EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 1, 2012, 30-44
ISSN 1307-5543 – www.ejpam.com

SPECIAL ISSUE FOR THE INTERNATIONAL CONFERENCE ON APPLIED ANALYSIS AND ALGEBRA
29 JUNE - 02 JULY 2011, ISTANBUL TURKEY

Practical Stability of Impulsive Differential Equations with "Supremum" by Integral Inequalities

Snezhana Hristova [1*], Kremena Stefanova [2]

[1] *Department of Applied Mathematics and Modeling, Faculty of Mathematics and Informatics, Plovdiv University, Plovdiv, Bulgaria*
[2] *Department of Computer Technologies, Faculty of Mathematics and Informatics, Plovdiv University, Plovdiv, Bulgaria*

Abstract. The paper deals with some stability properties of the solutions of impulsive differential equations with "supremum". Initially several integro-summation inequalities for piecewise continuous functions are solved. The main characteristic of the considered inequalities is the presence of the supremum of the unknown function over a past time interval. These inequalities are generalizations of Bihari's integral inequality. They are base of studying the practical stability as well as the uniform practical stability of the solutions of nonlinear impulsive differential equations with "supremum".

2000 Mathematics Subject Classifications: 26D99, 34A37, 34D99

Key Words and Phrases: Integral Inequality, Supremum, Practical Stability, Impulses

---

1. Introduction

In the last few decades great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity. Such kind of real world problems are adequately modeled by differential equations with "maxima" [16]. In connection with many possible applications it is absolutely necessary to be developed qualitative theory of differential equations with "maxima" (see the monograph [3] and papers [2, 4, 5, 6, 10, 11]). One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison results, perturbations, boundedness, and stability of solutions of differential and integral equations is the method of integral

*Corresponding author.

Email addresses: `snehri@uni-plovdiv.bg` (S. Hristova), `kstefanova@uni-plovdiv.bg` (K. Stefanova)

inequalities. Various types of integral inequalities are solved in the papers [1, 7, 9, 13, 15, 17, 20, 21, 22]. The involvement of maximum function in the equation requires application of a new type of integral inequalities. Additionally, if the unknown function is piecewise continuous then so called integro-summation inequalities with supremum have to be applied.

The purpose of the paper is studying some stability properties of the solutions of impulsive differential equations with "supremum". The main apparatus of investigation are integral inequalities which contain the supremum of the unknown scalar piecewise continuous function over a past time interval. Some nonlinear inequalities are solved and applied to investigate some properties of the solutions of the considered equation.

2. Mathematical Model

Let $\{t_i\}_1^\infty$ be a given sequence of points such that $t_i < t_{i+1}$, $\lim_{i\to\infty} t_i = \infty$.

Let the points t_0, T be fixed, $0 \le t_0 < T \le \infty$, and the following condition be satisfied:

H1 The functions $\sigma, \tau \in C^1([t_0, T), \mathbb{R}_+)$ are nondecreasing, $\tau(t) \le t$ for $t \in [t_0, T)$ and there exists a nonnegative constant h such that the inequalities $0 \le \tau(t) - \sigma(t) \le h$ hold for $t \in [t_0, T)$.

Denote by $\mathbb{Z}(t_0, T)$ the set of all natural numbers k such that $t_k \in (t_0, T)$. Consider the following impulsive differential equation with "supremum"

$$x' = f\Big(t, x(t), \sup_{s\in[\sigma(t),\tau(t)]} x(s)\Big), \text{ for } t \in [t_0, T),\ t \ne t_i, \tag{1}$$

$$\Delta x\Big|_{t=t_i} = I_i\big(x(t_i)\big), \text{ for } i \in \mathbb{Z}(t_0, T), \tag{2}$$

with initial condition

$$x(t) = \phi(t),\ t \in [\tau(t_0) - h, t_0] \tag{3}$$

where $x \in \mathbb{R}$, $\Delta x\Big|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$ for $i \in \mathbb{Z}(t_0, T)$.

Let $PC(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}$, be the set of all functions $u : \Omega \to \mathbb{R}$ which are piecewise continuous in Ω, i.e. there exist limits $\lim_{t\downarrow t_k} u(t) = u(t_k + 0) < \infty$ and $\lim_{t\uparrow t_k} u(t) = u(t_k - 0) = u(t_k) < \infty$, $t_k \in \Omega$.

Denote by $x(t; t_0, \phi)$ the solution of the initial value problem (1)–(3) and $|\phi|_0 = \max_{s\in[\tau(t_0)-h, t_0]} |\phi(s)|$.

Let the following conditions be satisfied:

H2 The function $f \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f(t, 0, 0) = 0$ and the inequality

$$|f(t, x, y)| \le A(t)|x|^p + B(t)|y|^p \text{ for } x, y \in \mathbb{R}$$

holds, where the functions A, $B \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $p = const > 0$.

