be written as

$$\begin{aligned} I_{csc}(p,a) \equiv \frac{1}{ap} &\left(1 + \int_{c-i\infty}^{c+i\infty} ds \, \frac{\Gamma(s+1)e^{-i\pi s/2}}{e^{-i\ pis/2} - e^{i\pi s/2}} \left(\frac{a}{p\pi}\right)^s (1-2^{1-s})\zeta(s) \right) \\ &- \ 2\pi i \left(\frac{p\pi}{a}\right) \frac{e^{-p\pi/a}}{(e^{-p\pi/a}+1)^2} \ , \end{aligned} \tag{114}$$

while for $N=1$ and $0 < \arg(a/p) < \pi/2$, we find that

$$\begin{aligned} I_{csc}(p,a) \equiv \frac{1}{ap} &\left(1 + \int_{c-i\infty}^{c+i\infty} ds \, \frac{\Gamma(s+1)e^{i\pi s/2}}{e^{-i\pi s/2} - e^{i\pi s/2}} \left(\frac{a}{p\pi}\right)^s (1-2^{1-s})\zeta(s) \right) \\ &+ \ 2\pi i \left(\frac{p\pi}{a}\right) \frac{e^{-p\pi/a}}{(e^{-p\pi/a}+1)^2} \ . \end{aligned} \tag{115}$$

By combining Equivalence (115) with Equivalence (110) or Equivalence (114) with Equivalence (111), and then comparing either result with the regularised value of a general Type II terminant, viz. Equivalence (102), we see that in this anomalous case where the asymptotic expansion has been derived initially in a Stokes sector rather than on a primary Stokes line, there is a discontinuity on reaching the line at $\arg(a/p)=0$, while on moving to the adjacent Stokes sector twice the discontinuity applies. Previously, we found that half the difference between the MB integrals was involved when moving in either direction off the primary Stokes line. Now half the discontinuity occurs on reaching the line and the entire discontinuity occurs when moving off the line into the adjacent Stokes sector. Whether half or the entire discontinuity is involved is connected to whether there is a semi-residue or full residue around the singularity.

Although $I_{csc}(p,a)$ is undefined along the real axis, its Cauchy principal value, however, exists. We can obtain this value simply by averaging Equivalences (110) and (111) or by evaluating either Equivalence (114) or (115) with only half of their second terms on the rhs. For more details the reader is referred to Ref. [19], where a spectacular numerical study involving these forms for the regularised value of $I_{csc}(p,a)$ is presented.

11. Examples

By presenting numerical examples to extremely high precision, we shall not only be able to verify the various expressions for the regularised value of both Type I and II series, but we shall also be in a position to see for ourselves whether Euler's "unorthodox" views are indeed valid or not. Underlying numerical demonstrations is the fact that numbers do not lie. This is generally ignored by practitioners in standard asymptotics, who instead rely on "proving theorems" by invoking such vague symbols as $\sim$, $\approx$, $O()$, $o()$, $+\ldots$, $\geq$, and $\leq$. As a consequence, no one knows exactly just how accurate the resulting expansions are or even the specific ranges over which they are valid. As indicated earlier, it is these deficiencies arising from the overly-permissive Poincaré prescription that are responsible for giving asymptotics a bad name. On the other hand, an effective numerical study such as that presented in this section is able to expose the deficiencies in standard asymptotics, where a so-called mathematical proof cannot. In fact, in these times where computing is continually being taken to new levels of accuracy, the reader will be surprised, or even alarmed, to see just how bad standard asymptotics is when compared with the results obtained from an accurate numerical investigation. Frequently, the situation is covered up by practitioners in asymptotics by using either considerably small values in their studies of small variable expansions or large values when dealing with large variable expansions. A typical example is the recent study by Paris into the asymptotics of n-dimensional Faxén-type integrals [26]. Although this reference considers values of $|z|$ that are not very large such as $|z| = 15$, the main asymptotic expansion is in powers of $(5/48)(z^3/3)^{4/5}$. When $z = 15$ is introduced into the "actual variable", it becomes quite large resulting in a large optimal point of truncation. Because small values of $|z|$, say less than unity, have not been considered, where the optimal point of truncation is non-existent, the reader is misled as to the accuracy of the asymptotic expansion. Finally, we have seen throughout this work that the most important problem in asymptotics is regularising the remainder when it becomes divergent. This, too, is completely disregarded in Ref. [26]. Instead, the author is content to truncate the main expansion, albeit to the optimal point of truncation and then introduce the ubiquitous tilde or $\sim$ into the main results.

Here we shall present a numerical demonstration involving a particular Type I terminant since spectacular demonstrations involving Type II generalised terminants have already been presented in Chs. 9 and 10 of Ref. [17]. Specifically, it was found to astonishing accuracy that the Borel-summed and MB-regularised forms for the regularised value of a Type II series agreed with each other, but no such analysis was ever applied to Type I series. We shall rectify the situation here, whilst at the same time providing the reader with a clearer idea of exactly the type of numerics we have in mind in carrying out such an investigation. At the end of this section, however, we shall explain how Borel summation can be extended to general Type II series by discussing the final example in Ref. [17].

The first point to be made here is that z will be replaced by z^3 in the general forms for the regularised value presented in the previous sections. This is necessary so that we can observe the effect of other Stokes sectors and domains of convergence within the principal branch of the complex plane for z. Next, we shall let $\alpha = 3/7$ rather than a simple value like unity or a half, so that the terminant can no longer be identified with a known special function. That is, this is a situation where only an asymptotic solution exists. After all, there is little point in developing a new approach to handling divergent series if all it does is provide another explanation of existing problems without possessing the capacity to explore the unknown. Since the forms for both types of terminant as given by Eqs. (63) and (64) represent small z

asymptotic series, we shall consider values of $|z|$ or rather $|z^3|$, where truncation is unable to provide an accurate estimate. Therefore, in the first instance, $|z^3|$ will be set equal to a value lying in the intermediate region of $0.1 < |z^3| < 2$ and in the second instance to a "large" value, where $|z^3| > 2$. Both of these situations would never be considered in standard asymptotics. The numerical study will also consider a wide range of values for the truncation parameter. As we shall see, altering the truncation parameter is effectively employing a different method for evaluating the regularised value. The regularised value of the particular terminant mentioned in the preceding paragraph can be obtained by substituting the appropriate values into the Borel-summed forms given by Equivalences (82) and (85). Then we find that

$$T_I(N,3/7,z^3) \equiv (-z^3)^N \int_0^\infty dt\, \frac{t^{N-4/7}\, e^{-t}}{1+z^3\, t} \mp\; 2\pi i\, z^{-9/7} e^{1/z^3}\, e^{3i\pi/7}\, \frac{\sin(3l\pi/7)}{\sin(3\pi/7)}\ , \tag{116}$$

