On the Vanishing Properties of Local Cohomology Modules Defined by a Pair of Ideals

M. Lotfi Parsa*, Sh. Payrovi

Department of Mathematics, I. K. International University, Qazvin, Iran

Abstract. As a generalization of the ordinary local cohomology modules, recently some authors introduced the local cohomology modules with respect to a pair of ideals. In this paper, we get some results on Artinianness, vanishing, finiteness and other properties of these modules. Let \( R \) be a commutative Noetherian ring, \( I, J \) two ideals of \( R \) and \( M \) a finitely generated \( R \)-module such that \( \dim_R M = n \). We prove that \( H^n_{I,J}(M)/JH^n_{I,J}(M) \) is \( I \)-cofinite Artinian and \( H^n_{I,J}(M)/IH^n_{I,J}(M) \) has finite length. Also we show that, if \( R \) is local with \( \dim R/I + J = 0 \) and \( \dim_R M/JM = d > 0 \), then \( H^d_{I,J}(M) \) is not finitely generated.

2000 Mathematics Subject Classifications: 13D45, 13E05, 13E10.

Key Words and Phrases: Artinian module, Cofinite module, Local cohomology, Noetherian module.

1. Introduction

Throughout this paper, \( R \) is a commutative Noetherian ring with non-zero identity, \( I, J \) are two ideals of \( R \) and \( M \) is an \( R \)-module. For notations and terminologies not given in this paper, the reader is referred to [1] and [6], if necessary.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa, in [6], introduced the local cohomology modules with respect to a pair of ideals \((I, J)\). To be more precise, let \( W(I,J) = \{ p \in \text{Spec}(R) : I^t \subseteq p + J \text{ for some positive integer } t \} \). The set of elements \( x \) of \( M \) such that \( \text{Supp}_R Rx \subseteq W(I,J) \), is said to be \((I,J)\)-torsion submodule of \( M \) and is denoted by \( \Gamma_{I,J}(M) \). It is easy to see that \( \Gamma_{I,J} \) is a covariant, \( R \)-linear functor from the category of \( R \)-modules to itself. For an integer \( i \), the local cohomology functor \( H^i_{I,J} \) with respect to \((I, J)\), is defined to be the \( i \)-th right derived functor of \( \Gamma_{I,J} \). Also \( H^i_{I,J}(M) \) is called

*Corresponding author.

Email addresses: lotfi.parsa@ikiu.ac.ir (M. Parsa), shpayrovi@ikiu.ac.ir (Sh. Payrovi)
the $i$-th local cohomology module of $M$ with respect to $(I, J)$. If $J = 0$, then $H_{I,J}^i$ coincides with the ordinary local cohomology functor $H_i^\Gamma$.

Some authors studied the properties of these extended modules; see, for example, [2, 3, 5, 7]. In this direction, we study Artinianness, vanishing and finiteness of the local cohomology modules defined by a pair of ideals. Suppose that $M$ is finitely generated with $\dim_R M = n$. It is well known that $H_i^\Gamma(M)$ is $I$-cofinite Artinian; [see 4, Proposition 5.1]. We generalize this result and prove that $H_{I,J}^n(M)/J H_{I,J}^n(M)$ is $I$-cofinite Artinian.

Let $R$ be local and $M$ finitely generated with $\dim_R M = n > 0$. It follows by Grothendieck’s Non-vanishing Theorem that $H_{I,J}^n(M)$ is not finitely generated, whenever $\dim R/I = 0$. As a generalization of this result, we show that if $\dim R/I + J = 0$ and $\dim_R M/JM = d > 0$, then $H_{I,J}^d(M)$ is not finitely generated.

2. Main Results

Recall that $R$ is a Noetherian ring, $I, J$ are two ideals of $R$ and $M$ is an $R$-module. The following result improves [5, Corollary 3.5].

**Theorem 1.** Let $M$ be finitely generated with $\dim_R M = n$. Then $H_{I,J}^n(M)/J H_{I,J}^n(M)$ is $I$-cofinite Artinian.

**Proof.** We use induction on $n$. If $n = 0$, then $M$ has finite length. Therefore $\Gamma_{I,J}(M)/J \Gamma_{I,J}(M)$ has finite length and so $\Gamma_{I,J}(M)/J \Gamma_{I,J}(M)$ is $I$-cofinite Artinian. Now suppose, inductively, that $n > 0$, and the result has been proved for all $R$-modules of dimensions smaller than $n$ satisfying the hypothesis. Since $H_{I,J}^n(M/\Gamma_{I,J}(M)) \cong H_{I,J}^n(M)$ by [6, Corollary 1.13(4)], we may assume in addition that $M$ is an $(I, J)$-torsion free $R$-module. Thus $I$ contains an element $a$ which is not zero-divisor on $M$. Since $\dim M/aM \leq n - 1$, it follows by the inductive hypothesis that $H_{I,J}^{n-1}(M/aM)/J H_{I,J}^{n-1}(M/aM)$ is $I$-cofinite Artinian. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ induces an exact sequence

$$\cdots \to H_{I,J}^{n-1}(M/aM) \to H_{I,J}^n(M) \xrightarrow{a} H_{I,J}^n(M) \to 0$$

of local cohomology modules. Now the exact sequence

$$H_{I,J}^{n-1}(M/aM)/J H_{I,J}^{n-1}(M/aM) \to H_{I,J}^n(M)/J H_{I,J}^n(M) \xrightarrow{a} H_{I,J}^n(M)/J H_{I,J}^n(M) \to 0$$

implies that $0 : H_{I,J}^n(M)/J H_{I,J}^n(M) a$ is $I$-cofinite Artinian. Therefore $H_{I,J}^n(M)/J H_{I,J}^n(M)$ is $I$-cofinite Artinian, by [4, Proposition 4.1]. This completes the inductive step. The result follows by induction.