H3 The functions $I_i : \mathbb{R} \to \mathbb{R}$, $I_i(0) = 0$ and the inequalities

$$|I_i(x)| \le \beta_i |x|^p \text{ for } x \in \mathbb{R},$$

hold, where $\beta_i = const > 0$ for $i \in \mathbb{Z}(0, \infty)$.

H4 For any point $t_0 \in \mathbb{R}_+$ and any initial function $\phi \in C([\tau(t_0) - h, t_0], \mathbb{R})$ the initial value problem (1)–(3) has a solution $x(t; t_0, \phi) \in PC([\tau(t_0) - h, \infty), \mathbb{R})$.

The solution $x(t) = x(t; t_0, \phi)$ of the initial value problem (1)–(3) satisfies the following integral equation

$$x(t) = \phi(t_0) + \sum_{t_0 < t_i < t} I_i\big(x(t_i)\big) + \int_{t_0}^{t} f\Big(s, x(s), \sup_{\xi \in [\sigma(s), \tau(s)]} x(\xi)\Big) ds, \; t \in [t_0, T). \quad (4)$$

3. Integro-summation Inequalities with "Supremum"

We will solve some nonlinear integro-summation inequalities which contain the supremum of the unknown scalar nonnegative piecewise continuous function over a past time interval.

In the proof of the main results we will use the following results:

Lemma 1 ([9, Corollary 1, p.16]). *Let the following conditions be satisfied:*

1. *The function $v(t) \in PC([0, \infty), [0, \infty))$.*
2. *The function $u(t) \in PC([0, \infty), [0, \infty))$ satisfies the inequality*

$$u(t) \le c + \sum_{0 < t_i < t} \beta_i u(t_i) + \int_0^t v(s) u(s) ds,$$

where $c \ge 0$, $\beta_i \ge 0$, $(i = 1, 2, \dots)$ are constants.

Then for $t \ge 0$ the inequality

$$u(t) \le c \Big(\prod_{0 < t_i < t} (1 + \beta_i) \Big) \exp \Big(\int_0^t v(s) ds \Big)$$

holds.

Lemma 2 ([8, Corollary 2.2.]). *Let the nonnegative piecewise continuous function $V(t)$ at $t \ge t_0 \ge 0$, with discontinuities of the first kind in the points t_k $(t_0 < t_1 < t_2 < \dots < \lim_{i \to \infty} t_i = \infty)$ satisfies the inequality*

$$V(t) \le c + \sum_{t_0 < t_i < t} a_i V^m(t_i - 0) + \int_{t_0}^{t} q(s) V^m(s) ds,$$

where $q(s) \in C([t_0, \infty), \mathbb{R}_+)$ and m is a positive constant. Then for $t \ge t_0$ the following estimates hold:

(i) for $m \in (0,1)$

$$V(t) \leq \prod_{t_0<t_i<t} (1+a_i c^{m-1}) \Big[c^{1-m} + (1-m) \int_{t_0}^{t} q(\tau) d\tau \Big]^{\frac{1}{1-m}}; \tag{5}$$

(ii) for $m > 1$

$$V(t) \leq c \prod_{t_0<t_i<t} (1+a_i m c^{m-1}) \times \Big[1-(m-1) \Big[c \prod_{t_0<t_i<t} (1+a_i m c^{m-1}) \Big]^{m-1} \int_{t_0}^{t} q(\tau) d\tau \Big]^{-\frac{1}{m-1}}, \tag{6}$$

where

$$\int_{t_0}^{t} q(\tau) d\tau \leq \frac{c^{1-m}}{m}, \text{ and } \prod_{t_0<t_i<t} (1+a_i m c^{m-1}) < \Big(\frac{m}{m-1} \Big)^{\frac{1}{m-1}}.$$

In the case when the supremum of the unknown nonnegative scalar piecewise continuous function is involved in the integrals we obtain the following result:

Theorem 1. *Let the following conditions be fulfilled:*

1. *The function $\alpha \in C^1([t_0, T), \mathbb{R}_+)$ is a nondecreasing function and $\alpha(t) \leq t$ for $t \in [t_0, T)$.*
2. *The functions $a, b \in C([\alpha(t_0), T), \mathbb{R}_+)$.*
3. *The function $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R}_+)$, where $h = const \geq 0$.*
4. *The function $u \in PC([\alpha(t_0) - h, T), \mathbb{R}_+)$ satisfies the following inequalities*

$$u(t) \leq \gamma + \sum_{t_0<t_i<t} \beta_i u^p(t_i) + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\big(u(s)\big)^p + b(s) \Big(\sup_{\xi \in [s-h,s]} u(\xi) \Big)^p \Big] ds \text{ for } t \in [t_0, T), \tag{7}$$

$$u(t) \leq \phi(t) \text{ for } t \in [\alpha(t_0) - h, t_0], \tag{8}$$

where the constants $p > 0$, $\beta_i \geq 0$ for $i \in \mathbb{Z}(t_0, T)$ and $\gamma \leq \max_{s \in [\alpha(t_0)-h, t_0]} \phi(s) = M$.

