EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 2, 2012, 116-128
ISSN 1307-5543 – www.ejpam.com

Singular Ideals of Ternary Semirings

Tapan K. Dutta[1], Kar Ping Shum [2,*], Shobhan Mandal[3]

[1] *Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-710019,India*
[2] *Institute of Mathematics, Yunnan University, Kunming, 650091, People Republic of China*
[3] *Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-710019, India*

Abstract. The notion of singular ideal in a ternary semiring is introduced. The notions of singular ternary semirings and nonsingular ternary semirings are also defined. Some properties of singular ideals in ternary semirings are given. Our results obtained can be used to study some radical classes related to singular ideals.

2010 Mathematics Subject Classifications: 16Y60, 16Y99.

Key Words and Phrases: Singular ideal, Singular ternary semiring and non-singular ternary semirings.

1. Introduction

It was remarked by M. Ferrero and E. R. Puczylowski in [10] "Studying properties of rings one can usually say more assuming that the considered rings are either singular or non-singular." The same remark is equally true in the case of a ternary semiring which was first introduced by T. K. Dutta and S. Kar in [1]. The notion of ternary semiring was introduced in 2003. The introduction of ternary algebra dated back to 1932 when Lehmer [13] studied certain ternary system called triplexes which turn out to be a generalization of abelian groups. Later on, Banach [cf. Los 15] also studied such algebraic structure and gave some examples of a ternary semigroup which does not reduce to a semigroup. In addition, W. G. Lister [14] introduced the notion of ternary ring. Abstractly, a ternary ring T is an abelian group in which a ternary product tuv is given which is right,center and left distributive and which satisfies $(tuv)xy = t(uvx)y = tu(vxy)$. In this paper, our ternary semiring is a generalized ternary ring investigated by W. G. Lister in 1971. Though the notion of ternary semiring generalizes the notion of semiring but it is not merely a generalization of semiring because there are certain notions, for example, the lateral ideals which have no analogue in semirings. Some

*Corresponding author.

Email addresses: duttatapankumar@yahoo.co.in (T. Dutta), kpshum@ynu.edu.cn (K. Shum), shobhan139@gmail.com (S. Mandal)

earlier works on ternary semiring may be found in [1, 2, 3, 4, 5, 6, 7, 8, 11].

In this paper, the singular ideals in ternary semirings, singular ternary semirings and non-singular ternary semirings will be considered. We will investigate the properties of the singular ideals of a ternary semiring. We will give some examples of singular ternary semirings and non-singular ternary semirings. We first show that the class of singular ternary semirings with identity as well as the class of non-singular ternary semirings with identity is closed under direct products and direct sums. Then we study the image and preimage of singular ternary semiring and nonsingular ternary semiring under semi-isomorphism. Finally, we show that the class of semiprime non-singular ternary semirings is hereditary. Our results in this paper can be applied to study the special radical class of ternary semirings and upper radical class determined by the above radical classes which are called the singular radical and special singular radical of ternary semirings respectively. For terminologies and notions not given in this paper, the reader is referred to W . G. Lister [14].

2. Preliminaries

We first give the following definitions.

Definition 1. *[1] A non-empty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:*

(i) $(abc)de = a(bcd)e = ab(cde)$;

(ii) $(a+b)cd = acd + bcd$,

(iii) $a(b+c)d = abd + acd$,

(iv) $ab(c+d) = abc + abd$ *for all* $a, b, c, d, e \in S$.

Definition 2. *[1] Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0+x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in S$ then "0" is called the zero element or simply the zero of the ternary semiring S. In this case, S is called a ternary semiring with zero.*

It is noted that a ternary semiring does not necessarily contain an identity but there are certain ternary semirings which contain generalized identity in the sense defined below.

