EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 2, No. 2, 2009, (250-267)
ISSN 1307-5543 – www.ejpam.com

Some Characterizations of Weighted Holomorphic Bloch Space

R. A. Rashwan[1], A. El-Sayed Ahmed[2*] and A. Kamal[3]

[1] *Assiut University, Faculty of Science, Department of Mathematics, Assiut, Egypt*

[2] *Sohag University, Faculty of Science, Department of Mathematics, Sohag 82524, Egypt*

Current Address: Taif University, Faculty of Science, Mathematics Department, El-Taif P.O.Box 888, El-Hawiyah, Kingdom of Saudi Arabia

[3] *Institute of Computer Science, Al-Kawser city at Sohag Egypt*

Abstract. In this paper we introduce a new space, the so called $Q_{K,\omega}$ space of analytic functions on the unit disk in terms of nondecreasing functions. The relation between integral norm of $Q_{K,\omega}$ space and integral norm of the weighted Bloch space $\mathscr{B}^{\alpha}_{\omega}$ is also given.

AMS subject classifications: 30D45, 46E15

Key words: $\mathbf{Q}_{\mathbf{K},\,\omega}$ spaces, weighted Bloch functions.

---

*Corresponding author.

Email addresses: r_rashwan@yahoo.com (R. Rashwan), ahsayed80@hotmail.com (A. Ahmed), alaa_mohamed1@yahoo.com (A. Kamal)

1. Introduction

Let $\Delta = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [2]) is defined as follows:

$$\mathscr{B} = \{f : f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty\};$$

the little Bloch space $\mathscr{B}_0$ (cf. [2]) is a subspace of $\mathscr{B}$ consisting of all $f \in \mathscr{B}$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

The Dirichlet space is defined by

$$\mathscr{D} = \{f : f \text{ analytic in } \Delta \text{ and } \int_\Delta \left|f'(z)\right|^2 d\sigma_z < \infty\},$$

where $d\sigma_z$ is the Euclidean area element $dxdy$. Let $0 < q < \infty$. Then the Besov-type spaces

$$\mathbf{B^q} = \left\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_\Delta \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-2}(1-|\varphi_a(z)|^2)^2 d\sigma_z < \infty\right\}$$

are introduced and studied intensively by Stroethoff (cf. [11]). Here, $\varphi_a(z)$ stands for the Möbius transformation of Δ given by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \text{ where } a \in \Delta.$$

In 1994, Aulaskari and Lappan [2] introduced a class of holomorphic functions, the so called $\mathbf{Q_p}$-spaces as follows:

$$\mathbf{Q_p} = \left\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_\Delta \left|f'(z)\right|^2 g^p(z,a) d\sigma_z < \infty\right\},$$

where $0 < p < \infty$ and the weight function

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$$

is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. The weight function $g(z,a)$ is actually Green's function in Δ with pole at $a \in \Delta$.

For $0 < p < \infty, -2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ (cf. [14]), if

$$\|f\|_{K,p,q} = \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p (1-|z|^2)^q K(g(z,a))d\sigma_z < \infty.$$

Recall that the analytic function

$$f(z) = \sum_k^\infty a_k z^{n_k} \ \ (\text{with } \ n_k \in \mathbb{N}\,; \text{ for all } \ k \in \mathbb{N} = \{1,2,3,\ldots\}\,)$$

is said to belong to the Hadamard gap class (also known as lacunary series) if there exists a constant $c > 1$ such that $\frac{n_{k+1}}{n_k} \geq c$ for all $k \in \mathbb{N}$ (see e.g. [17]).

Two quantities A_f and B_f, both depending on an analytic function f on Δ, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on Δ we have:

$$\frac{1}{C}B_f \leq A_f \leq CB_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

Now, given a reasonable function $\omega : (0,1] \to [0,\infty)$, the weighted Bloch space $\mathscr{B}_\omega$ (see [4]) is defined as the set of all analytic functions f on Δ satisfying

$$(1-|z|)|f'(z)| \leq C\omega(1-|z|), \quad z \in \Delta,$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathscr{B}_\omega$ reduces to the classical Bloch space $\mathscr{B}$. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.

Now, we introduce the following definitions:

Definition 1.1. *For a given reasonable function* $\omega : (0,1] \to [0,\infty)$ *and for* $0 < \alpha < \infty$. *An analytic function* f *on* Δ *is said to belong to the* $\alpha-$*weighted Bloch space* $\mathscr{B}^{\alpha}_{\omega}$ *if*

$$\|f\|_{\mathscr{B}^{\alpha}_{\omega}} = \sup_{z\in\Delta} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)} |f'(z)| < \infty.$$

Definition 1.2. *For a given reasonable function* $\omega : (0,1] \to [0,\infty)$ *and for* $0 < \alpha < \infty$. *An analytic function* f *on* Δ *is said to belong to the little weighted Bloch space* $\mathscr{B}^{\alpha}_{\omega,0}$ *if*

$$\|f\|_{\mathscr{B}^{\alpha}_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)} |f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$. Now, we introduce the following new definition:

Definition 1.3. *For a nondecreasing function* $K : [0,\infty) \to [0,\infty), 0 < p < \infty$, *and for a given reasonable function* $\omega : (0,1] \to (0,\infty)$, *an analytic function* f *in* Δ *is said to belong to the space* $Q_{K,\omega}$ *if*

$$\|f\|^p_{K,\omega} \;=\; \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^p (1-|z|)^p \frac{K(g(z,a))}{\omega^p(1-|z|)} \, d\sigma_z < \infty.$$

Remark 1.1. *It should be remarked that our* $Q_{K,\omega}$ *classes are more general than many classes of analytic functions. If* $\omega \equiv 1$, *we obtain* $Q_K(p,p)$ *type spaces (cf. [14] and [15]). If* $p = 2$, *and* $\omega(t) = t$, *we obtain* Q_K *spaces as studied recently in [5, 6, 9, 12, 13, 16] and others. If* $p = 2$, $\omega(t) = t$ *and* $K(t) = t^p$, *we obtain* Q_p *spaces as studied in [2, 3, 17] and others. If* $\omega \equiv 1$ *and* $K(t) = t^s$, *then* $Q_{K,\omega} = F(p,p,s)$ *classes (cf. [1, 18]).*

In this paper, we characterize the weighted Bloch space $\mathscr{B}^{\alpha}_{\omega}$ by our $Q_{K,\omega}$ spaces. One of the main results is a general Besov-type characterization for $\mathscr{B}^{\alpha}_{\omega}$ functions that extends and generalizes the Stroethoff's theorem [11]. Also, we extend and improve some results due to Essén et. al [6] using our new definitions.

2. Holomorphic $Q_{K,\omega}$ Classes

In this paper we show some relations between $Q_{K,\omega}$ norms and $\mathscr{B}^{\alpha}_{\omega}$ norms for a nondecreasing function K, , also we give a general way to construct different spaces Q_{K,ω_1} and $Q_{K_2,\omega}$ by using some functions K_1 and K_2. Before proving theorems we recall few facts about the Möbius function φ_a. First, the function φ_a is easily seen to be it own inverse under composition:

$$(\varphi_a \circ \varphi_a)(z) = z \quad \text{for all} \quad z \in \Delta$$

The following identity can be obtained by straight forward computation:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\,, \quad (a, z \in \Delta).$$

A slightly different form in which we will apply the above identity is:

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)|\,, \quad (a, z \in \Delta). \tag{2.1}$$

For $a \in \Delta$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $d\sigma_w = |\varphi_a'(z)|^2 d\sigma_z$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)} h(\varphi_a(w))d\sigma_w = \int_{\Delta(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 d\sigma_z\,. \tag{2.2}$$

We assume throughout this paper that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr < \infty\,. \tag{2.3}$$

We need the following lemmas in the sequel.

Lemma 2.1. *[17] Let* $\alpha \in (0,\infty)$ *and suppose that* $f(z) = \sum_{j=1}^{\infty} a_j z^{n_j}$ *belongs to Hadamard gap class. Then* $f \in \mathscr{B}^{\alpha}$ *if and only if*

$$\sup_{j\in\mathbb{N}} |a_j| n_j^{1-\alpha} < \infty\,, \ \ \textit{where} \ \ \mathbb{N} = \{1, 2, 3, \dots\}\,.$$

Lemma 2.2. *Let* $\omega : (0,1] \to (0,\infty)$ *be a nondecreasing function. Then there are two functions* f_1, $f_2 \in \mathscr{B}_\omega$ *such that*

$$|f_1'(z)| + |f_2'(z)| \approx \frac{\omega(1-|z|)}{(1-|z|)}, \quad z \in \Delta. \tag{2.4}$$

Proof. For a large number $q \in \mathbb{N}$, choose a gap series:

$$f_1(z) = \sum_{j=0}^{\infty} z^{q^j}, \quad z \in \Delta.$$

Then, apply lemma 2.1 to infer that $\frac{(1-|z|)|f_1'(z)|}{\omega((1-|z|))} \leq \lambda$ holds for all $z \in \Delta$, where λ is a constant. Furthermore, let us verify

$$\frac{(1-|z|)|f_1'(z)|}{\omega((1-|z|))} \geq \lambda, \quad 1-q^{-k} \leq |z| \leq 1-q^{-(k+\frac{1}{2})}, \ k \in \mathbb{N}. \tag{2.5}$$

And

$$q^{-(k+\frac{1}{2})} \leq 1-|z| \leq q^{-k} \Rightarrow \omega(q^{-(k+\frac{1}{2})}) \leq \omega(1-|z|) \leq \omega(q^{-k}).$$

Observe that for any $z \in \Delta$,

$$|f_1'(z)| \geq q^k|z|^{q^k} - \sum_{j=0}^{k-1} q^j|z|^{q^j} - \sum_{k+1}^{\infty} q^j|z|^{q^j} = T_1 - T_2 - T_3.$$

And then, fix a z with $|z| \in [1-q^{-k}, 1-q^{-(k+\frac{1}{2})}], k \in \mathbb{N}$, and put $x = |z|^{q^k}$.
Thus

$$(1-q^{-k})^{q^k} \leq x \leq [(1-q^{-(k+\frac{1}{2})})^{q^{k+\frac{1}{2}}}]^{q^{\frac{-1}{2}}}.$$