Then for $t \in [t_0, T)$ the following inequalities are fulfilled:

(i) for $p = 1$

$$u(t) \leq M \Big(\prod_{t_0<t_i<t} (1+\beta_i) \Big) \exp\Big(Q(t) \Big), \tag{9}$$

where

$$Q(t) = \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s) \Big] ds; \tag{10}$$

(ii) for $p \in (0,1)$

$$u(t) \leq \Big(\prod_{t_0<t_i<t} \big(1+\beta_i M^{p-1}\big) \Big) \Big[M^{1-p} + (1-p)Q(t) \Big]^{\frac{1}{1-p}}; \tag{11}$$

(iii) for $p > 1$

$$u(t) \leq M \Big(\prod_{t_0<t_i<t} \big(1+p\beta_i M^{p-1}\big) \Big) \times \Big\{ 1-(p-1) \Big[M \Big(\prod_{t_0<t_i<t} \big(1+p\beta_i M^{p-1}\big) \Big) \Big]^{p-1} Q(t) \Big\}^{\frac{1}{p-1}}, \tag{12}$$

where

$$Q(t) \leq \frac{M^{1-p}}{p}, \tag{13}$$

$$\prod_{t_0<t_i<t} \big(1+p\beta_i M^{p-1}\big) < \Big(\frac{p}{p-1}\Big)^{\frac{1}{p-1}}. \tag{14}$$

Proof. Define a function $z : [\alpha(t_0) - h, T) \to [M, \infty)$ by the equalities

$$z(t) = \begin{cases} M + \sum_{t_0<t_i<t} \beta_i u^p(t_i) & \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\big(u(s)\big)^p + b(s)\Big(\sup_{\xi\in[s-h,s]} u(\xi) \Big)^p \Big] ds & t \in [t_0, T) \\ M & t \in [\alpha(t_0) - h, t_0] \end{cases}$$

From the definition of the function $z(t)$ and the choice of the constant M it follows the validity of the inequalities

$$u(t) \leq z(t), \ t \in [\alpha(t_0) - h, T) \tag{15}$$

$$\sup_{\xi\in[s-h,s]} u(\xi) \leq \sup_{\xi\in[s-h,s]} z(\xi) = z(s), \ s \in [\alpha(t_0), T). \tag{16}$$

Then from (7), (15), (16) and the definition of the function $z(t)$ we get

$$z(t) \leq M + \sum_{t_0<t_i<t} \beta_i z^p(t_i) + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s) \Big] \big(z(s)\big)^p ds, \ t \in [t_0, T). \tag{17}$$

Consider the following three cases:

Case (i): Let $p = 1$. Then inequality (17) reduces to the following inequality

$$z(t) \leq M + \sum_{t_0<t_i<t} \beta_i z(t_i) + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s) \Big] z(s) ds, \ t \in [t_0, T). \tag{18}$$

From inequality (18) according to Lemma 1 it follows

$$z(t) \leq M \Big(\prod_{t_0<t_i<t} \big(1+\beta_i\big) \Big) \exp\Big(\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s) + b(s) \Big] \Big), \ t \in [t_0, T). \tag{19}$$

Inequalities (19) and (15) imply the validity of the required inequality (9).

Case (ii): Let $p \in (0,1)$. From inequality (17) according to Lemma 2 we obtain for $t \in [t_0, T)$

$$z(t) \le \left(\prod_{t_0<t_i<t} \left(1+\beta_i M^{p-1}\right) \right) \left[M^{1-p} + (1-p)Q(t) \right]^{\frac{1}{1-p}}, \tag{20}$$

where the function $Q(t)$ is defined by equality (10).

Substitute the bound (20) for the function $z(t)$ into the right hand-side of (15) and get the required inequality (11).

Case (iii): Let $p > 1$. As in the case (ii) from inequality (17) according to Lemma 2 we obtain for $t \in [t_0, T)$

$$\begin{aligned} z(t) \le M &\left(\prod_{t_0<t_i<t} \left(1+p\beta_i M^{p-1}\right) \right) \times \\ &\times \left\{ 1-(p-1)\left[M\left(\prod_{t_0<t_i<t} \left(1+p\beta_i M^{p-1}\right) \right) \right]^{p-1} Q(t) \right\}^{\frac{1}{p-1}}, \end{aligned} \tag{21}$$

where the function $Q(t)$ is defined by equality (10) and inequalities (13) and (14) hold.

Substitute the bound (21) for the function $z(t)$ into the right hand-side of (15) and get the required inequality (12).