Definition 3. *[4] A ternary semiring S admits an identity provided that there exist elements $\{(e_i, f_i) \in S \times S\ (i = 1, 2, \ldots, n)\}$ such that $\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x$ for all $x \in S$. In this case, the ternary semiring S is said to be a ternary semiring with identity $\{(e_i, f_i) : i = 1, 2, \ldots, n\}$. In particular, if there exists an element $e \in S$ such that $eex = exe = xee = x$ for all $x \in S$, then "e" is called a unital element of a ternary semiring S.*

It is easy to see that $xye = (exe)ye = ex(eye) = exy$ and $xye = x(eye)e = xe(yee) = xey$, for all $x, y \in S$. Hence, the following result follows.

Proposition 1. *If e is a unital element of a ternary semiring S, then $exy = xey = xye$, for all $x, y \in S$.*

We now state the definitions of ternary subsemiring and left (right, lateral) ideals of a ternary semiring.

Definition 4. *[1] An additive subsemigroup T of a ternary semiring S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.*

Definition 5. *[1] An additive subsemigroup I of a ternary semiring S is called a left (right, lateral) ideal of S if s_1s_2i (respectively is_1s_2, s_1is_2) $\in I$ for all $s_1, s_2 \in S$ and $i \in I$. If I is a left, a right and a lateral ideal of S, then I is called an ideal of S.*

In the following proposition, we describe the left(right, lateral) ideal of a ternary semiring.

Proposition 2. *[1] Let S be a ternary semiring and $a \in S$. Then the following statements hold:*

(i) *left ideal generated by "a" is given by $\langle a \rangle_l = SSa + na$*

(ii) *right ideal generated by "a" is given by $\langle a \rangle_r = aSS + na$*

(iii) *two-sided ideal generated by "a" is given by $\langle a \rangle_t = SSa + aSS + SSaSS + na$*

(iv) *lateral ideal generated by "a" is given by $\langle a \rangle_m = SaS + SSaSS + na$*

(v) *ideal generated by "a" is given by $\langle a \rangle = SSa + aSS + SaS + SSaSS + na$,*

where $n \in \mathbb{Z}_0^+$ (set of all positive integers with zero).

The following definitions are useful in the study of ternary semirings.

Definition 6. *An ideal I of a ternary semiring S is said to be a k-ideal if $x + y \in I$; $x \in S$, $y \in I$ imply $x \in I$.*

Definition 7. *[4] A ternary semiring (ring) S is said to be zero divisor free (ZDF) if for $a, b, c \in S$, $abc = 0$ implies $a = 0$ or $b = 0$ or $c = 0$.*

Definition 8. *[4] A ternary semiring S is said to be commutative if $abc = bac = bca$ for all $a, b, c \in S$.*

Definition 9. *[4] A commutative ternary semiring (ring) is called a ternary semi-integral (resp. integral) domain if it is zero divisor free (ZDF).*

Definition 10. *[2] A proper ideal P of a ternary semiring S is called a prime ideal of S if for any three ideals A, B, C of S; $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.*

Definition 11. *[2] A ternary semiring S is called a prime ternary semiring if the zero ideal $\{0\}$ is a prime ideal of S.*

Definition 12. *[3] A proper ideal P of a ternary semiring S is called a semiprime of S if for any ideal A of S; $A^3 \subseteq P$ implies $A \subseteq P$.*

Definition 13. *[3] A ternary semiring S is called a semiprime ternary semiring if the zero ideal* (0) *is a semiprime ideal of S.*

Definition 14. *[5] An additive commutative semigroup M with a zero element 0_M is called a right ternary semimodule over a ternary semiring S or simply a right ternary S-semimodule if there exists a mapping $M \times S \times S \longrightarrow M$ (images to be denoted by ms_1s_2 for all $m \in M$ and $s_1, s_2 \in S$) satisfying the following conditions:*

(i) $(m_1 + m_2)s_1s_2 = m_1s_1s_2 + m_2s_1s_2$

(ii) $m_1s_1(s_2 + s_3) = m_1s_1s_2 + m_1s_1s_3$

(iii) $m_1(s_1 + s_2)s_3 = m_1s_1s_3 + m_1s_2s_3$

(iv) $(m_1s_1s_2)s_3s_4 = m_1(s_1s_2s_3)s_4 = m_1s_1(s_2s_3s_4)$

(v) $0_Ms_1s_2 = 0_M = m_1s_10_S = m_10_Ss_2$ *for all* $m_1, m_2 \in M$ *and for all* $s_1, s_2, s_3, s_4 \in S$.