where the upper sign is valid for $(2l-1)\pi/3 < \arg z < (2l+1)\pi/3$ and the lower sign is valid for $-(2l+1)\pi/3 < \arg z < -(2l-1)\pi/3$. In both cases l is a non-negative integer, while the offset c is given by $\mathrm{Max}[N-1,-3/7] < c = \Re s < N$. For the Stokes lines, where $\arg z = \pm(2l+1)\pi/3$, the regularised value derived from Equivalences (83) and (86) is given by

$$T_I(N,3/7,z^3) \equiv |z|^{3N-3} P \int_0^\infty dt\, \frac{t^{N-4/7}\, e^{-t}}{t-|z|^3} - \pi i\, |z|^{-9/7} e^{-1/|z|^3} \times \left(2e^{\mp 3(l+1)i\pi/7}\, \frac{\sin(3l\pi/7)}{\sin(3\pi/7)} + 1 \right)\ . \tag{117}$$

The MB-regularised value of the series can be obtained by introducing the appropriate values into Equivalence (96). This yields

$$T_I(N,3/7,z^3) \equiv \int_{c-i\infty}^{c+i\infty} ds\, \frac{z^{3s}\, e^{\mp 2li\pi s}\,\Gamma(s+3/7)}{e^{-i\pi s}-e^{i\pi s}} \mp 2\pi i z^{-9/7} e^{1/z^3} \times\ e^{\pm 3li\pi/7}\, \frac{\sin(3l\pi/7)}{\sin(3\pi/7)}\ , \tag{118}$$

where $(\pm 2l - 3/2)\pi/3 < \arg z < (\pm 2l + 3/2)\pi/3$. Therefore, we find that there are three different forms covering the principal branch of the complex plane for z. The $l = 0$ form is valid over $-\pi/2 < \arg z < \pi/2$, while the $l = 1$ and $l = -1$ forms are valid over $\pi/6 < \arg z < 7\pi/6$ and $-7\pi/6 < \arg z < -\pi/6$, respectively. Hence, the $l = 0$ and $l = 1$ forms share a common region of $\pi/6 < \arg z < \pi/2$, which is where we expect both forms to yield identical results for the regularised value. If this does not occur, then we know that the results of the previous section are invalid. On the other hand, the common region for the $l = 0$ and $l = -1$ forms is $-\pi/2 < \arg z < -\pi/6$. Hence, these forms are expected to yield identical values for the regularised value when z is situated within this sector of the complex plane.

Table 1 in the appendix presents a small sample of the results obtained by programming Equivalence (118) as a module in Mathematica [34]. Only a summary of the various modules used to produce the numerical results in this work is presented. The actual modules will appear elsewhere [20]. Specifically, the table is composed of the various terms on the rhs of Equivalence (118) plus the truncated series up to $N-1$ since

$$T_I(0,\alpha,z) = T_I(N,\alpha,z) + \sum_{k=0}^{n_1} \Gamma(k+\alpha)(-z)^k \ . \tag{119}$$

That is, the regularised value of the entire series on the lhs of the above equation or $T_I(0,\alpha,z)$ is equivalent to the truncated series plus the regularised value on the rhs of Equivalence (118). Hence, Eq. (119) represents a method for checking the concept of regularisation since by varying the value of the truncation parameter N, we are calculating completely different values for both the truncated series and the MB integral in the regularised value of $T_I(0,\alpha,z)$.

The first column of Table 1 displays the value of the truncation parameter. Because $\arg z$ has been set equal to $\pi/4$, the regularised value of this particular terminant can be determined by setting $l=0$ and $l=1$ in the rhs of Equivalence (118). Therefore, the second column displays the value of l used to obtain the regularised value for the entire series via Eq. (119). The next column presents the value of the truncated series or the first term on the rhs of Eq. (119) with z equal to $(4/5)\exp(i\pi/4)$. The fourth column lists the value of the MB integral on the rhs of Equivalence (118), while the fifth column labelled Discontinuity displays the values for the second term on the rhs of the equivalence statement. This term vanishes for $l=0$, but remains fixed for $l=1$. The final column displays the regularised value of the entire series, which is determined by summing the quantities in the third, fourth and fifth columns of the table.

The results in Table 1 have been obtained by running Mathematica 4.1 on a Pentium computer. The numerical integration routine in this software package known as NIntegrate was used to evaluate the MB integral in Equivalence (118). This was achieved by expressing each MB integral as the sum of two separate integrals ranging from zero to infinity. Because the NIntegrate routine can miss sudden peaks occurring in the integrand, it is advisable to divide the range of integration into several smaller intervals. In addition, the routine can be fine-tuned by setting the options of AccuracyGoal, PrecisionGoal and WorkingPrecision to high values. Because Mathematica 4.1 is limited by the machine precision of the computer, which in this case was 16 decimal places, Working Precision was set equal to 16, while the other options were set equal to 14. This limitation in the precision of the results due to machine precision does not apply to more recent versions of the software package such as Versions 6.0 to 8.0. Provided the input variables are not expressed as decimal numbers in these versions, the results can be obtained to unlimited precision, but it will come at a cost in the time taken to carry out the computation of the integrals. Finally, in obtaining the results for the MB integrals, the options of MinRecursion and MaxRecursion, which determine the minimum and maximum number of sample points used in the NIntegrate routine, were set equal to 3 and 10 respectively. Again, these can be adjusted, but at the expense of the CPU time.

Because of the limitation on the working precision in the evaluation of the MB integral in Equivalence (118) and the fact that the accuracy and precision goals were set to 14, we can at best expect that the above results will only be accurate to about 14 significant figures. This situation applies to all the values in the table for $N \le 5$. The $N=1$ and $N=2$ values in the third column represent the values one would obtain by adopting "standard asymptotics". Here, we see that the truncated values are quite inaccurate and can only be regarded as estimates despite the fact that we are effectively carrying out our investigation for $|z^3|$, which is much smaller than $|z|$ or 4/5. The inaccuarcy is expected because for this value of z, the optimal point of truncation N_T is equal to unity.