Similarly to the proof of Theorem 1 we can obtain the following result:

Theorem 2. *Let the following conditions be fulfilled:*

1. *The functions $\alpha_j \in C^1([t_0, T), \mathbb{R}_+)$ are nondecreasing and the inequalities $\alpha_j(t) \le t$ hold for $t \in [t_0, T)$, $j = 1, 2, \ldots, m$.*
2. *The functions $a_j, b_j \in C([A, T), \mathbb{R}_+)$ for $j = 1, 2, \ldots, m$, where $A = \min_{1\le j\le n} \alpha_j(t_0)$.*
3. *The function $\phi \in C([A-h, t_0], \mathbb{R}_+)$, where $h = const \ge 0$.*
4. *The function $u \in PC([A-h, T), \mathbb{R}_+)$ satisfies the following inequalities*

$$u(t) \le \gamma + \sum_{t_0<t_i<t} \beta_i u^p(t_i) + \sum_{j=1}^{m} \int_{\alpha_j(t_0)}^{\alpha_j(t)} \left[a_j(s)\big(u(s)\big)^p + b_j(s)\Big(\sup_{\xi\in[s-h,s]} u(\xi) \Big)^p \right] ds, \ t \in [t_0, T), \tag{22}$$

$$u(t) \le \phi(t), \ t \in [A-h, t_0], \tag{23}$$

where the constants $p > 0$, $\beta_i \ge 0$ for $i \in \mathbb{Z}(t_0, T)$ and $\gamma \le \max_{s\in[A-h,t_0]} \phi(s) = \tilde{M}$.

Then for $t \in [t_0, T)$ the following inequalities are fulfilled:

(i) for $p = 1$

$$u(t) \le \tilde{M}\Big(\prod_{t_0<t_i<t}(1+\beta_i)\Big)\exp\Big(\tilde{Q}(t)\Big), \tag{24}$$

where

$$\tilde{Q}(t) = \sum_{j=1}^{m}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\Big[a_j(s)+b_j(s)\Big]ds; \tag{25}$$

(ii) for $p \in (0,1)$

$$u(t) \le \Big(\prod_{t_0<t_i<t}(1+\beta_i\tilde{M}^{p-1})\Big)\Big[\tilde{M}^{1-p}+(1-p)\tilde{Q}(t)\Big]^{\frac{1}{1-p}}; \tag{26}$$

(iii) for $p > 1$

$$u(t) \le \tilde{M}\Big(\prod_{t_0<t_i<t}(1+p\beta_i\tilde{M}^{p-1})\times\Big\{1-(p-1)\Big[\tilde{M}\Big(\prod_{t_0<t_i<t}(1+p\beta_i\tilde{M}^{p-1})\Big)\Big]^{p-1}\tilde{Q}(t)\Big\}^{\frac{1}{p-1}}, \tag{27}$$

where

$$\tilde{Q}(t) \le \frac{\tilde{M}^{1-p}}{p}, \tag{28}$$

$$\prod_{t_0<t_i<t}(1+p\beta_i\tilde{M}^{p-1}) < \Big(\frac{p}{p-1}\Big)^{\frac{1}{p-1}}. \tag{29}$$

4. Practical Stability

Now we will use the solved above inequalities to investigate some stability properties of the solutions of impulsive differential equation with "supremum" (1), (2). Note that stability properties of solutions of various types of differential equations are very intensively studied because of its applications to many models of real world problems [14, 18, 19]. The main object of the paper is practical stability. We will extend the concept of boundedness as well as practical stability to the considered nonlinear system of impulsive differential equation with "supremum" (1), (2), based on the definitions for ordinary differential equations given in [14].

Definition 1. *We will say that the solution $x(t;t_0,\phi)$ of the initial value problem (1), (2), (3) is* bounded *if for any number $\alpha > 0$ there exists $\beta = \beta(\alpha, t_0) > 0$ such that inequality $|\phi|_0 < \alpha$ implies $|x(t;t_0,\phi)| < \beta$, $t \ge t_0$, where $\phi \in C([\tau(t_0)-h, t_0], \mathbb{R})$.*

Definition 2. *We will say that the solutions of the initial value problem (1), (2), (3) are* uniformly bounded *if for any number $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that inequality $|\phi|_0 < \alpha$ implies $|x(t;t_0,\phi)| < \beta$, $t \ge t_0$ for all $t_0 \in \mathbb{R}_+$, where $\phi \in C([\tau(t_0)-h, t_0], \mathbb{R})$.*

Let the constants $\lambda, \Lambda : 0 < \lambda < \Lambda$ be given.

Definition 3. *We will say that the system of impulsive differential equation with "supremum" (1), (2) is*

- practically stable *with respect to* (λ, Λ) *if the inequality* $|\phi|_0 < \lambda$ *implies* $|x(t; t_0, \phi)| < \Lambda$, $t \geq t_0$ *for some* $t_0 \in \mathbb{R}_+$, *where* $\phi \in C([\tau(t_0) - h, t_0], \mathbb{R})$;
- uniformly practically stable *with respect to* (λ, Λ) *if the inequality* $|\phi|_0 < \lambda$ *implies* $|x(t; t_0, \phi)| < \Lambda$, $t \geq t_0$ *for all* $t_0 \in \mathbb{R}_+$, *where* $\phi \in C([\tau(t_0) - h, t_0], \mathbb{R})$.

Now will obtain some stability properties of the solutions of the impulsive differential equation with "supremum" (1), (2). We will consider the case when the right part of the equations satisfy the conditions H2 and H3 for different values of the power p.