A left ternary S-semimodule can be similarly defined.

Definition 15. *[5] A non-empty subset N of a right ternary S-semimodule M is said to be a ternary subsemimodule of M if (i) $a + b \in N$ and (ii) $ast \in N$ for all $a, b \in N$ and $s, t \in S$.*

Definition 16. *A nonzero right ideal I of a ternary semiring S with zero is called an essential right ideal of S if for any nonzero right ideal J of S, $I \cap J \neq (0)$.*

Definition 17. *A class ρ of ternary semirings is called hereditary if I is an ideal of a ternary semiring S and $S \in \rho$ implies $I \in \rho$.*

Definition 18. *[1] Let S and T be two ternary semirings and f be a mapping which maps S into T. Then the mapping $f : S \longrightarrow T$ is called a homomorphism of S into T if the following conditions hold:*
(i) $f(a + b) = f(a) + f(b)$ and (ii) $f(abc) = f(a)f(b)f(c)$ for all $a, b, c \in S$.

Moreover, if both the ternary semirings S and T have zeros 0_S and 0_T, respectively, then the following condition: (iii) $f(0_S) = 0_T$ also holds.

We define $ker f = \{x \in S : f(x) = 0_T\}$.

Throughout this paper, we use S to denote a ternary semiring with zero and $S^* = S \setminus \{0\}$.
We now use M to denote a right ternary S-semimodule with zero.

3. Singular Ideals

In this section, S is a ternary semiring and M a right ternary S-semimodule. Let $m \in M$. Then the right annihilator of m in S, denoted by $r_S(m)$, is defined by $\{x \in S : mxs = 0_M$ for all $s \in S\}$.

The right annihilator $r_S(m)$ has the following property.

Proposition 3. *Let S be a ternary semiring and M a right ternary S-semimodule. Then $r_S(m)$ is a right k-ideal of the ternary semiring S.*

Proof. Obviously, the set $r_S(m)$ is non-empty, since $0_S \in r_S(m)$. Let $a, b \in r_S(m)$. Then $a+b \in r_S(m)$. Again let $a \in r_S(m)$ and $s_1, s_2 \in S$. Then $mas = 0_M$ for all $s \in S \Rightarrow mas_1s_2s = 0_M$ for all $s \in S$. This leads to $as_1s_2 \in r_S(m)$. Hence, $r_S(m)$ is a right ideal of S.

Let $a, a+b \in r_S(m)$. Then $mas = 0_M = m(a+b)s$ for all $s \in S$. This implies $mbs = 0_M$ for all $s \in S$. This shows that $b \in r_S(m)$. Hence, $r_S(m)$ is a right k-ideal of S.

Let S be a ternary semiring and M a right ternary S-semimodule. We define $Z_S(M)$ as follows:

$$Z_S(M) = \{m \in M : r_S(m) \text{ is an essential right ideal of } S\}$$

Proposition 4. *$Z_S(M)$ is a ternary subsemimodule of M.*

Proof. Clearly, $0_M \in Z_S(M)$ as $r_S(0_M) = S$ is an essential right ideal of S. So $Z_S(M)$ is a non empty subset of M. Let $m_1, m_2 \in Z_S(M)$. Then, it follows that $r_S(m_1)$ and $r_S(m_2)$ are two essential right ideals of S. Therefore, $r_S(m_1) \cap r_S(m_2)$ is also an essential right ideal of S. Let H be a nonzero right ideal of S. Now $r_S(m_1) \cap r_S(m_2) \cap H \subseteq r_S(m_1 + m_2) \cap H$ since $r_S(m_1) \cap r_S(m_2) \subseteq r_S(m_1 + m_2)$. Therefore, $r_S(m_1 + m_2)$ is an essential right ideal of S. Hence, we have $m_1 + m_2 \in Z_S(M)$. Let $m \in Z_S(M)$ and $s_1, s_2 \in S$. Then, $r_S(m)$ is an essential right ideal of S. Let H be any nonzero right ideal of S. Now if $s_1s_2H = 0$, then $ms_1s_2Hs = 0_M$ for all $s \in S$. This leads to $H \subseteq r_S(ms_1s_2)$. Hence, $r_S(ms_1s_2) \cap H = H \neq 0$. If s_1s_2H is a nonzero right ideal of S, then $r_S(m) \cap s_1s_2H \neq 0$. Let $s_1s_2h_1(\neq 0) \in r_S(m) \cap s_1s_2H$ for some $h_1(\neq 0) \in H$. Then, $ms_1s_2h_1s = 0_M$, for all $s \in S$. Therefore, $h_1 \in r_S(ms_1s_2) \cap H$, and so $r_S(ms_1s_2) \cap H \neq 0$. Thus, in any case $ms_1s_2 \in Z_S(M)$. This Shows that $Z_S(M)$ is a subsemimodule of M.

We now give the definition of a singular subsemimodule over a ternary semiring.

Definition 19. *The ternary subsemimodule $Z_S(M)$ of M is called a singular ternary subsemimodule of the right ternary S-semimodule M.*

The singular ternary subsemimodule $Z_S(S)$ is a right ideal of the ternary semiring S and is called the (right) singular ideal of the ternary semiring S which is denoted by $Z(S)$, i.e. $Z(S) = \{t \in S : r_S(t) \text{ is an essential right ideal of } S\}$.

A ternary semiring S is said to satisfy the condition α if for any nonzero element a in S, $r_S(a) \neq S$ or equivalently $aSS = 0$ implies $a = 0$.

We now give some examples of ternary semirings satisfying the condition α.

Example 1. *Let S be a ternary semi-integral domain. Then S satisfies the condition α.*

Example 2. *Let S be a prime (semiprime) ternary semiring. Then S satisfies the condition α.*

Example 3. *Let S be a ternary semiring with a unital element e. Then S satisfies the condition α.*

Example 4. *Let S be a ternary semiring with identity. Then S satisfies the condition α.*

In the following proposition, we study the ternary semirings with condition α.

Proposition 5. *Let S be a ternary semiring with condition α. Then the (right) singular ideal $Z(S)$ is a k-ideal of S.*

Proof. From Proposition 4, it follows that $Z(S)$ is a right ideal of S. Now let $a \in Z(S)$, $s_1, s_2 \in S$. Then $r_S(a)$ is an essential right ideal of S. and so $r_S(a) \cap H \neq 0$ for any nonzero right ideal H of S. Now $r_S(a) \subseteq r_S(s_1 s_2 a)$. It is clear that $(0) \neq r_S(a) \cap H \subseteq r_S(s_1 s_2 a) \cap H$. Therefore, $r_S(s_1 s_2 a) \cap H \neq 0$. Thus, $s_1 s_2 a \in Z(S)$. This shows that $Z(S)$ is a left ideal of S.