For values of N greater than 5, we find that the regularised value in the final column is not as accurate as the $N \le 5$ results. This is because the truncated series begins to grow dramatically or rather, diverges. To compensate for the divergence of the truncated series, the value of the MB integral diverges in the opposite sense such that when the latter is combined with the value for the truncated series, it yields a less accurate regularised value due to the cancellation of redundant decimal places. Hence, by the time the truncation parameter reaches a value of 20, we find that the truncated series is of the order of 10^{10}, which means that the first 10 significant figures will be cancelled before the regularised value for the series on the lhs of Eq. (119) can be obtained. Consequently, the regularised value in the final column for the $l=0$ form will only be accurate to 5 decimal places, courtesy of the machine precision of the computing system. It should also be noted that all the values displayed in this and subsequent tables are not rounded off at any stage. That is, the final digit presented in the tables represents in all cases the output as derived from running the computer programs.

Another point requiring mention is that the values of the $l=1$ MB integrals for $N \ge 5$ in the table have been asterisked because problems occurred during their computation. In fact, for $N=20$ the regularised value in the final column is not even correct. This is an indication that numerical integration is largely a "black art" relying on the intuition of the programmer to gauge the limitations of the software being used. In this instance, the integrand outside the Mathematica module has been set equal to

```
Intgrd[z_, s_, l_] := (z ∧ 3) ∧ s Exp[−2 I l Pi s] Gamma[s + 3/7]/
(Exp[−I Pi s] − Exp[I Pi s])   .
```

The main module requires the value of $|z|$ and $\arg z$ as input before calculating the value of $z(=|z|\exp(i\arg z))$. The problem with the above form for the integrand is that z^{3s} may become very large or very small before it can be countered by the factor of $\exp(2i\pi s)$ and/or the other factors in the integrand. When this occurs, Mathematica alerts the user that it is experiencing convergence problems in the numerical integration. The asterisked results in Table 1 are examples of this type of occurrence. In fact, what is surprising about these results is that although convergence problems did arise in the evaluation of the MB integral, the software package was still able to give accurate values for the regularised value in all, but the last calculation, i.e. for $N=20$ and $l=1$.

Let us raise the ante by carrying out calculations using a later version of the software, namely Version 7.0, on a PowerMac G5 with 1.25 GB of RAM. We shall also carry out calculations for both $|z|=4/5$ and $|z|=2$. Two separate tables of results will be presented in the appendix: the first displaying the results obtained for $|z| = 4/5$ and $\arg z$ situated in the upper half of the principal branch, viz. $0 < \arg z < \pi$, and the second table displaying the results for $|z| = 2$ and $-\pi < \arg z < 0$. It would simply be inconceivable to consider the second lot of values for $|z|$ in standard asymptotics as truncation would result in extremely inaccurate results. Furthermore, as a result of the discussion in the preceding paragraph, we shall re-write the integrand for the NIntegrate routine in the Mathematica 7.0 module as

$$\begin{aligned}&\mathrm{Intgrd[modz\_, argz\_, s\_, l\_] := modz \wedge (3}s\mathrm{)Exp[(3\,argz - 2\,l\,Pi)\,I\,s]}\\&\mathrm{Gamma[s + 3/7]/(Exp[-I\,Pi\,s] - Exp[I\,Pi\,s])} \quad .\end{aligned}$$

This provides us with the best opportunity to avoid convergence problems that arose in presenting the results in Table 1.

Before the code was used to generate the results for various values of $\arg z$, it was rerun for the same set of values in Table 1 in order to enable a comparison between the two computing systems. Not only was the second computing system much quicker, it was also able to generate values for much larger values of the truncation parameter, i.e. for values of N where the Pentium plus Mathematica 4.1 system experienced convergence problems. In addition, for $N < 10$, many of the results took less than 30 seconds of CPU time with the more powerful computing system, whereas they took several minutes to compute using the first system despite the fact that the precision and accuracy goals were much lower in the former system. Nevertheless, irrespective of the value of z, it must be emphasised that problems with convergence as well as with precision and accuracy goals will also arise in the more powerful system, once the truncation parameter becomes sufficiently large. This will be discussed shortly when we consider the case of $|z| = 2$ and $\arg z = -\pi/4$.

Table 2 presents a sample of the results obtained by running the modified Mathematica module on the Power Mac G5 plus Mathematica 7.0 system. Not all the decimal places for the various results are displayed in the table due to limited space. Because the truncation parameter was not very large, i.e. N was generally taken to be less than 25, the regularised values in the final column are accurate to at least 30 decimal places. As a consequence, one is not able to observe any variation in any of the regularised values appearing in the final column.

From the table we see that for those values of $\arg z$, where both the $l = 0$ and $l = 1$ forms of Equivalence (118) are valid, we ultimately obtain the same regularised value. Therefore, we have two completely different forms for the regularised value yielding identical results, which is in accordance with Euler's so-called "unorthodox" views about divergent series. Moreover, we see that altering the truncation parameter for the same value of z yields the same regularised value even though the truncated series and MB integrals are different for each value of N. This also vindicates Euler's second view that one should obtain the same value irrespective of what method or approach is used.

Table 3 presents a sample of the results obtained by running the Mathematica module used to obtain Table 2 again on the same Power Mac G5 plus Mathematica 7.0 system, but on this occasion, $|z|$ has been set equal to 2 and $\arg z$ is less than zero. This means that the MB-regularised values can only be evaluated by using the $l = 0$ and $l = -1$ forms of Equivalence (118). As in the case of the previous table not all the decimal places of the values in the various columns were able to be displayed here. The major difference between this and the previous table is that the truncation parameter need not be reasonably large before the truncated series and the MB integral begin to diverge rapidly. E.g., for $N=6$, the magnitude of both the truncated series and MB integral is of the order of 10^5. Nevertheless, as they diverge in opposite directions, there is a great cancellation of decimal places in the process of arriving at the regularised value of the series.

The main characteristics or features of Table 2 are also evident in Table 3. Although the common region is different, viz. $-\pi/2 < \arg z < -\pi/6$, both the $l = 0$ and $l = -1$ forms of Equivalence (118) yield identical results for the regularised value of $T_I(0, 3/7, z^3)$. This is despite the fact that the jump discontinuity is different for both forms. In addition, altering the truncation parameter for a fixed value of $\arg z$ always yields the same value for the regularised value of $T_I(0, 3/7, z^3)$ even though the truncated series and MB integrals vary for each value of N. In fact, the only difference between choosing a value of $|z|$ in the intermediate region and one in the large region is that the truncated series and MB integrals do not diverge as rapidly in the former case as they do in the latter case.