If q is large enough, then for $k \geq 1$ one has

$$\frac{1}{3} \leq x \leq (\frac{1}{2})^{q^{\frac{-1}{2}}}, \tag{2.6}$$

and hence $T_1 \geq \frac{q^k}{3}$. Since it is easy to establish

$$T_2 \leq \sum_{j=0}^{k-1} q^j \leq \frac{q^k}{q-1},$$

it remains to deal with the third term T_3. Noting that

$$|z|^{q^n(q-1)} \leq |z|^{q^{k+1}(q-1)}, \quad n \geq k+1,$$

namely, in T_3 the quotient of two successive terms is not greater than the ratio of the first two terms, one finds that the series of T_3 is controlled by the geometric series having the same first two terms. Accordingly (2.6) is applied to produce

$$\begin{aligned} T_3 &\leq q^{k+1}|z|^{q^{k+1}} \sum_{j=0}^{\infty} \left(q|z|^{q^{k+2}-q^{k+1}} \right)^j \\ &= \frac{q^{k+1}|z|^{q^{k+1}}}{1-q|z|^{(q^{k+2}-q^{k+1})}} = q^k \frac{qx^q}{1-qx^{q^2-q}} \\ &\leq q^k \frac{q(\frac{1}{2})^{q^{\frac{1}{2}}}}{1-q(\frac{1}{2})q^{\frac{3}{2}}-q^{\frac{1}{2}}}. \end{aligned}$$

The preceding estimates for T_1, T_2 and T_3 imply

$$\begin{aligned} |f_1'(z)| &\geq \frac{q^k}{4} \frac{\omega(1-|z|)}{\omega(1-|z|)} = \frac{q^{k+\frac{1}{2}}}{4q^{\frac{1}{2}}} \frac{\omega(1-|z|)}{\omega(1-|z|)} \\ &\geq \frac{\omega(1-|z|)}{4q^{\frac{1}{2}}(1-|z|) \times \omega(1-|z|)} \\ &\geq \frac{\omega(1-|z|)}{4q^{\frac{1}{2}}\omega(q^{-k}) \times (1-|z|)}; \quad \omega(q^{-k}) \not\to \infty. \end{aligned}$$

Reaching (2.5).

In a completely similar manner one can prove that if q is a large natural number, for example $q = m^2$ where m is a large natural number, and if

$$f_2(z) = \sum_{j=0}^{\infty} z^{q^j}, \quad z \in \Delta,$$

then $(1-|z|^2)|f_2'(z)| \leq \lambda$ for all $z \in \Delta$ (owing to Lemma 2.1) and

$$\frac{(1-|z|)|f_1'(z)|}{\omega((1-|z|))} \leq \lambda, \quad 1-q^{-(k+\frac{1}{2})} \leq |z| \leq 1-q^{-(k+1)}, \quad k \in \mathbb{N}. \tag{2.7}$$

Of course, (2.5) and (2.7) yield (2.4) unless it occurs that f_1' and f_2' have common zero in $\{z \in \Delta : |z| < 1 - q^{-1}\}$ in which case one can replace f_2 with $f_2(\zeta z)$ for appropriate $\zeta \in \partial\Delta$, where $\partial\Delta$ is the boundary of the unit disk (note that $f'(0) = 1$). Our lemma is therefore proved .

Using the same steps of Lemma 2.2, it is not hard to prove the following lemma.

Lemma 2.3. *Let* $\omega : (0,1] \to (0,\infty)$ *be a nondecreasing function and let* $1 \le \alpha < \infty$. *Then there are two functions* $f_1, f_2 \in \mathscr{B}_\omega^\alpha$ *such that*

$$|f_1'(z)| + |f_2'(z)| \approx \frac{\omega(1-|z|)}{(1-|z|)^\alpha}, \qquad z \in \Delta. \tag{2.8}$$

Proof. The proof is very similar to the proof of Lemma 2.2 and lemma 3.1 in [7], so it will be omitted.

Theorem 2.1. *For each non-decreasing function* $K : [0,\infty) \to [0,\infty)$, $0 < p < \infty$ *and for a given reasonable non-decreasing function* $\omega : (0,1] \to (0,\infty)$ *with* $\omega(\alpha t) \approx \omega(t)$, $\alpha > 0$, *we have that*

(i) $Q_{K,\omega} \subset \mathscr{B}_\omega^{\frac{p+2}{p}}$ *and*

(ii) $Q_{K,\omega} = \mathscr{B}_\omega^{\frac{p+2}{p}}$, *iff*

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr < \infty.$$

Proof. For a fixed $r \in (0,1)$ and $a \in \Delta$, let

$$E(a,r) = \left\{z \in \Delta\,,\ |z-a| < r(1-|a|)\right\}.$$

We know that $E(a,r) \subset \Delta(a,r)$ and for any $z \in E(a,r)$, we have

$$(1-r)(1-|a|) \le 1-|z| \le (1+r)(1-|a|),$$

which means that $1-|z|^2 \simeq 1-|a|^2$ for any $z \in E(a,r)$. Denote

$$F_{\omega,p}(f)(z) = \left|f'(z)\right|^p \frac{(1-|z|)^p}{\omega^p(1-|z|)}$$

Then, we obtain

$$\int_{\Delta} F_{\omega,p}(f)(z)K\big(g(z,a)\big)\,d\sigma_z \geq \int_{\Delta(a,r)} F_{\omega,p}(f)(z)K\big(g(z,a)\big)\,d\sigma_z$$

$$\geq K\left(\log\frac{1}{r}\right)\int_{\Delta(a,r)} F_{\omega,p}(f)(z)\,d\,\sigma_z$$

$$\geq K\left(\log\frac{1}{r}\right)\int_{E(a,r)} F_{\omega,p}(f)(z)\,d\sigma_z.$$