Next, let $s_1, s_2 \in S$ and $a \in Z(S)$. Now, if $s_2 HS = (0)$, then $H \subseteq r_S(s_2) \subseteq r_S(s_1 a s_2)$. Thus $r_S(s_1 a s_2) \cap H = H \neq (0)$. If $s_2 HS \neq (0)$, then $s_2 HS$ is a nonzero right ideal of S. So $r_S(a) \cap s_2 HS \neq 0$. Let $s_2 h_1 s_3 (\neq 0) \in r_S(a) \cap s_2 HS$ for some $h_1(\neq 0) \in H$ and $s_3(\neq 0) \in S$. So $a s_2 h_1 s_3 s = 0$ for all $s \in S \Rightarrow s_1 a s_2 h_1 s_3 s t = 0$ for all $s, t \in S$. As $h_1 s_3 s \in H$ for all $s \in S$, $h_1 s_3 s \in r_S(s_1 a s_2) \cap H$ for all $s \in S$. Now, $h_1 s_3 s = 0$ for all $s \in S \Rightarrow s_2 h_1 s_3 s t = 0$ for all $s, t \in S \Rightarrow S \subseteq r_S(s_2 h_1 s_3)$ i.e. $r_S(s_2 h_1 s_3) = S$. Now by condition (α) this implies that $s_2 h_1 s_3 = 0$, a contradiction. Hence $h_1 s_3 s \neq 0$ for some $s \in S$. Thus $r_S(s_1 a s_2) \cap H \neq (0)$. Therefore in any case, $s_1 a s_2 \in Z(S)$. Consequently, $Z(S)$ is a lateral ideal of S and whence, $Z(S)$ is an ideal of S.

Finally, let $a + b, a \in Z(S)$. Then $r_S(a)$ and $r_S(a + b)$ are both essential right ideals of S. Therefore, $r_S(a + b) \cap r_S(a) \cap H \neq 0$ for any nonzero right ideal H of S.

Suppose that $p(\neq 0) \in r_S(a+b) \cap r_S(a) \cap H$. Then, $(a + b)ps = 0 = aps$, for all $s \in S$ and $p \in H$, which implies $bps = 0$ for all $s \in S$. This leads to $p \in r_S(b) \cap H$ i.e. $r_S(b) \cap H \neq 0$. Hence, $b \in Z(S)$. Therefore, $Z(S)$ is a k-ideal of S.

Remark 1. *The condition "α" is assumed only to show that Z(S) is a lateral ideal of S. In order to show that Z(S) is an ideal of a commutative ternary semiring, it is not necessary to assume the condition "α".*

Remark 2. *On assuming the condition "α", we do not lose the generality because our aim is to study the weakly special radical class and the special radical class of ternary semiring which are the radical classes of nonsingular semiprime ternary semirings and nonsingular prime ternary semirings respectively in which the condition α holds which is evident from the above Example 2.*

Proposition 6. *Let S be a ternary semiring. Then $Z(S) = \{x \in S : xIS = 0$ for some essential right ideal I of $S\}$*

Proof. Let $Z'(S) = \{x \in S : xIS = 0$ for some essential right ideal I of $S\}$. Suppose $x \in Z(S)$ and $J = r_S(x)$. Then J is an essential right ideal of S. Also $xJS = 0$. Hence, $x \in Z'(S)$.

Conversely, let $x \in Z'(S)$. Then $xIS = 0$ for some essential right ideal I of S. Consequently, $I \subseteq r_S(x)$. Let H be any nonzero right ideal of S. Then $0 \neq I \cap H \subseteq r_S(x) \cap H$ which implies $x \in Z(S)$ and hence the result.

Proposition 7. *Let S be a ternary semiring. Then $Z(S) = \{x \in S : xIS = 0$ for some essential right k-ideal I of $S\}$*

Proof. Let $Z''(S) = \{x \in S : xIS = 0$ for some essential right k-ideal I of $S\}$. From Proposition 6, it follows that $Z''(S) \subseteq Z(S)$.

Conversely, let $x \in Z(S)$ and $J = r_S(x)$. Then J is an essential right ideal of S. Now let $a, a+b \in J$. Then $xas = 0 = x(a+b)s$ for all $s \in S$, this implies $xbs = 0$ for all $s \in S$, and so $b \in r_S(x) = J$. Thus, J is a right k-ideal of S. Also, $xJS = 0$. This leads to $x \in Z''(S)$ and hence $Z(S) \subseteq Z''(S)$. The proposition is hence proved.