Table 4, which also appears in the appendix, presents a small sample of the results for the regularised value of $T_I(0, 3/7, z^3)$ obtained from the Borel-summed forms given by Equivalence (116). This means that another Mathematica module was created, which evaluates all the quantities on the rhs of this equivalence. The values in the table have been obtained by running the new Mathematica module with $|z|=4/5$ and $\arg z > 0$ on the same Power Mac G5 computer plus Mathematica 7.0 system. In this code the options in the call to the NIntegrate routine were set to the same values as in the module that was used to obtain the results in Tables 2 and 3. Hence, where the same value of z is involved, the results in Table 4 can be compared directly with those in Table 2. As was the case in the two preceding tables, not all the decimal places for the results were able to be displayed due to limited space. One interesting feature about these results is that they took considerably less time to compute than their MB-regularised counterparts. In fact, they generally took only a few CPU seconds to compute compared with the MB-regularised values, which took between 20 and 90 seconds and even longer on the Pentium computer plus Mathematica 4.1 system. This is quite surprising because the opposite was found to apply when determining the regularised values from the MB-regularised and Borel-summed forms for the complete asymptotic expansion of the generalised Euler-Jacobi series for $p/q=3$ in Ref. [21].

The first column in Table 4 presents the value of the truncation parameter or N that was used to evaluate the various quantities on the rhs of Equivalence (116). The next column displays the values of l used in evaluating the Stokes discontinuity term appearing in the equivalence statement. As indicated previously, these integers are dependent on the value of $\arg z$, which appear in the third column of the table. The next column displays the values of the truncated series, viz. the second term on the rhs of Eq. (119), while the fifth and sixth columns display the values corresponding to the other terms on the rhs of Equivalence (116). The integral on the rhs of Equivalence (116) is referred to here as the Borel integral. The final column presents the Borel-summed regularised values of $T_I(0, 3/7, z^3)$, which have been calculated by summing the respective quantities in the three preceding columns.

From Table 4 it can be seen that the regularised value of $T_I(0, 3/7, z^3)$ remains invariant for each value of $\arg z$. That is, irrespective of the value selected for the truncation parameter, we end up with the same regularised value for the entire series. Each value of N results in a completely different integrand being computed by the NIntegrate routine, which means effectively that different methods are being employed to evaluate the regularised value. Nevertheless, the regularised value remains invariant as it did when the truncation parameter was altered in the MB-regularised forms of the regularised value. From the table it can be seen that for $N > 10$, the truncated series diverges rapidly, while the Borel integral obliges by diverging in the opposite direction. Even for the smaller values of the truncation parameter the truncated series represents a poor approximation to the regularised value, which emphasises the fact that the truncated series is only a good approximation when $|z|$ is very small, namely less than 0.01.

As mentioned previously, because the regularised values in Table 4 have been evaluated by using the same options in the NIntegrate routine as those in Table 2, we can compare corresponding results. As a result, we find that for the same value of $\arg z$ that the regularised values are identical to one another within the precision and accuracy goals set in both programs. This means that the regularised value of $T_I(0, 3/7, z^3)$ can be evaluated by combining Eq. (119) with either the Borel-summed form given by Equivalence (116) or the MB-regularised form given by Equivalence (118). In other words, we have two totally different methods yielding the same regularised value, which is, again, in accordance with Euler's second view on divergent series.

Table 5 in the appendix presents another small sample of the results obtained by running the second module on the same Power Mac G5 with Mathematica 7.0, but now we set $|z| = 2$ and consider the lower half of the principal branch of the complex plane, viz. $\arg z < 0$. The new table is composed of the same quantities appearing in Table 4. As was found to be the case for the results in the previous table, they were computed far more quickly than their MB-regularised counterparts displayed in Table 3. Once again, we see that for fixed values of $\arg z$, the regularised value of $T_I(0, 3/7, z^3)$ remains invariant despite the variation in the truncation parameter. Of course, this is provided that N is not sufficiently large to cause convergence problems when the Nintegrate routine is called.

As can be seen from the table, the truncated series diverges rapidly as soon as N becomes greater than 3. As expected, the divergence is much greater and more rapid than the truncated series for $|z| = 4/5$ again confirming that the truncated series will only be accurate for very small values of $|z|$, where an optimal point of truncation exists. Furthermore, we find that for the same value of $\arg z$ that the regularised value given in the final column of Table 5 is identical to the corresponding value in Table 3. Hence, both MB-regularised and Borel-summed forms again yield identical values for the regularised value of $T_I(0, 3/7, z^3)$.

Now let us examine the evaluation of the regularised values of $T_I(0, 3/7, z^3)$ via the Borel-summed forms given in Equivalence (116). Although the Stokes discontinuity term or the second term on the rhs of Equivalence (116) is identical to the extra term on the rhs of the MB-regularised value given by Equivalence (118), it appears for different values of $\arg z$ in the principal branch of the complex plane for z. We have already noticed that the MB-regularised value for $\pi/6 < |\arg z| < \pi/2$ can be written in terms of an MB integral without the extra term, i.e. $l = 0$ in Equivalence (118), or it can be expressed in terms of another MB integral with the $l=1$ value of the extra term on the rhs of Equivalence (118). However, the extra term or Stokes discontinuity term is zero for Borel-summed regularised values when $|\arg z|<\pi/3$, but yields a contribution when $\pi/3<|\arg z|<\pi$. That is, $l=0$ in Equivalence (116) for $|\arg z|<\pi/3$, while for $\pi/3<\arg z<\pi$, we put $l=1$ in the upper-signed version and for $-\pi/<\arg z<-\pi/3$, we put $l=1$ in the lower-signed version. Notwithstanding, we find that whichever form is used to evaluate the regularised value, we get the same result despite varying the truncation parameter.

All that remains is to describe the evaluation of the Borel-summed forms for the regularised value of $T_I(0,3/7,z^3)$ along the Stokes lines of $\arg z=\pm\pi/3$. To accomplish this, another Mathematica module is required in order to evaluate all the terms on the rhs of Equivalence (117), particularly the Cauchy principal value. In Mathematica 7.0 this is achieved by specifying Method→'PrincipalValue' in the call to the NIntegrate routine. Unfortunately, this can fail, as described in Sec. 11.3 of Ref. [17]. A better approach is either to evaluate the Cauchy principal value by using the special add-on package in Mathematica 4.1 or to specify the singularity in the range of integration. The second option has been adopted here, which means that separate modules are required for $|z|=4/5$ and $|z|=2$. For the sake of brevity we shall only consider the latter case. Because the singularity in the Cauchy integral occurs at $t=1/8$ in this case, $1/8$ has to be introduced into the subdivision of the range of integration in the NIntegrate routine. The other options of WorkingPrecision, AccuracyGoal, PrecisionGoal, MinRecursion and MaxRecursion in the NIntegrate routine were set to the same values used to obtain the results displayed in Tables 2 to 5.