For every $z \in E(a,r)$, we have that

$$(1-r)(1-|a|) \leq 1-|z| \leq (1+r)(1-|a|),$$

Then,

$$(1-|z|)^p \geq (1-r)^p(1-|a|)^p\ ,\ \forall\ \ p>0.$$

Now, since we assume that ω is non-decreasing, then we obtain that

$$\int_{E(a,r)} F_{\omega,p}(f)(z)\,d\sigma_z \geq \frac{(1-r)^p(1-|a|)^p}{\omega^p((1-r)(1-|a|))}\int_{E(a,r)} \left|f'(z)\right|^p\,d\sigma_z.$$

Since $|f'(z)|^p$ is a subharmonic function, then

$$\int_{E(a,r)} \left|f'(z)\right|^p\,d\sigma_z \geq |E(a,r)|\,.\,|f'(a)\big|^p = r^2(1-|a|)^2|f'(a)\big|^p.$$

Then we obtain

$$\int_{\Delta} F_{\omega,p}(f)(z)K\big(g(z,a)\big)\,d\sigma_z \geq K\left(\log\frac{1}{r}\right)\frac{(1-r)^p(1-|a|)^{p+2}}{\omega^p((1-r)(1-|a|))}|f'(a)\big|^p$$

$$\geq \lambda K\left(\log\frac{1}{r}\right)\frac{(1-r)^p(1-|a|)^{p+2}}{\omega^p(1-|a|)}|f'(a)\big|^p$$

where λ is a constant. If $f \in Q_{K,\omega}$, then by the above estimate we have that

$$\sup_{a\in\Delta}\frac{(1-|a|)^{p+2}|f'(z)|^p}{\omega^p(1-|a|)} < \infty.$$

The proof of (i) is therefore completed.

Now, we show that $\mathscr{B}_{\omega}^{\frac{p+2}{p}} \subset Q_{K,\omega}$ provided that K satisfies condition (2.3). For $f \in \mathscr{B}_{\omega}^{\frac{p+2}{p}}$, we have that,

$$\int_{\Delta} F_{\omega,p}(f)(z)K\big(g(z,a)\big)\,d\sigma_z \leq \big\|f\big\|^p_{\mathscr{B}_{\omega}^{\frac{p+2}{p}}} \int_{\Delta}(1-|z|^2)^{-2}K\big(g(z,a)\big)\,d\,\sigma_z$$
$$= 2\pi\big\|f\big\|^p_{\mathscr{B}_{\omega}^{\frac{p+2}{p}}} \int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr < \infty,$$

which shows that

$$\mathscr{B}_{\omega}^{\frac{p+2}{p}} \subset Q_{K,\omega}.$$

Now we assume that $\mathscr{B}_{\omega}^{\frac{p+2}{p}} = Q_{K,\omega}$ and we verify (2.3) holds. From Lemma 2.3, for f_1 and f_2 in $\mathscr{B}_{\omega}^{\frac{p+2}{p}}$, we have that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{\omega(1-|z|)}{(1-|z|)^{\frac{p+2}{p}}}. \tag{2.9}$$

Then $f_1, f_2 \in Q_{K,\omega}$ and

$$\infty > \sup_{a\in\Delta}\int_{\Delta}\left(\left|f_1'(z)\right|^p + \left|f_2'(z)\right|^p\right)(1-|z|)^p\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z$$
$$\geq \int_{\Delta}\left(\left|f_1'(z)\right| + \left|f_2'(z)\right|\right)^p(1-|z|)^p\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\,d\sigma_z \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\int_{\Delta}\left(\left|f_1'(z)\right|^p + \left|f_2'(z)\right|^p\right)(1-|z|)^p\,\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\,d\sigma_z \approx 2\pi\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr.$$

Thus (2.3) holds, and this completes the proof.

3. The Classes $Q_{K,\omega,0}$ and $\mathscr{B}^{\alpha}_{\omega,0}$

We say that $f \in Q_{K,\omega,0}$ if

$$\lim_{|a|\to 1^-}\int_{\Delta}\left|f'(z)\right|^p(1-|z|)^p\,\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z = 0. \tag{3.1}$$

Also, as a subspace of $\mathscr{B}^{\alpha}_{\omega}$, we define the little weighted Bloch space $\mathscr{B}^{\alpha}_{\omega,0}$ as the space which consists of analytic functions f on Δ such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}|f'(z)|}{\omega(1-|z|)} = 0$$

where $0 < \alpha < \infty$. Thus we can obtain the following theorem:

Theorem 3.1. *For each nondecreasing function* $K : [0,\infty) \to [0,\infty)$, $0 < p < \infty$, *for a given reasonable non-decreasing function* $\omega : (0,1] \to (0,\infty)$ *with* $\omega(\alpha\, t) \approx \omega(t)$, $\alpha > 0$. *Then*

(i) $\quad Q_{K,\omega,0} \subset \mathscr{B}^{\frac{p+2}{p}}_{\omega,0}$ *and*

(ii) $\quad Q_{K,\omega,0} = \mathscr{B}^{\frac{p+2}{p}}_{\omega,0}$, *if and only if (2.3) holds.*

Proof. Without loss of generality, we assume that $K(1) > 0$. From the proof of Theorem 2.1, we have that

$$\begin{aligned}
\pi(\tfrac{1}{e})^2 K(1)\tfrac{(1-|a|)^{p+2}}{\omega^p(1-|a|)}|f'(a)|^p \quad &\le K(1)\int_{E(a)} F_{\omega,p}(f)(z)\, d\sigma_z \\
&\le K(1)\int_{\Delta(a,\frac{1}{e})} F_{\omega,p}(f)(z)\, d\sigma_z \\
&\le \int_{\Delta} F_{\omega,p}(f)(z) K\big(g(z,a)\big) \ d\sigma_z \ ,
\end{aligned}$$

where

$$E(a) = \left\{ z \in \Delta \, , \, |z-a| < \frac{1}{e}(1-|a|) \right\}.$$

If $f \in Q_{K,\omega,0}$, we obtain that

$$\lim_{|a|\to 1^-} \frac{(1-|a|)^{p+2}|f'(a)|^p}{\omega^p(1-|a|)} = 0.$$

(ii) We only need to prove that $\mathscr{B}^{\frac{p+2}{p}}_{\omega,0} \subset Q_{K,w,0}$. Assume that

$$A = \int_0^1 K\left(\log\frac{1}{r}\right) \frac{r}{(1-r^2)^2}\, dr < \infty.$$

For a given $\epsilon > 0$ there exists an r_1, $0 < r_1 < 1$, such that

$$\int_{r_1}^{1} K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr < \epsilon. \tag{3.2}$$

Then we have that,

$$\begin{aligned}
&\int_{\Delta\setminus\Delta(a,r_1)} \left|f'(z)\right|^p (1-|z|)^p \frac{K(g(z,a))}{\omega^p(1-|z|)}\,d\sigma_z \le \left\|f\right\|^p_{\mathscr{B}^{\frac{p+2}{p}}_{\omega,0}} \int_{\Delta\setminus\Delta(a,r_1)} \frac{K(g(z,a))}{(1-|z|^2)^2}\,d\sigma_z\\
= \;& \left\|f\right\|^p_{\mathscr{B}^{\frac{p+2}{p}}_{\omega,0}} \int_{r_1<|w|<1} K\left(\log\frac{1}{|w|}\right)\frac{1}{(1-|w|^2)^2}\,d\sigma_w\\
= \;& \left\|f\right\|^p_{\mathscr{B}^{\frac{p+2}{p}}_{\omega,0}} \int_{r_1}^{1} K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr \le 2\pi\epsilon \left\|f\right\|^p_{\mathscr{B}^{\frac{p+2}{p}}_{\omega,0}}\,. \qquad (3.3)
\end{aligned}$$

Similarly, if $f \in \mathscr{B}^{\frac{p+2}{p}}_{\omega,0}$, we obtain that

$$|f'(\varphi_a(w))|^p \frac{(1-|\varphi_a(w)|^2)^{\frac{p+2}{p}}}{\omega^p(1-|\varphi_a(w)|)} \longrightarrow 0$$

converges uniformly for $|w| \le r$ if $|a| \to 1^-$, where r is fixed and $0 < r < 1$. Then, we obtain that

$$\begin{aligned}
&\lim_{|a|\to 1^-}\int_{\Delta} \left|f'(z)\right|^p (1-|z|)^p \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z\\
&= \lim_{|a|\to 1^-}\int_{|w|<r} \left|f'(\varphi_a(w))\right|^p (1-|\varphi_a(w)|)^p \frac{K\big(\log\frac{1}{|w|}\big)}{\omega^p(1-|\varphi_a(w)|)}\frac{1}{(1-|w|^2)^2}\,d\sigma_w\,.\\
&\le A \lim_{|a|\to 1^-}\sup_{|w|\le r_1} \left|f'(\varphi_a(w))\right|^p \frac{(1-|\varphi_a(w)|)^{p+2}}{\omega^p(1-|\varphi_a(w)|)} = 0 \qquad (3.4)
\end{aligned}$$

where By (3.2) and (3.3) it is easy to obtain that

$$\lim_{|a|\to 1^-}\int_{\Delta} \left|f'(z)\right|^p (1-|z|)^p \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z = 0. \tag{3.5}$$

Conversely, suppose that (2.3) does not hold; that is

$$\int_{0}^{1} K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr = \infty.$$

Thus we find a continuous strictly decreasing function $g : [0,1) \longrightarrow [0,\infty)$ tending to zero at 1 such that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{g(r)}{(1-r^2)^2}\,r\,dr = \infty. \tag{3.6}$$