Theorem 3. *Let the following conditions be fulfilled:*

1. *The conditions H1–H4 are satisfied for* $p = 1$.
2. *For any* $t_0 \in \mathbb{R}_+$ *there exist* $\lim_{t\to\infty} \Psi(t_0, t) = \eta_1(t_0)$ *and* $\lim_{t\to\infty} \Phi(t_0, t) = \eta_2(t_0)$ *where the functions* $\Psi(t_0, t)$ *and* $\Phi(t_0, t)$ *are defined by the equalities*

$$\Psi(t_0, t) = \prod_{t_0 < t_i < t} \left(1 + \beta_i\right), \tag{30}$$

$$\Phi(t_0, t) = \int_{t_0}^{t} \Big[A(s) + B(s)\Big] ds, \tag{31}$$

and the functions $\eta_1, \eta_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$.

Then:

- *(i) any solution of the impulsive differential equation with "supremum" (1), (2) is bounded;*
- *(ii) if the functions* $\eta_1(t)$ *and* $\eta_2(t)$ *are bounded, i.e. there exist constants* $\mu_1, \mu_2 > 0$ *such that* $\eta_k(t) \leq \mu_k$, $(k = 1, 2)$ *for* $t \in \mathbb{R}_+$, *then all solutions of the impulsive differential equation with "supremum" (1), (2) are uniformly bounded;*
- *(iii) if for the given constants* $0 < \lambda < \Lambda$ *there exists a point* $t_0 \in \mathbb{R}_+$ *such that*

$$\lambda \eta_1(t_0) e^{\eta_2(t_0)} < \Lambda, \tag{32}$$

then the trivial solution of the impulsive differential equation with "supremum" (1), (2) is practically stable with respect to (λ, Λ);

- *(iv) if the functions* $\eta_1(t)$ *and* $\eta_2(t)$ *are bounded, i.e. there exist constants* $\mu_1, \mu_2 > 0$ *such that* $\eta_k(t) \leq \mu_k$, $(k = 1, 2)$ *for* $t \in \mathbb{R}_+$, *and*

$$\lambda \mu_1 e^{\mu_2} < \Lambda, \tag{33}$$

then the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).

Proof. According to conditions H1–H4 from integral equation (4) we get

$$
\begin{aligned}
|x(t)| \le & |\phi|_0 + \sum_{t_0<t_i<t} \left| I_i\big(x(t_i)\big)\right| + \int_{t_0}^{t} \left[A(s)|x(s)|^p + B(s)\Big| \sup_{\xi\in[\sigma(s),\tau(s)]} x(\xi)\Big|^p \right] ds \\
\le & |\phi|_0 + \sum_{t_0<t_i<t} \beta_i |x(t_i)|^p + \int_{t_0}^{t} A(s)|x(s)|^p ds \\
& + \int_{\tau(t_0)}^{\tau(t)} B(\tau^{-1}(\eta))(\tau^{-1}(\eta))' \sup_{\xi\in[\eta-h,\eta]} \Big|x(\xi)\Big|^p d\eta,\ t\in[t_0,T)
\end{aligned}
\tag{34}
$$

$$
|x(t)| \le |\phi|_0,\ t\in[\tau(t_0)-h,t_0], \tag{35}
$$

where $x(t) = x(t;t_0,\phi)$.

From inequalities (34), (35) according to Theorem 2 for $m=2$, $\alpha_1(t)\equiv t$, $\alpha_2(t)\equiv\tau(t)$, $u(t)=|x(t)|$, $\tilde{M}=|\phi|_0$, $a_1(t)\equiv A(t)$, $a_2(t)\equiv 0$, $b_1(t)\equiv 0$, $b_2(t)\equiv B(\tau^{-1}(\eta))(\tau^{-1}(\eta))'$ for $t\in[\tau(t_0),T)$, $p=1$ and $t\in[t_0,T)$ we obtain

$$
\begin{aligned}
|x(t)| \le & |\phi|_0 \Big(\prod_{t_0<t_i<t} \big(1+\beta_i\big) \Big) \exp\Big(\int_{t_0}^{t} \Big[A(s)+B(s)\Big] ds \Big) \\
= & |\phi|_0 \Psi(t_0,t) e^{\Phi(t_0,t)}.
\end{aligned}
\tag{36}
$$

Since the functions $\Psi(t_0,t)$ and $\Phi(t_0,t)$ are nondecreasing in their second arguments, from inequality (36) and condition 2 of Theorem 3 it follows

$$
|x(t)| \le |\phi|_0 \eta_1(t_0) e^{\eta_2(t_0)}. \tag{37}
$$

The inequality (37) proves the claim of Theorem 3.