Table 6, also in the appendix, presents a sample of the results obtained by setting $\arg z=-\pi/3$ and then varying the truncation parameter N. Again, all the values appearing in the table were computed in only a few CPU seconds. As we have seen in the other tables with $|z|=2$, the truncated series begins to diverge rapidly for fairly small values of the truncation parameter, but is countered by the divergence in the value of the Cauchy principal value integral, whose values appear in the column denoted by PV Integral. As is typical for this Stokes line, the Stokes discontinuity term is purely imaginary. In Ch. 1 of Ref. [8], Dingle gives a rule based on this behaviour for continuing asymptotic expansions across a Stokes line. In particular, he states that an asymptotic series generates a discontinuity that is $\pi/2$ out of phase or imaginary with the series. Although this occurs in the fourth column of Table 6, it only occurs because $l=0$ for the Stokes discontinuity. For other values of l, this need not necessarily be the case.

For all values of the truncation parameter we obtain the same regularised value, which can be checked with the value obtained from the MB-regularised form displayed in Table 3. Although the correct value was obtained for $N=30$, Mathematica did indicate problems with internal precision when calculating the Cauchy principal value. This is presumably due to the fact that 56 decimal places had to be cancelled before yielding the regularised value. Consequently, this value is asterisked in the table. Again, the results in this table vindicate Euler's view that the value assigned to an infinite series should be independent of the method used to determine it.

At the beginning of this section it was mentioned that a numerical study of generalised Type II terminants had already been carried out in Ref. [17]. So, there is no need to present a numerical study of this type of terminant here. Nevertheless, it was found that both the MB-regularised and Borel-summed forms for the regularised value of a generalised Type II terminant yielded identical results for all values of variable over the principal branch of the complex plane as we have observed here with Type I terminants. Since the regularised value remains invariant when both regularisation techniques are applied to Type II terminants, this means that Euler's views hold regardless of the type of terminant.

Ref. [17] also concludes with a numerical study of the following series:

$$P(z) = \sum_{k=1}^{\infty} \frac{\Gamma(2k)\Gamma(k+\nu/2)}{\Gamma(\nu/2-k+1)} \, z^{7k/3} \ . \tag{120}$$

This example of a general Type II series can be MB-regularised by following the approach presented in the previous section, but it is not Borel-summable. However, at the end of Sec. 9 it was mentioned that Borel summation can be extended to general Type I and Type II series provided $f(k)$ can be expressed as a Mellin transform. To see how this applies to Eq. (120), we note that the quotient of the gamma functions in the coefficients of $P(z)$ appears in a more general form in the integral given by No. 1.16.21(1) in Ref. [28]. This is

$$\begin{aligned}\int_0^{\infty} dx\, x^{pk+q-1} J_\nu(ax) K_\nu(ax) = & \frac{2^{pk+q-3}}{a^p k+q} \, \frac{\Gamma((pk+q)/4+\nu/2)}{\Gamma(1-(pk+q)/4+\nu/2)} \\ & \times \ \Gamma((pk+q)/2) \ .\end{aligned} \tag{121}$$

Hence, putting $k=2k$, $p=2$, $q=2$ and $a=2$ in the above result produces the precise form for the quotient of gamma functions appearing in Eq. (120). By introducing the resulting integral into Eq. (120) and interchanging the order of the summation and integration, we arrive at a series where the summation over k is only in powers of k. Since the resulting series is a variant of the geometric series, it can be regularised. E.g., for the Stokes sector of $0<\arg z<6\pi/7$, the extended Borel-summed form of the regularised value of $P(z)$ is found to be

$$\begin{aligned}P(z) \equiv & -2\int_0^{\infty} dt\, \frac{J_\nu\left(2t^{1/4}\right) K_\nu\left(t^{1/4}\right)}{t-z^{-7/3}} \\ & - \ 2\pi i\, J_\nu\left(2z^{-7/12}\right) K_\nu\left(2z^{-7/12}\right) \ .\end{aligned} \tag{122}$$

The above result, which is composed of a Cauchy integral and a jump discontinuity term, is typical of the extended Borel-summed forms presented in Sec. 9. Therefore, we see that Borel summation can be extended to more complicated series other than those with gamma function growth in their coefficients as in generalised terminants, but only on the condition that the coefficients must be expressible in terms of a Mellin transform. By using the Borel-summed forms such as Equivalence (122), an extensive numerical study is undertaken in conjunction with the MB-regularised forms for the regularised value of $P(z)$ in Ch. 11 of Ref. [17]. Again, it is found that the regularised values evaluated by the different forms agree with each other for a great number of values of z situated in the principal branch of the complex plane. Hence, the two entirely different methods for regularising asymptotic series yield the same regularised values of this general Type II series. As a consequence, we can be confident that the MB-regularised forms will yield the correct regularised value of the asymptotic expansion for the original function even when extended Borel-summed forms cannot be determined. More importantly, we find that Euler's views hold yet again. Because on this occasion a far more complicated series than a terminant is being considered, we can see that his views are going to hold for all divergent series when a theory of divergent series is finally realised.

12. Conclusion

This article has been concerned with re-evaluating Euler's views on infinite series as a result of recent developments aimed at obtaining meaningful results for divergent series. Basically, Euler believed that all series, whether they are convergent or divergent, could be summed to a particular value and that this value should remain invariant whatever method was employed. In relation to divergent series these views are not only regarded as unorthodox, but also totally unfounded by the mathematical community today. To emphasise this point, Varadarajan [32] wrote on the occasion of the tercentenary of Euler's birth in 2007 that although Euler was unsure about calling the limit value a sum, he was unable to appreciate just how subtle divergent series are. Yet in this article we have seen the opposite, namely that Euler's views are indeed valid and that the current dogma is, therefore, misguided.

The concept that Euler seems to have missed or not been aware of is regularisation. Even here the situation is uncanny because he was the first mathematician to uncover the concept in the course of calculating the constant that now bears his name from the logarithmically divergent harmonic series [13]. Nevertheless, it is true that like so many others after him he did not apply the concept to more complicated divergent series, particularly those appearing in asymptotic expansions. In short, the concept was left for others to enunciate. Instead of referring to a limit sum for a divergent series, we now refer to a regularised value, which is defined as the removal of the infinity in the remainder so as to make the entire series summable. On its own, regularisation represents a mathematical abstraction, but it is necessary in asymptotics for correcting the improprieties due to the various asymptotic methods that are used to derive asymptotic power series expansions from their original functions or integrals.

From the material presented in this article, it is obvious that Euler was clearly well ahead of his time, whilst those following him such as Abel and Cauchy were simply wrong to ridicule his views on divergent series. Unfortunately, because their attitudes prevailed, Weierstrass only concentrated upon convergence when laying down the foundations of classical analysis. Consequently, a vast and important frontier in mathematics was largely ignored for about a century. Today, understanding divergent series and developing techniques for obtaining meaningful values from them have become a top priority in mathematics because in general, the most important and difficult problems in applied mathematics and modern theoretical physics are either asymptotic or divergent in nature.