Let $\tilde{W}(I,J)$ denote the set of ideals $a$ of $R$ such that $I^t \subseteq a + J$ for some positive integer $t$. It is easy to see that, for any $a \in \tilde{W}(I,J)$, $\Gamma_a(M)$ is a subset of $\Gamma_{I,J}(M)$.

**Theorem 2.** Let $M$ be finitely generated with $\dim_R M = n$ and $t$ a positive integer. If $H_{I,J}^i(M) = 0$, for all $i > t$, then $H_{I,J}^i(M)/aH_{I,J}^i(M) = 0$, for any $a \in \tilde{W}(I,J)$.
Proof. Let a ∈ Wię(I, J) be fixed. We prove the claim by using induction on n. If n = 0, then the claim is clear. Assume, inductively, that n > 0 and the result has been proved for any R-module of dimension less than n satisfying the hypothesis. Since H_{i,J}^i(M/Γ_{i,J}(M)) ≅ H_{i,J}^i(M) for all i > 0, by [6, Corollary 1.13(4)], we may assume in addition that Γ_{i,J}(M) = 0. We have Γ_a(M) ⊆ Γ_{i,J}(M), thus Γ_a(M) = 0, and therefore a contains an element a which is non zero-divisor on M. The exact sequence 0 → M → M → M/aM → 0 induces the following exact sequence

\[ \cdots \rightarrow H_{i,J}^i(M) \xrightarrow{a} H_{i,J}^i(M) \rightarrow H_{i,J}^i(M/aM) \rightarrow H_{i,J}^{i+1}(M) \rightarrow \cdots \]

of local cohomology modules. In view of the hypothesis and the above exact sequence, H_{i,J}^i(M/aM) = 0 for all i > t. Since a is non zero-divisor on M, we have dim M/aM ≤ n − 1, and therefore the inductive hypothesis implies that H_{i,J}^i(M/aM)/aH_{i,J}^i(M/aM) = 0. The above exact sequence implies that H_{i,J}^i(M)/aH_{i,J}^i(M) ≅ H_{i,J}^{i+1}(M/aM). Since a ∈ a, therefore

\[ H_{i,J}^i(M)/aH_{i,J}^i(M) \cong H_{i,J}^i(M/aM)/aH_{i,J}^i(M/aM). \]

The inductive step is complete. The result follows by induction.

Corollary 1. Let M be a finitely generated module such that dim_R M = n. Then H_{i,J}^n(M)/aH_{i,J}^n(M) has finite length, for any a ∈ Wię(I, J). Specially, H_{i,J}^n(M)/IH_{i,J}^n(M) has finite length.

Proof. Let a ∈ Wię(I, J) be fixed. If n = 0, then M has finite length and so Γ_{i,J}(M)/aΓ_{i,J}(M) has finite length. Now assume that n > 0. It follows by [6, Theorem 4.7(1)] and Theorem 2, that H_{i,J}^n(M)/aH_{i,J}^n(M) = 0.

Corollary 2. Let M be finitely generated of finite dimension such that dim_R M/JM = d. Then H_{i,J}^{d+1}(M)/aH_{i,J}^{d+1}(M) is finitely generated, for any a ∈ Wię(I, J). Specially, H_{i,J}^{d+1}(M)/IH_{i,J}^{d+1}(M) is finitely generated.

Proof. Let a ∈ Wię(I, J) be fixed. If d = −1, then the claim is trivial. Now assume that d ≥ 0. It follows by [6, Theorem 4.7(2)] and Theorem 2, that H_{i,J}^{d+1}(M)/aH_{i,J}^{d+1}(M) = 0.

Corollary 3. Let R be local and M a finitely generated module such that dim_R M/JM = d. Then H_{i,J}^d(M)/aH_{i,J}^d(M) is finitely generated, for any a ∈ Wię(I, J). In particular, H_{i,J}^d(M)/IH_{i,J}^d(M) is finitely generated.

Proof. Let a ∈ Wię(I, J) be fixed. If d = 0, then the claim is trivial. Now assume that d > 0. It follows by [6, Theorem 4.3] and Theorem 2, that H_{i,J}^d(M)/aH_{i,J}^d(M) = 0.

Proposition 1. Let R be local, M finitely generated and t a non-negative integer. If H_{i,J}^i(M) is finitely generated, for all i > t, then H_{i,J}^i(M) = 0, for all i > t.
Proof. We may assume that $I \neq R$, otherwise $\Gamma_{I,J}$ is identity functor. Proposition 4.10, in [6], says that $H^i_{I,J}(M) = 0$, for all $i > \text{ara}(I\overline{R})$, where $\overline{R} = R/\sqrt{J + \text{Ann}_R(M)}$. Let $s = \text{ara}(I\overline{R})$. When $t \geq s$, there is nothing to prove. Now, assume that $t < s$. In view of Theorem 2, we have $H^i_{I,J}(M)/IH^i_{I,J}(M) = 0$, so Nakayama’s Lemma shows that $H^i_s(M) = 0$. By keeping this process, we deduce that $H^i_{I,J}(M) = 0$, for all $i > t$.

**Corollary 4.** Let $R$ be local with $\dim R/I + J = 0$ and $M$ finitely generated. Then $H^d_{I,J}(M)$ is not finitely generated, where $\dim_R M/JM = d > 0$.

**Proof.** Note that $\sup\{i : H^i_{I,J}(M) \neq 0\} = d$, by [6, Theorem 4.5]. Now the claim follows by Proposition 1.

**References**