Remark 1. *If the conditions 1 and 2 of Theorem 3 are satisfied, then the trivial solution of the impulsive differential equation with "supremum" (1), (2) is stable in the sense of Lyapunov.*

Theorem 4. *Let the following conditions be fulfilled:*

1. *The conditions H1–H4 are satisfied for $p\in(0,1)$.*
2. *For any $t_0\in\mathbb{R}_+$ there exist $\lim_{t\to\infty}\Psi(t_0,t)=\eta_1(t_0)$ and $\lim_{t\to\infty}\Phi(t_0,t)=\eta_2(t_0)$ where the functions $\Psi(t_0,t)$ and $\Phi(t_0,t)$ are defined by (30) and (31), correspondingly, and the functions $\eta_1,\eta_2\in C(\mathbb{R}_+,\mathbb{R}_+)$.*

Then for the given constants $0<\lambda<\Lambda$ such that $\lambda\in(0,1)$ and

(i) *there exists a point $t_0\in\mathbb{R}_+$ such that*

$$
\eta_1(t_0)\Big[\lambda^{1-p}+(1-p)\eta_2(t_0)\Big]^{\frac{1}{1-p}} < \Lambda,
$$

then the impulsive differential equation with "supremum" (1), (2) is practically stable with respect to (λ,Λ);

(ii) if the functions $\eta_1(t)$ and $\eta_2(t)$ are bounded, i.e. there exist constants $\mu_1, \mu_2 > 0$ such that $\eta_k(t) \le \mu_k$, $(k = 1, 2)$ for $t \in \mathbb{R}_+$, and

$$\mu_1 \Big[\lambda^{1-p} + (1-p)\mu_2 \Big]^{\frac{1}{1-p}} < \Lambda,$$

then the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).

Proof. From inequalities (34), (35) according to Theorem 2 for $m = 2$, $\alpha_1(t) \equiv t$, $\alpha_2(t) \equiv \tau(t)$, $u(t) = |x(t)|$, $\tilde{M} = |\phi|_0$, $a_1(t) \equiv A(t)$, $a_2(t) \equiv 0$, $b_1(t) \equiv 0$, $b_2(t) \equiv B(\tau^{-1}(t))(\tau^{-1}(t))^{'}$ for $t \in [\tau(t_0), T)$, $p \in (0, 1)$ and $t \in [t_0, T)$ we obtain

$$|x(t)| \le \Bigg(\prod_{t_0 < t_i < t} \left(1 + \beta_i |\phi|_0^{p-1}\right) \Bigg) \times \Bigg\{ |\phi|_0^{1-p} + (1-p) \int_{t_0}^{t} \Big[A(s) + B(s) \Big] ds \Bigg\}^{\frac{1}{1-p}}. \tag{38}$$

Let $|\phi|_0 < \lambda$. Then from $1 + \beta_i |\phi|_0^{p-1} \le 1 + \beta_i \lambda^{p-1} \le 1 + \beta_i$, the monotonic property of functions $\Psi(t_0, t)$ and $\Phi(t_0, t)$, condition 2 of Theorem 4 and inequality (38) we get

$$|x(t)| \le \Psi(t_0, t) \Big[\lambda^{1-p} + (1-p)\Phi(t_0, t) \Big]^{\frac{1}{1-p}}. \tag{39}$$

From inequality (39) according to condition 2 of the theorem it follows

$$|x(t)| \le \eta_1(t_0) \Big[\lambda^{1-p} + (1-p)\eta_2(t_0) \Big]^{\frac{1}{1-p}}. \tag{40}$$

The inequality (40) proves the claim of Theorem 4.

5. Applications

Now we will apply some of the obtained sufficient conditions for special types of impulsive differential equations with "supremum" (1), (2).

Theorem 5. *Let the following conditions be fulfilled:*

1. *The conditions H1 and H4 are satisfied.*
2. *The function $f \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f(t, 0, 0) = 0$ and*

$$|f(t, x, y)| \le e^{-t} \Big[|x| + |y| \Big] \text{ for } x, y \in \mathbb{R}.$$

3. *The functions $I_i : \mathbb{R} \to \mathbb{R}$, $I_i(0) = 0$ and*

$$|I_i(x)| \le \frac{1}{4i^2 - 1} |x| \text{ for } x \in \mathbb{R}, i \in \mathbb{Z}(t_0, T).$$

Then:

(i) all solutions of the system of impulsive differential equation (1), (2) are uniformly bounded;

(ii) if, additionally, the given positive constants λ and Λ are such that $\lambda\pi e^2 < 2\Lambda$, then the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).

Proof. According to the notations in Theorem 3 we have

$$\Phi(t_0, t) = \int_{t_0}^{t} \left[e^{-s} + e^{-s} \right] ds = 2(e^{-t_0} - e^{-t})$$

and

$$\Psi(t_0, t) = \prod_{t_0 < t_i < t} \left(1 + \frac{1}{4i^2 - 1}\right) \leq \prod_{i=1}^{\infty} \left(1 + \frac{1}{4i^2 - 1}\right).$$

Using the convergence of Wallis product $\prod\limits_{n=1}^{\infty} \left(1 + \frac{1}{4n^2-1}\right) = \frac{\pi}{2}$, it follows $\Psi(t_0, t) \leq \frac{\pi}{2}$ for any $t, t_0 \in \mathbb{R}_+$. Also $\lim\limits_{t\to\infty} \Phi(t_0, t) = 2e^{-t_0} \leq 2$. Therefore the conditions of Theorem 3 are satisfied for $\mu_1 = \frac{\pi}{2}$ and $\mu_2 = 2$.