Definition 20. *A ternary semiring S is said to be singular if $Z(S) = S$ and is said to be non-singular if $Z(S) = 0$.*

We now study the singular ternary semirings.

Proposition 8. *Let S be a non-singular ternary semiring, then S satisfies the condition α.*

Proof. Let $aSS = 0$. Then $r_S(a) = S$ is an essential right ideal of S. Hence $a \in Z(S)$. Since S is non-singular, $Z(S) = 0$ and so $a = 0$. Thus, S satisfies the condition α.

Definition 21. *[5] An element s of a ternary semiring S is said to be nilpotent if for each $t \in S$, there exists a positive integer n(depending on t) such that $(st)^n s = 0$.*

Definition 22. *A ternary semiring S is said to be a nil ternary semiring if each a in S is nilpotent.*

As examples of singular ternary semirings and non-singular ternary semirings we have the following propositions.

Proposition 9. *Every commutative nil ternary semiring S with a unital element e is singular.*

Proof. Pick $a(\neq 0) \in S$. Let H be a nonzero right ideal of S and $b(\neq 0) \in H$. As S is nil, a is clearly nilpotent. Then, for each t in S, there exists a positive integer n(depending on t) such that $(at)^n a = 0$, or equivalently, we have $a(ta)^n = 0$. Thus, in particular, $a(ea)^n = 0$ for some positive integer n. This implies $a(ea)^n bS = 0$. Now let m be the least positive integer such that $a(ea)^m bS = 0$. Then $(ea)^m b \in r_S(a)$. Also $(ea)^m b \in H$. Now $(ea)^m b = a(ea)^{m-1} be \neq 0$ by minimality of m, for $a(ea)^{m-1} be = 0 \Rightarrow a(ea)^{m-1} bese = 0$ for all $s \in S$, that is, $a(ea)^{m-1} bS = 0$, a contradiction. Hence, $r_S(a) \cap H \neq 0$ and so $a \in Z(S)$. This proves that $Z(S) = S$.

We give below the definition of a right strongly prime ternary semiring.

Definition 23. *[12] A ternary semiring S is called right strongly prime if for every nonzero element $a \in S$, there exist some finite subsets F_1, F_2, F_3 of S such that $aF_1F_2F_3y = \{0\} \Rightarrow y = 0$ for all $y \in S$.*

We study below the properties of a right strongly prime ternary semiring.

Lemma 1. *Let S be a right strongly prime ternary semiring with identity. Then every nonzero ideal of S obtains a finite subset G such that $r_S(G) = 0$, where $r_S(G) = \{t \in S : Gts = 0$ for all $s \in S\}$.*

Proof. Since S admits identity, there exist elements $\{(e_i, f_i) \in S \times S\ (i = 1, 2, \ldots, n)\}$ such that $\sum_{i=1}^{n} e_i f_i x = \sum_{i=1}^{n} e_i x f_i = \sum_{i=1}^{n} x e_i f_i = x$ for all $x \in S$. Suppose I is any nonzero ideal of S. Let $a(\neq 0) \in I$. Since S is a right strongly prime ternary semiring, there exist finite subsets F_1, F_2, F_3 of S such that $aF_1F_2F_3y = \{0\} \Rightarrow y = 0$ for all $y \in S$. Now, set $E = \{e_i : i = 1, 2, \ldots, n\}$ and $F = \{f_i : i = 1, 2, \ldots, n\}$. Let $G = aF_1F_2F_3E$. Then G is a finite subset of I. It is clear that $y \in r_S(G) \Rightarrow aF_1F_2F_3Eys = 0$ for all
$s \in S \Rightarrow aF_1F_2F_3EyF = 0 \Rightarrow aF_1F_2F_3 \sum_{i=1}^{n} e_i y f_i = 0 \Rightarrow aF_1F_2F_3y = 0 \Rightarrow y = 0$. Thus, $r_S(G) = 0$.