The indifference towards divergent series in the nineteenth century was also responsible for the limited and inadequate Poincaré prescription or definition being applied universally in asymptotics. Over the past two decades the subject of asymptotics beyond all orders or exponential asymptotics [4, 5, 21, 27, 29] has evolved with researchers actively engaged in the derivation and formulation of methods aimed at isolating subdominant exponential terms in asymptotic expansions. For these problems the Poincaré prescription is basically useless, but in order to obtain meaningful numerical values for these terms, which become dominant with further movement in the complex plane, again a theory of divergent series is required as can be seen by the subdominant series appearing in Eq. (7).

Although this article has described the initial steps and presented many examples for developing a fully-fledged theory of divergent series, more complicated examples will need to be studied in the future before such a theory can be realised. For example, extending the asymptotics of the gamma function to the entire complex plane involves further development of the material presented here so that an infinite number of singularities situated on Stokes lines can be handled rather than a single singularity. Another problem is whether the techniques of Borel summation and MB regularisation can be used to develop the complete asymptotic forms for the confluent hypergeometric functions throughout the entire complex plane. The subdominant terms in these expansions are expected to become divergent series similar in form to $T_3^K(a)$ in Eq. (7). Whilst such series are not expected to pose a problem for MB regularisation as a result of the numerical study in Ref. [21], the question is whether Borel summation can be extended even further not only to produce such series, but also to handle them. Furthermore, this problem has the advantage that for particular values of their parameters the confluent hypergeometric functions reduce to the family of Bessel and Hankel functions. So far, MB regularisation has only been applied to positive real values of the variable in the asymptotic expansions of these special functions [15]. Nevertheless from the material presented in this work, we have seen that Euler's so-called unorthodox views on divergent series hold true. With his reputation restored, perhaps he can now be regarded as the greatest of all mathematicians.

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Appendix: Tables

Table 1: MB-regularised values of $T_I(0, 3/7, z^3)$ for various values of N with $z = (4/5)\exp(i\pi/4)$.

N	l	Truncated Series	MB Integral	Discontinuity	Regularised Value
0	0	0	2.11636689711647	0	2.11636689711647
			$-0.47053628409637\,i$	0	$-0.47053628409637\,i$
0	1	0	3.93536614918420	-1.81899925206772	2.116366897116477
			$+0.58629220846559\,i$	$-1.05682849256197\,i$	$-0.47053628409637\,i$
1	0	2.06751172656022	0.04885517055624	0	2.11636689711647
			$-0.47053628409637\,i$		$-0.47053628409637\,i$
1	1	2.06751172656022	1.867854422623970	-1.81899925206772	2.11636689711647
			$+0.58629222084655\,i$	$-1.05682849256197\,i$	$-0.47053628409637\,i$
2	0	2.38830566931499	-0.27193877219851	0	2.11636689711647
		$-0.32079394275467\,i$	$-0.14974234134161\,i$		$-0.47053628409637\,i$
2	1	2.38830566931499	1.54706047986920	-1.81899925206772	2.11636689711647
		$-0.32079394275467\,i$	$+0.90708615122036\,i$	$-1.05682849256197\,i$	$-0.47053628409637\,i$
5	0	1.37225259445007	0.74411430266640	0	2.11636689711647
		$-0.94437739311064\,i$	$+0.47384110901427\,i$		$-0.47053628409637\,i$
5	1	1.37225259445007	2.56311355473412*	-1.81899925206772	2.11636689711647
		$-0.94437739311064\,i$	$+1.53066960157624\,i$*	$-1.05682849256197\,i$	$-0.47053628409637\,i$
10	0	242.349288466045	-240.232921568929	0	2.11636689711576
		$-158.966005072758\,i$	$+158.495468788662\,i$		$-0.47053628409614\,i$
10	1	242.349288466045	-238.413922316848*	-1.81899925206772	2.11636689712853
		$-158.966005072758\,i$	$+159.552297281241\,i$*	$-1.05682849256197\,i$	$-0.47053628407897\,i$
15	0	-208800.24691375	208802.36328065	0	2.11636690574232
		$+1.788757490 \times 10^6\,i$	$-1.788757960 \times 10^6\,i$		$-0.47053628531284\,i$
15	1	-208800.24691375	208804.182279090*	-1.81899925206772	2.11636608755697
		$+1.788757490 \times 10^6\,i$	$-1.788756903 \times 10^6\,i$*	$-1.05682849256197\,i$	$-0.47053533281075\,i$
20	0	$-4.688631796 \times 10^{10}$	$4.688631796 \times 10^{10}$	0	2.11630249023437
		$-5.523649527 \times 10^{10}\,i$	$+5.523649527 \times 10^{10}\,i$		$-0.47086334228515\,i$
20	1	$-4.6886317961 \times 10^{10}$	$4.6886317966 \times 10^{10}$*	-1.81899925206772	3.23580268885024
		$-5.5236495270 \times 10^{10}\,i$	$+5.5236495272 \times 10^{10}\,i$*	$+0.20513439806302\,i$	$+0.31231213243802\,i$

Table 2: MB-regularised values of $T_I(0, 3/7, z^3)$ for $|z| = 4/5$ and $\arg z > 0$.