According to claim (ii) of Theorem 3 all solutions of the system (1), (2) are uniformly bounded, i.e. for any number $\alpha > 0$ the inequality $|\phi|_0 < \alpha$ implies $|x(t; t_0, \phi)| < e^2\alpha\frac{\pi}{2}$ for all $t_0 \in \mathbb{R}_+$.

If, additionally, the inequality $\lambda\pi e^2 < 2\Lambda$ holds, then according to claim (iv) of Theorem 3 the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).

Example 1. *Consider the initial value problem for the scalar impulsive differential equation*

$$\begin{cases} x' = e^{-t} x & \text{for } t \neq n, \\ x(n+0) - x(n-0) = \frac{1}{4n^2-1} x(n-0) & \text{for } n \in \mathbb{Z}(t_0, \infty), \\ x(t_0) = x_0 \end{cases} \tag{41}$$

where $x \in \mathbb{R}$ and $t_0 \in \mathbb{R}_+$.

The solution of the initial value problem (41) is

$$x(t; t_0, x_0) = \Big(\prod_{i=j}^{k} \frac{4i^2}{4i^2 - 1}\Big) x_0 e^{e^{-t_0} - e^{-t}} \quad \text{for } t \in (k, k+1],$$

where j is a natural number such that $j - 1 \leq t_0 < j$ and $k = j, j = 1, j+2, \ldots$. It is easy to see the solution is uniformly bounded and stable.

Now we will perturb the equation (41) by the maximum function of the unknown function, i.e. consider the following impulsive differential equation with "supremum"

$$\begin{cases} x' = e^{-t}\left(x + \sup_{s\in[t-h,t]} x(s)\right) & \text{for } t \geq t_0,\ t \neq n, \\ x(n+0) - x(n-0) = \frac{1}{4n^2-1} x(n-0) & \text{for } n \in \mathbb{Z}(t_0,\infty), \\ x(t) = \varphi(t) & \text{for } t \in [t_0 - h, t_0], \end{cases} \tag{42}$$

where $x \in \mathbb{R}$ and $h > 0$ is a given constant.

The initial value problem (42) is not possible to be solved in analytical form, but according to Theorem 5 its solutions are uniformly bounded. i.e. the perturbation as well as impulsive conditions could save the property boundedness.

Also, if the positive constants λ and Λ satisfy $\lambda\pi e^2 < 2\Lambda$, then the solution of the impulsive differential equation with "supremum" (42) is uniformly practically stable with respect to (λ, Λ).

Theorem 6. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 5 are satisfied.*
2. *The functions $I_i : \mathbb{R} \to \mathbb{R}$, $I_i(0) = 0$ and*

$$|I_i(x)| \leq \frac{1}{2^i}|x| \text{ for } x \in \mathbb{R},\ i \in \mathbb{Z}(t_0, T).$$

Then:

- (i) *all solutions of the system of impulsive differential equation (1), (2) are uniformly bounded;*
- (ii) *if, additionally, the given positive constants λ and Λ are such that $\lambda e^3 < \Lambda$, then the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).*

Proof. The proof of the claim follows by the fact that

$$\prod_{i=1}^{\infty}\left(1 + \frac{1}{2^i}\right) \leq e^{\sum_{i=1}^{\infty}\frac{1}{2^i}} = e$$

and Theorem 3.

Theorem 7. *Let the following conditions be fulfilled:*

1. *The conditions H1 and H4 are satisfied.*
2. *The function $f \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f(t,0,0) = 0$ and*

$$|f(t,x,y)| \leq e^{-t}\left[|x|^p + |y|^p\right] \text{ for } x, y \in \mathbb{R},$$

where the constant $p \in (0,1)$.

3. *The functions* $I_i : \mathbb{R} \to \mathbb{R}$, $I_i(0) = 0$ *and*

$$|I_i(x)| \leq \frac{1}{4i^2 - 1}|x|^p \text{ for } x \in \mathbb{R},\ i \in \mathbb{Z}(t_0, T).$$

Then if the given constants $\lambda \in (0,1)$ *and* $\Lambda > 0$ *are such that* $\frac{\pi}{2}\left[\lambda^{1-p} + 2(1-p)\right]^{\frac{1}{1-p}} < \Lambda$, *then the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to* (λ, Λ).

Proof. As in the proof of Theorem 5 we prove the conditions of Theorem 4 are satisfied and therefore if $\lambda \in (0,1)$ and $\frac{\pi}{2}\left[\lambda^{1-p} + 2(1-p)\right]^{\frac{1}{1-p}} < \Lambda$, then according to claim (ii) of Theorem 4 the impulsive differential equation with "supremum" (1), (2) is uniformly practically stable with respect to (λ, Λ).