Proposition 10. *Every right strongly prime ternary semiring S with identity is non-singular.*

Proof. If possible, let $Z(S) \neq 0$. Then $Z(S)$ is a nonzero ideal of S. Since S is a right strongly prime ternary semiring, by Lemma 1, there exists a finite subset $F = \{t_1, t_2, \ldots, t_k\}$ of $Z(S)$ such that $r_S(F) = 0$ i.e. $r_S(t_1) \cap r_S(t_2) \cap \ldots, \cap r_S(t_k) = 0$. Now $t_i \in Z(S)$ implies that $r_S(t_i)$ is an essential right ideal of S for $i = 1, 2, \ldots, k$. Therefore $r_S(F)$ is an essential right ideal of S. Consequently, $r_S(F) \neq 0$, a contradiction. The proposition is hence proved.

We now give the definition of a strongly nilpotent element in a ternary semiring.

Definition 24. *An element a in a ternary semiring S is said to be strongly nilpotent if there exists a positive integer n such that $a^{2n+1} = 0$.*

Definition 25. *A ternary semiring S is said to be a reduced ternary semiring if it does not contain any nonzero strongly nilpotent elements.*

In the following proposition, we describe the reduced ternary semirings.

Proposition 11. *Every reduced ternary semiring S with identity is non-singular.*

Proof. Take any $a \in S^*$. If $x \in r_S(a) \cap aSS$, then for some $y_1, y_2 \in S, x = ay_1y_2$ and $a^2y_1y_2s = axs = 0$ for all $s \in S$. This implies that $(ay_1y_2Sa)^3 = 0$. Hence, since S is reduced, $ay_1y_2Sa = 0$.

Consequently, $xSx = ay_1y_2Say_1y_2 = 0$. Thus, $xSxSx = 0, xSSxSxS = 0$ and $SxSxSSx = 0$. Also, $(xSSxSSa)^3 = 0$. Since S is reduced, $xSSxSSa = 0$. Thus, $xSSxSSx = xSSxSSay_1y_2 = 0$. However, the ternary semiring S, being reduced, is semiprime, and so $x = 0$.

Consequently, $r_S(a) \cap aSS = 0$. Now,because S admits an identity, aSS is a nonzero right ideal of S. This means that $r_S(a)$ is not an essential right ideal of S, for any $a \in S^*$. This implies that $Z(S) = 0$.

Lemma 2. *Let $\{S_\alpha : \alpha \in \Lambda$, where Λ is a index set$\}$ be a family of ternary semirings. Then, $r_{\prod_{\alpha\in\Lambda} S_\alpha}((a_\alpha)_{\alpha\in\Lambda}) = \prod_{\alpha\in\Lambda} r_{S_\alpha}(a_\alpha)$.*

Proof. Let $(x_\alpha)_{\alpha\in\Lambda} \in r_{\prod_{\alpha\in\Lambda} S_\alpha}((a_\alpha)_{\alpha\in\Lambda}) \Leftrightarrow (a_\alpha)_{\alpha\in\Lambda}(x_\alpha)_{\alpha\in\Lambda}(s_\alpha)_{\alpha\in\Lambda} = (0_\alpha)_{\alpha\in\Lambda}$ for all $(s_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} S_\alpha \Leftrightarrow (a_\alpha x_\alpha s_\alpha)_{\alpha\in\Lambda} = (0_\alpha)_{\alpha\in\Lambda}$ for all $s_\alpha \in S_\alpha$, $\alpha \in \Lambda \Leftrightarrow a_\alpha x_\alpha s_\alpha = 0_\alpha$ for all $s_\alpha \in S_\alpha, \alpha \in \Lambda \Leftrightarrow x_\alpha \in r_{S_\alpha}(a_\alpha)$ for each $\alpha \in \Lambda$. Hence, the result.