It is easy to see that

$$r^{2^{k+1}-2} \geq \exp\{-2^{k+2}(1+r)\}, \quad r\in[0.5,1). \tag{3.7}$$

We know for $\beta > 0$ that, $t^{2\beta}\exp\{-4t\}_{t=\frac{\beta}{2}} = \left(\frac{\beta}{2}\right)^{2\beta}\exp\{-2\beta\}$. Then, there exists an integer k for $\frac{3}{4}\leq r<1$ such that $\frac{\beta}{2}\leq 2^k(1-r)<\frac{\beta+1}{2}$ and

$$\begin{aligned} 2^{\beta k}\exp\{-2^{k+2}(1-r)\} &= (1-r)^{-2\beta}\left(2^k(1-r)\right)^{2\beta}\exp\{-2^{k+2}(1-r)\} \\ &> \left(\frac{1+\beta}{2}\right)^{2\beta}(1-r)^{-2\beta}\exp\{-2(\beta+1)\}. \end{aligned} \tag{3.8}$$

For $\frac{3}{4}\leq r<1$ we define

$$f_0(z)=\sum_{k=0}^{\infty}a_k\,2^{\frac{2k}{p}}\,z^{2^k},$$

where $a_k = g\left(1-\frac{(p+1)}{p}2^k\right)$, $k=0,1,2,\dots$. By (3.7) and (3.8), we deduce that

$$\begin{aligned} M_2^2(r,f_0') &= \int_0^{2\pi}|f'_0(r\,e^{i\theta})|^2\,d\theta = 2\pi\sum_{k=0}^{\infty}a_k^2\,2^{\frac{2k(p+2)}{p}}\,z^{2^k-2} \\ &\geq 2\pi g^{\frac{2}{p}}(r)\,2^{\frac{2k(p+2)}{p}}\exp\{-2^{k+2}(1-r)\} \geq \lambda\,g^{\frac{2}{p}}(r)(1-r)^{\frac{-2(p+2)}{p}}, \end{aligned} \tag{3.9}$$

where λ is a constant. Since f_0 is defined by a gap series with Hadamard condition, we have

$$M_2(r,f_0')\approx M_p(r,f_0'),\ \text{where}\quad M_p(r,f_0')=\left(\int_0^{2\pi}|f'_0(r\,e^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned}\sup_{a\in\Delta}\int_{\Delta}\left|f_0'(z)\right|^p(1-|z|)^p\frac{K(g(z,a))}{\omega^p(1-|z|)}\,d\sigma_z &\geq \int_0^1 M_p^p(r,f_0')(1-r^2)^pK\left(\log\frac{1}{r}\right)r\,dr\\ &\approx \int_0^1 M_2^p(r,f_0')(1-r^2)^pK\left(\log\frac{1}{r}\right)r\,dr\\ &\geq \int_{\frac{3}{4}}^1 K\left(\log\frac{1}{r}\right)\frac{g(r)}{(1-r^2)^2}\,r\,dr=\infty.\end{aligned}$$

This means that $f_0 \in \mathscr{B}_{\omega,0}^{\frac{p+2}{p}} \setminus Q_{K,w,0}$, which is a contraction. Hence (2.3) holds. This completes the proof of our theorem.

4. More Results on $Q_{K,\omega}$-spaces

The following result means that the kernel function K can be chosen as bounded.

Theorem 4.1. *Assume that $K(1)>0$. Let $K_1(r)=\inf\{K(r),K(1)\}$, then*

$$Q_{K,w}=Q_{K_1,w}.$$

Proof. Since $K_1 \leq K$ and K_1 is nondecreasing, it is clear that $Q_{K,\omega}\subset Q_{K_1,w}$. It remains to prove that $Q_{K_1,\omega}\subset Q_{K,\omega}$. We note that

$$g(z,a)>1,\quad z\in\Delta(a,\frac{1}{e})\ \text{ and}$$

$$g(z,a)\leq 1,\quad z\in\Delta\setminus\Delta(a,\frac{1}{e}).$$

Thus $K(g(z,a))=K_1(g(z,a))$ in $\Delta\setminus\Delta(a,\frac{1}{e})$. It suffices to deal with integrals over $\Delta(a,\frac{1}{e})$. If $f\in Q_{K_1,\omega}$ and f is a weighted Bloch function i.e, $f\in\mathscr{B}_\omega$ then by Theorem 2.1, it follows that

$$\int_{\Delta(a,\frac{1}{e})}|f'(z)|^p(1-|z|)^p\,\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z\leq\left\|f\right\|^p_{\mathscr{B}_\omega^{\frac{p+2}{p}}}\int_{\Delta(a,\frac{1}{e})}K\big(g(z,a)\big)\frac{1}{(1-|z|^2)^2}\,d\sigma_z$$

$$=\ \left\|f\right\|^p_{\mathscr{B}_\omega^{\frac{p+2}{p}}}\int_{\Delta(0,\frac{1}{e})}K\left(\log\frac{1}{|w|}\right)\frac{1}{(1-|z|^2)^2}\,d\sigma_w\leq C\left\|f\right\|^p_{\mathscr{B}_\omega^{\frac{p+2}{p}}}$$

Thus, $f \in Q_{K,\omega}$ and Theorem 4.1 is proved.