Example 2. *Consider the initial value problem for the scalar impulsive differential equation with "supremum"*

$$\begin{cases} x' = e^{-t}\left(\sqrt{x} + \sqrt{\sup_{s\in[t-h,t]} x(s)}\right) & \text{for } t \geq t_0,\ t \neq n, \\ x(n+0) - x(n-0) = \frac{1}{4n^2-1}\sqrt{x(n-0)} & \text{for } n \in \mathbb{Z}(t_0, \infty), \\ x(t) = \varphi(t) & \text{for } t \in [t_0 - h, t_0], \end{cases} \tag{43}$$

where $x \in \mathbb{R}$, $h > 0$ *is a given constant and* $\varphi \in C([t_0 - h, t_0], \mathbb{R}_+)$.

The conditions of Theorem 5 are satisfied for $p = \frac{1}{2}$. Then if the positive constants $\lambda \in (0,1)$ and Λ satisfy $\frac{\pi}{2}\left[\sqrt{\lambda} + 1\right]^2 < \Lambda$, then the solution of the impulsive differential equation with "supremum" (43) is uniformly practically stable with respect to (λ, Λ).

ACKNOWLEDGEMENTS: Research was partially supported by Fund "Scientific Research" MU11FMI005/29.05.2011, Plovdiv University.

References

[1] R Agarwal, S Deng and W Zhang. Generalization of a retarded Gronwall-like inequality and its applications. *Appl. Math. Comput.* 165(3):599–612, 2005.

[2] V Angelov and D Bainov. On the functional differential equations with "maximums". *Appl. Anal.* 16:187–194, 1983.

[3] D Bainov and S Hristova. *Differential Equations with Maxima.* Chapman and Hall/CRC, USA,ISBN-10: 1439867577, 2011.

[4] D Bainov and S Hristova. Monotone-iterative techniques of Lakshmikantham for a boundary value problem for systems of differential equations with "maxima". *J. Math. Anal. Appl.* 190(2):391–401, 1995.

[5] D Bainov, V Petrov and V Proytcheva. Existence and asymptotic behavior of nonoscillatory solutions of second-order neutral differential equations with “maxima”. *J. Comput. Appl. Math.,* 83(2):237–249, 1997.

[6] D Bainov, V Petrov and V Proytcheva. Asymptotic behavior of second order neutral differential equations with “maxima”. *Tamkang J. Math.*, 26(3):267–275, 1995.

[7] W Cheung. Some new nonlinear inequalities and applications to boundary value problems. *Nonlinear Analysis*, 64:2112–2128, 2006.

[8] A Gallo and A Piccirillo. About new analogies of Gronwall–Bellman–Bihari type inequalities for discontinuous functions and estimated solutions for impulsive differential systems. *Nonlinear Analysis*, 67:1550–1559, 2007.

[9] S Hristova. *Qualitative investigations and Approximate Methods for Impulsive Equations.* Nova Science Publ., New York, 2009.

[10] S Hristova and D Bainov. Application of the monotone-iterative techniques of V. Lakshmikantham to the solution of the initial value problem for impulsive differential equations with “supremum”. *J. Math. Phys. Sci.*, 25(1):69–80, 1991.

[11] S Hristova and L Roberts. Boundedness of the solutions of differential equations with “maxima”. *Int. J. Appl. Math.* 4(2):231–240, 2000.

[12] S Hristova and K Stefanova. Some integral inequalities with maximum of the unknown functions. *Adv. Dyn. Sys. Appl.*, 6(1):57–69, 2011.

[13] Y Kim. On some new integral inequalities for functions in one and two variables. *Acta Math. Sin., English Series*, 21(2):423–434, 2005.

[14] V Lakshmikantham, S. Leela and A. Martynyuk. *Practical stability of nonlinear systems.* World Scientific, 1990.

[15] Q Ma and J Pecaric. On certain new nonlinear retarded integral inequalities for functions in two variables and their applications. *J. Korean Math. Soc.*, 45(1):121–136, 2008.

[16] E Popov. *Automatic Regulation and Control.* Moscow 1966 (in Russian).

[17] D Snow. Gronwall’s inequality for systems of partial differential equations in two independent variables. *Proc. Amer. Math. Soc.*, 33(1):46–54, 1972.

[18] C Tunc. On the instability of solutions of some fifth order nonlinear delay differential equations. *Appl. Math. Inf. Sci.*, 5(1):112-121, 2011.

[19] C Tunc. Further results on the instability of solutions of certain nonlinear vector differential equations of fifth order. *Appl. Math. Inf. Sci.*, 2(3):51–60, 2008.

[20] W Wang. A generalized retarded Gronwall-like inequality in two variables and applications to BVP. *Appl. Math. Comput.*, 191:144-154, 2007.

[21] W Wang and C Shen. On a generalized retarded integral inequality with two variables. *J. Ineq. Appl.*, Article ID 518646, 9 p, 2008.

[22] K Zheng. Some retarded nonlinear integral inequalities in two variables and applications. *J. Ineq. Pure Appl. Math.*, 9, Iss. 2, Article 57, 11 pp, 2008.