N	l	$\arg z$	Truncated Series	MB Integral	Discontinuity	Regularised Value
0	0	$\pi/5$	0	$1.971898996789572 - 0.37794459349277\,i$	0	$1.971898996789572 - 0.37794459349277\,i$
0	1	$\pi/5$	0	$6.405352445877721 + 0.7627141840391\,i$	$-4.43345344908814 - 1.14065877753194\,i$	$1.971898996789572 - 0.37794459349277\,i$
5	0	$\pi/5$	$1.8292655633621052 + 0.3048603039562\,i$	$0.1426334334274671 - 0.68280489744902\,i$	0	$1.971898996789572 - 0.37794459349277\,i$
5	1	$\pi/5$	$1.8292655633621052 + 0.3048603039562\,i$	$4.576086882515615 + 0.4578538800829\,i$	$-4.43345344908814 - 1.14065877753194\,i$	$1.971898996789572 - 0.37794459349277\,i$
1	0	$3\pi/7$	2.0675117265602293	$2.2614145153684305 - 0.5800987411949\,i$	0	$4.32892624192865 - 0.5800987411949\,i$
1	1	$3\pi/7$	2.0675117265602293	$0.0083835743576715 + 0.4490098958443\,i$	$2.2530309410107589 - 1.02910863703932\,i$	$4.32892624192865 - 0.5800987411949\,i$
4	0	$3\pi/7$	$1.904788598635045 + 0.8572251658410\,i$	$2.424137643293614 - 1.43732390703605\,i$	0	$4.32892624192865 - 0.5800987411949\,i$
2	1	$2\pi/3$	1.6138405819892990	0.164958026798924	$-57.54165792587062 - 13.1335079271702\,i$	$-55.76285931708239 - 13.1335079271702\,i$
8	1	$2\pi/3$	-9.844470293491122	11.6232689022793457	$-57.54165792587062 - 13.1335079271702\,i$	$-55.76285931708239 - 13.133507927170\,i$
2	1	$8\pi/9$	$2.2943472988456945 - 0.3928907361623\,i$	$-0.266077718034366 - 0.02681592216461\,i$	$2.2479719678953279 + 2.2102553462406\,i$	$4.276241548706656 + 1.7905486879136\,i$
6	1	$8\pi/9$	$2.17482659335582993 + 1.36926557120305\,i$	$-0.146557012544501 - 1.78897222953006\,i$	$2.2479719678953279 + 2.2102553462406\,i$	$4.276241548706656 + 1.7905486879136\,i$

Table 3: MB-regularised values of $T_l(0, 3/7, z^3)$ for $|z| = 2$ and $\arg z < 0$.

N	l	$\arg z$	Truncated Series	MB Integral	Discontinuity	Regularised Value
1	0	$-\pi/9$	2.067751172656022	-1.07962608371176	0	0.98788564284846
				$+0.3210639387498\,i$		$+0.321063938749\,i$
6	0	$-\pi/9$	-784806.14583354	784807.1337191856	0	0.987885642848465
			$-1.287266362678\,i$	$+1.287266683742\,i$	0	$+0.321063938749\,i$
2	0	$-\pi/7$	0.490147245326580	0.483064672608320	0	0.973211917934900
			$+6.910885340526\,i$	$-6.491669723693\,i$	0	$+0.419215616832\,i$
5	0	$-\pi/7$	27825.84192190162	-27824.86870998368	0	0.973211917934900
		i	$+32494.03006569\,i$	$-32493.610850078\,i$		$+0.419215616832\,i$
2	0	$-\pi/4$	7.07991708210346	-6.2105272239808399	0	0.869389858122629
			$+5.012405355432\,i$	$-4.220185540933908\,i$		$+0.792219814609\,i$
2	-1	$-\pi/4$	7.07991708210346	-6.7900986207555987	0.579571396774	0.869389858122629
			$+5.012405355432\,i$	$-6.50703153290034\,i$	$+2.286845991\,i$	$+0.792219814609\,i$
5	0	$-\pi/4$	-442777.383495959	44278.252885817824	0	0.869389858122629
			$+1198.983686647\,i$	$-1198.191466832895\,i$		$+0.792219814609\,i$
5	-1	$-\pi/4$	-442777.383495959	44277.673314421049	0.579571396774	0.869389858122629
			$+1198.983686647\,i$	$-1200.47831282486\,i$	$+2.286845991\,i$	$+0.792219814609\,i$
5	0	$-\pi/3$	44835.63477770317	-44834.957245040938	0	0.677532662231576
				$+1.13716765042504\,i$	i	$+1.137167650425\,i$
5	-1	$-\pi/3$	44835.63477770317	-44834.957245040938	0.0	0.677532662231576
				$-1.137167650425048\,i$	$+2.274335300\,i$	$+1.137167650425\,i$
4	0	$-3\pi/8$	668.2300900870667	-667.7147676640257	0	0.515241344680886
			$-1514.1477848902\,i$	$+1515.4681865276\,i$		$+1.3204016374113\,i$
4	-1	$-3\pi/8$	668.2300900870667	-667.4388408829950	-0.27592678103	0.515241344680886
			$-1514.1477848902\,i$	$+1513.1887472848\,i$	$+2.279439242\,i$	$+1.32040163374113\,i$
9	0	$-3\pi/8$	$-2.06217683 \times 10^{11}$	$2.0621768337 \times 10^{11}$	0	0.515241344680886
			$-1.35571241 \times 10^{9}\,i$	$+1.35571241 \times 10^{9}\,i$		$+1.3204016374113\,i$
9	-1	$-3\pi/8$	$-2.06217683 \times 10^{11}$	$2.0621768337 \times 10^{11}$	-0.27592678103	0.515241344680886
			$-1.35571241 \times 10^{9}\,i$	$+1.35571241 \times 10^{9}\,i$	$+2.2794392427\,i$	$+1.3204016374113\,i$
3	-1	$-\pi/2$	-78.9451926611059	79.90366381612858	-1.34308796448	-0.38461680945944
			$-7.0886116339207\,i$	$+6.592650733540\,i$	$+2.199514935\,i$	$+1.70355403523184\,i$
7	-1	$-\pi/2$	$-6.638111341 \times 10^{7}$	$6.63811143703 \times 10^{7}$	-1.34308796448	-0.38461680945944
			$-1.52793787 \times 10^{6}\,i$	$+1.52793737 \times 10^{6}\,i$	$+2.199514935\,i$	$+1.70355403523184\,i$
2	-1	$-3\pi/4$	-2.9448936289830	3.9418113902568939	-2.75887232413	-1.76195456286419
			$+5.012405355543\,i$	$-4.77410100772261\,i$	$-0.56094042932\,i$	$-0.32263608150788\,i$
9	-1	$-3\pi/4$	$2.005995535 \times 10^{11}$	$-2.00599553541 \times 10^{11}$	-2.75887232413	-1.76195456286419
			$-2.3502051 \times 10^{9}\,i$	$+2.350205185 \times 10^{9}\,i$	$-0.56094042932\,i$	$-0.3226360815078\,i$
0	-1	$-7\pi/9$	0	0.9878856428484654	-2.59297885441	-1.60509321156175
				$+0.32106393874989\,i$	$-0.8958519939\,i$	$-0.5747880552394\,i$
2	-1	$-7\pi/9$	-20053.774450535	20054.762336178323	-2.59297885441	-1.60509321156175
			$+37323.59976542\,i$	$-37323.2787014826\,i$	$-0.8958519939\,i$	$-0.5747880552394\,i$

Table 4: Borel-regularised values of $T_l(0, 3/7, z^3)$ for $|z| = 4/5$ and $\arg z > 0$.