Corollary 4.1. *Let* $0 < p < \infty$, $\omega : (0,1] \to (0,\infty)$. *Then* $f \in Q_{K,w}$ *if and only if*

$$\sup_{a\in\Delta} \int_{\Delta} |f'(z)|^p (1-|z|)^p \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} d\sigma_z < \infty.$$

For the application of the above results, we state the following lemma which is needed later.

Lemma 4.1. *Let* $K : [0,\infty) \to [0,\infty)$, $0 < p < \infty$, *for a given reasonable function* $\omega : (0,1] \to (0,\infty)$. *Then*

(i) $f \in \mathcal{B}_{\omega}^{\frac{p+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\sup_{a\in\Delta} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^p \frac{(1-|z|)K(g(z,a))}{\omega^p(1-|z|)} d\sigma_z < \infty, \tag{4.1}$$

(ii) $f \in \mathcal{B}_{\omega,0}^{\frac{p+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\lim_{|a|\to 1^-} \int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^p \frac{K(g(z,a))}{\omega^p(1-|z|)} d\sigma_z = 0. \tag{4.2}$$

Proof. (i) Assume $f \in \mathcal{B}_{\omega}^{\frac{p+2}{p}}$. For any $R \in (0,1)$ and $a \in \Delta$, we have

$$\begin{aligned}
&\int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^p \frac{K(g(z,a))}{\omega^p(1-|z|)} d\sigma_z \\
&= \int_{\Delta(0,R)} |f'(\varphi_a(z))|^p \frac{(1-|\varphi_a(z)|^2)^{p+2}}{(1+|\varphi_a(z)|)^{p+2}} \frac{K\left(\frac{1}{|z|}\right)}{(1-|z|^2)^2\omega^p(1-|z|)} d\sigma_z \\
&\leq \|f\|^p_{\mathcal{B}_{\omega}^{\frac{p+2}{p}}} \int_{\Delta(0,R)} K\left(\log\frac{1}{|z|}\right) \frac{1}{(1-|z|^2)^2} d\sigma_z \\
&\leq \lambda_1 \|f\|^p_{\mathcal{B}_{\omega}^{\frac{p+2}{p}}},
\end{aligned}$$

where $1 < (1+|\varphi_a(z)|)^{p+2} < 2^{p+2}$ and λ_1 is a constant. Conversely, suppose that (4.1) holds for some $R, 0 < R < 1$, by the proof of Theorem 2.1 (i) with $1-|a| \approx 1-|z|$ on

$E(a,R)$; $a,z \in \Delta$, we obtain

$$\int_{\Delta(a,R)} |f'(z)|^p (1-|z|)^p \frac{K(g(z,a))}{\omega^p(1-|z|)} d\sigma_z \geq K(\log\frac{1}{R}) \int_{\Delta(a,R)} |f'(z)|^p \frac{(1-|z|)^p}{\omega^p(1-|z|)} d\sigma_z$$
$$\geq \lambda_2 K\left(\log\frac{1}{R}\right)\omega^{-p}(1-|a|)\int_{E(a,R)} |f'(z)|^p (1-|z|)^p \, d\sigma_z$$
$$\geq \pi\lambda_2 R^2 K\left(\log\frac{1}{R}\right) \frac{(1-|a|)^p}{\omega^p(1-|a|)}|f'(a)|^p ,$$

where λ_2 is a constant. The last inequality shows that $f \in \mathscr{B}_\omega^{\frac{p+2}{p}}$ The proof of (ii) is similar to proof (i) by taking the limit when $|a| \longrightarrow 1^-$ in (i), hence it can be omitted.

Theorem 4.2. *Let* $0 < p < \infty$, $\omega : (0,1] \to (0,\infty)$. *Assume* $K_1(r) \leq K_2(r)$ *for* $r \in (0,1)$ *and* $\frac{K_1(r)}{K_2(r)} \to 0$ *as* $r \to 0$. *If the integral in (2.3) is divergent for* K_2, *then*

$$Q_{K_2,\omega} \subsetneqq Q_{K_1,\omega} .$$

Proof. It is clear that $Q_{K_2,\omega} \subset Q_{K_1,\omega}$. Suppose that

$$Q_{K_2,\omega} = Q_{K_1,\omega}.$$

By the open mapping theorem (see [8]), we know that the identity map from one of these spaces into the other one is continuous. Thus there exists a constant C such that

$$\|f\|_{K_2,\omega} \leq C\|f\|_{K_1,\omega} .$$

Since $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$, then there exists $r_0 \in (0,1)$ such that $K_1(r) \leq (2C)^{-1}K_2(r)$ for $0 < r \leq r_0$. Choose $t_0 = e^{-r_0}$ and we deduce that if $f \in Q_{K_2,\omega}$, then

$$\sup_{a\in\Delta}\int_{\Delta} |f'(z)|^p (1-|z|)^p \frac{K_2(g(z,a))}{\omega^p(1-|z|)} d\sigma_z \leq C \sup_{a\in\Delta}\int_{\Delta(a,t_0)} |f'(z)|^p (1-|z|)^p \frac{K_1(g(z,a))}{\omega^p(1-|z|)} d\sigma_z$$
$$+\frac{1}{2}\sup_{a\in\Delta}\int_{\Delta} |f'(z)|^p (1-|z|)^p \frac{K_2(g(z,a))}{\omega^p(1-|z|)} d\sigma_z .$$