N	l	$\arg z$	Truncated Series	Borel Integral	Discontinuity	Regularised Value
3	0	0	1.9456686191611794	-0.166870010372956	0	1.778798608788223
7	0	0	5.1797052832766771	-3.400906674488453	0	1.778798608788223
0	0	$\pi/5$	0	1.971898996789572 $-0.37794459349277\,i$	0	1.9718989967895723 $-0.37794459349277\,i$
9	0	$\pi/5$	-53.5026687306015 $+20.720679253054\,i$	55.4745677273911721 $-21.0986238465476\,i$	0 0	1.971898996789572 $-0.3779445934927\,i$
15	0	$\pi/5$	284459.67049219113 $+1.659066369 \times 10^6\,i$	-284457.6985931943 $-1.65906674 \times 10^6\,i$	0 0	1.971898996789572 $-0.3779445934927\,i$
1	0	$\pi/4$	2.0675117265602293	0.04885517055624873 $-0.470536284096377\,i$	0	2.116366897116478 $-0.47053628409637\,i$
10	0	$\pi/4$	242.34928846604510 $-158.966005072758\,i$	-240.2329215689286 $+158.495468788662\,i$	0	2.116366897116478 $-0.4705362840963\,i$
4	1	$3\pi/7$	1.9047885986350452 $+0.85722516584108\,i$	0.1711067022828556 $-0.40821526999672\,i$	2.25303094101075 $-1.029108637039\,i$	4.32892624192865 $-0.5800987411949\,i$
8	1	$3\pi/7$	18.756954328980284 $-4.62692996459794\,i$	-16.681059028062383 $+5.07593986044230\,i$	2.25303094101075 $-1.029108637039\,i$	4.32892624192865 $-0.5800987411949\,i$
2	1	$2\pi/3$	1.6138405819892990	0.164958026798924	-57.541657925870 $-13.13350792717\,i$	-55.7628593170823 $-13.13350792717\,i$
17	1	$2\pi/3$	$8.436158292140 \times 10^7$	$-8.43615811426 \times 10^7$	-57.541657925870 $-13.133507927170\,i$	-55.76285931708239 $-13.133507927170\,i$
6	1	$8\pi/9$	2.174826593355829 $+1.3692655712030\,i$	-0.146557012544501 $-1.78897222953006\,i$	2.24797196789532 $+2.210255346240\,i$	4.276241548706656 $+1.7905486879136\,i$
15	1	$8\pi/9$	-655770.33200706 $-1.607729425 \times 10^6\,i$	655772.36027664411 $+1.607729005 \times 10^6\,i$	2.24797196789532 $+2.21025534624\,i$	4.276241548706656 $+1.7905486879136\,i$

Table 5: Borel-regularised values of $T_l(0, 3/7, z^3)$ for $|z| = 2$ and $\arg z < 0$.

N	l	$\arg z$	Truncated Series	Borel Integral	Discontinuity	Regularised Value
1	0	$-\pi/9$	2.06775117265602	−1.07962608371176 0.32106393874989 i	0	0.9878856428484 +0.32106393874 i
6	0	$-\pi/9$	−784806.145833542 $-1.28726636 \times 10^6\, i$	784807.13371918 $+1.28726668 \times 10^6\, i$	0	0.9878856428484 +0.32106393874 i
2	0	$-\pi/7$	0.490147245326580 +6.9108853405261 i	0.48306467260832 −6.49166972369336 i	0	0.9732119179349 +0.41921561683 i
5	0	$-\pi/7$	27825.84192190162 +32494.030065694 i	−27824.8687099836 −32493.6108500780 i	0	0.9732119179349 +0.41921561683 i
0	0	$-\pi/4$	0	0.869389858122629 +0.79221981460933 i	0	0.8693898581226 +0.79221981460 i
5	0	$-\pi/4$	−44277.383495959 +1198.98368664 i	44278.2528858178 −1198.19146683289 i	0	0.8693898581226 +0.79221981460 i
4	−1	$-3\pi/8$	668.230090087066 −1514.147784890 i	−667.438840882995 +1513.188747284 i	−0.275926781030 +2.2794392427 i	0.5152413446808 +1.320401637411 i
9	−1	$-3\pi/8$	$-2.062176833 \times 10^{11}$ $-1.355712417 \times 10^9\, i$	$2.0621768337 \times 10^{11}$ $+1.355712418 \times 10^9\, i$	−0.275926781030 +2.27943924273 i	0.5152413446808 +1.320401637411 i
3	−1	$-\pi/2$	−78.9451926611059 −7.0886116339207 i	79.90366381612858 +6.5926507335409 i	−1.343087964482 +2.1995149356 i	−0.3846168094594 +1.70355403523 i
7	−1	$-\pi/2$	-6.638111341×10^7 $-1.527937870 \times 10^6\, i$	6.6381114370×10^7 $+1.527937374 \times 10^6\, i$	−1.343087964482 +2.1995149356 i	−0.3846168094594 +1.70355403523 i
2	−1	$-3\pi/4$	−2.94489362898301 +5.0124053555432 i	3.941811390256893 −4.77410100772261 i	−2.758872324138 −0.56094042932 i	−1.7619545628641 −0.322636081507 i
9	−1	$-3\pi/4$	$2.00599553 \times 10^{11}$ $-2.3502051 \times 10^9\, i$	$-2.0059955354 \times 10^{11}$ $+2.350205185 \times 10^9\, i$	−2.758872324138 −0.56094042932 i	−1.7619545628641 −0.322636081507 i
0	−1	$-7\pi/9$	0	0.987885642848465 +0.32106393874989 i	−2.592978854410 −0.89585199398 i	−1.605093211561 −0.574788055239 i
3	−1	$-7\pi/9$	−41.983146284233 −64.02014227646 i	42.971031927081694 +64.3412062152105 i	−2.592978854410 −0.89585199398 i	−1.605093211561 −0.574788055239 i

Table 6: Borel-regularised value of $T_I(0, 3/7, z^3)$ for $|z| = 2\exp(i\pi/3)$.

N	Truncated Series	Borel Integral	Discontinuity	Regularised Value
0	0	0.6775326622315765205	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
1	2.06751172656022935	−1.389979064328652832	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
2	9.15612336048101570	−8.478590698249391857	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
5	44835.6347777031703	−44834.95724504093874	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
10	$1.38954464105 \times 10^{13}$	$-1.38954464105 \times 10^{13}$	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
15	$8.49177284753 \times 10^{20}$	$-8.49177284753 \times 10^{20}$	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
20	$3.260190269339 \times 10^{33}$	$-3.260190269339 \times 10^{33}$	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$
30	$1.997889102631 \times 10^{56}$	$-1.99788910263 \times 10^{56}*$	0 $+1.1371676504250487\,i$	0.67753266223157652 $+1.137167650425048\,i$