Therefore,

$$\sup_{a\in\Delta}\int_{\Delta} |f'(z)|^p (1-|z|)^p \frac{K_2(g(z,a))}{\omega^p(1-|z|)} d\sigma_z \leq 2C \sup_{a\in\Delta}\int_{\Delta(a,t_0)} |f'(z)|^p (1-|z|)^p \frac{K_1(g(z,a))}{\omega^p(1-|z|)} d\sigma_z .$$

By Lemma 4.1 and for $f \in Q_{K_2,\omega}$, there exists a constant C_1 such that

$$\sup_{a\in\Delta}\int_{\Delta} |f'(z)|^p (1-|z|)^p \frac{K_2(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z \leq C_1 \|f\|^p_{\mathscr{B}_\omega^{\frac{p+2}{p}}}. \tag{4.3}$$

If $g \in \mathscr{B}_\omega^{\frac{p+2}{p}}$ and $g_r(z) = g(rz)$, $0 < r < 1$, then $\|g_r\|_{\mathscr{B}_\omega^{\frac{p+2}{p}}} \leq \|g\|_{\mathscr{B}_\omega^{\frac{p+2}{p}}}$. Since $g_r \in Q_{K_2,\omega}$, $0 < r < 1$, we can choose $f = g_r$ in the inequality (4.3). Using Fatou's lemma (see [10]), we deduce that

$$\sup_{a\in\Delta}\int_{\Delta} |g'(z)|^p (1-|z|)^p \frac{K_2(g(z,a))}{\omega^p(1-|z|)} d\sigma_z < C_1 \|g\|^p_{\mathscr{B}_\omega^{\frac{p+2}{p}}} .$$

We have proved that $g \in Q_{K_2,\omega}$. It means that $Q_{K_2,\omega} = \mathscr{B}_\omega^{\frac{p+2}{p}}$. It follows from Theorem 2.1 that the integral in (2.3) with $K = K_2$ must be convergent, a contradiction. We obtain that

$$Q_{K_2,\omega} \subsetneqq Q_{K_1,\omega}.$$

Now, the proof of Theorem 4.2 is completed.

ACKNOWLEDGEMENTS. The authors would like to thank the referee for his careful reading of the manuscript and for the useful suggestions.

References

[1] A. El-Sayed Ahmed and M. A. Bakhit, Composition operators on some holomorphic Banach function spaces, Mathematica Scandinavica, Vol 104 (2)(2009),275-295.

[2] R. Aulaskari and P. Lappan, Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, Complex Analysis and its Applications (Eds Y. Chung-Chun et al.), Pitman Research Notes in Mathematics 305, Longman (1994), 136-146.

[3] R. Aulaskari, P. Lappan, Peter and R. Zhao, On harmonic normal and $Q_p^{\#}$ functions, Ill. J. Math. 45, No.2(2001), 423-440.

[4] K. M. Dyakonov, Weighted Bloch spaces, H^p, and BMOA, J. Lond. Math. Soc. II. Ser. 65 No. 2(2002), 411-417.

[5] M. Essén and H. Wulan, On analytic and meromorphic functions and spaces of Q_K type, Illinois J. Math. 46(2002), 1233-1258.

[6] M. Essén, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant Q_K spaces, J. Funct. Anal. 230, No. 1 (2006), 78-115.

[7] P. Galanopoulos, On $\mathscr{B}_{\log}$ to $\mathscr{Q}^p_{\log}$ pullbacks, J. Math. Anal. Appl. Vol 337(2008), 712-725.

[8] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons (1978).

[9] S. Li, Songxiao and H. Wulan, Composition operators on Q_k spaces, J. Math. Anal. Appl. 327(2)(2007), 948-958.

[10] H. L. Royden, Real analysis. 2nd ed. New York: Macmillan, 1968.

[11] K. Stroethoff, Besov-type characterisations for the Bloch space, Bull. Austral. Math. Soc. 39(1989), 405-420.

[12] H. Wulan and P. Wu, Characterizations of Q_T spaces, J. Math. Anal. Appl. 254 No. 2(2001), 484-497.

[13] H. Wulan and Y. Zhang, Hadamard products and Q_K spaces, J. Math. Anal. Appl. 337(2008), 1142-1150.

[14] H. Wulan and K. Zhu, Q_K type spaces of analytic functions, J. Funct. spaces Appl. 4(2006), 73-84.

[15] H. Wulan and J. Zhou, The higher order derivatives of Q_K type spaces, J. Math. Anal. Appl. 332, No. 2(2007), 1216-1228.

[16] H. Wulan and K. Zhu, Derivative-free characterizations of Q_K spaces, J. Aust. Math. Soc. 82, No. 2(2007), 283-295.

[17] J. Xiao, Holomorphic Q Classes, Springer LNM 1767, Berlin, 2001.

[18] R. Zhao, On a general family of function spaces, Ann. Acad. Sci. Fenn. Math. Diss. 105, 1996.