Lemma 3. *Let $\{S_\alpha : \alpha \in \Lambda$ where Λ is an index set$\}$ be a family of ternary semirings with identity. Let P be a right ideal of $\prod_{\alpha\in\Lambda} S_\alpha$. Let $\pi_\alpha : \prod_{\alpha\in\Lambda} S_\alpha \to S_\alpha$ be the projection map. Suppose $P_\alpha = \pi_\alpha(P)$ for each $\alpha \in \Lambda$. Then $P = \prod_{\alpha\in\Lambda} P_\alpha$.*

Proof. Obviously, $P \subseteq \prod_{\alpha\in\Lambda} P_\alpha$. Now let $(a_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} P_\alpha$. Then, we have $a_\alpha \in P_\alpha$. As P_α is the αth projection map, there exists a $(b_\gamma)_{\gamma\in\Lambda-\{\alpha\}} \in \prod_{\alpha\in\Lambda-\{\alpha\}} S_\gamma$ such that $(a_\alpha, b_\gamma)_{\gamma\in\Lambda-\{\alpha\}} \in P$. Let $\{(e_{i_\alpha}, f_{i_\alpha}) \in S_\alpha \times S_\alpha \ (i_\alpha = 1, 2, \ldots, n_\alpha)\}$ be the identity of S_α. Now, we consider the sum of products

$$\sum_{i=1}^{n_\alpha}(a_\alpha, b_\gamma)_{\gamma\in\Lambda-\{\alpha\}}(e_{i_\alpha}, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}}(f_{i_\alpha}, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}} = \sum_{i=1}^{n_\alpha}(a_\alpha e_{i_\alpha} f_{i_\alpha}, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}} = (a_\alpha, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}} \in P$$

as P is a right ideal of $\prod_{\alpha\in\Lambda} S_\alpha$.

Thus, $(a_\alpha, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}} \in P$ for each $\alpha \in \Lambda$. Hence, $\sum_{\alpha\in\lambda}(a_\alpha, 0_\gamma)_{\gamma\in\Lambda-\{\alpha\}} = (a_\alpha)_{\alpha\in\Lambda} \in P$ as P is a right ideal of $\prod_{\alpha\in\Lambda} S_\alpha$. Hence $P = \prod_{\alpha\in\Lambda} S_\alpha$.

Proposition 12. *For every family $\{S_\alpha : \alpha \in \Lambda$ where Λ is a index set$\}$ of ternary semirings with identity $Z(\prod_{\alpha\in\Lambda} S_\alpha) = \prod_{\alpha\in\Lambda} Z(S_\alpha)$ and $Z(\bigoplus_{\alpha\in\Lambda} S_\alpha) = \bigoplus_{\alpha\in\Lambda} Z(S_\alpha)$.*

Proof. Suppose that $S = \prod_{\alpha\in\Lambda} S_\alpha$. Let $(x_\alpha)_{\alpha\in\Lambda} \in Z(S)$. Then, $r_S((x_\alpha)_{\alpha\in\Lambda}$ is an essential right ideal of S. Let P_α be a nonzero right ideal of S_α, $\alpha \in \Lambda$. Then $\prod_{\alpha\in\Lambda} P_\alpha$ is a nonzero right ideal of S. This shows that

$$r_S((x_\alpha)_{\alpha\in\Lambda}) \cap \prod_{\alpha\in\Lambda} P_\alpha \neq (0_\alpha)_{\alpha\in\Lambda} \Rightarrow \prod_{\alpha\in\Lambda} r_{S_\alpha}(x_\alpha) \cap \prod_{\alpha\in\Lambda} P_\alpha \neq (0_\alpha)_{\alpha\in\Lambda},$$

by Lemma 2. We claim that $r_{S_\alpha}(x_\alpha) \cap P_\alpha \neq 0_\alpha$, for each $\alpha \in \Lambda$. If the assertion is not true. Then there exists $I \subseteq \Lambda$ such that $r_{S_\alpha}(x_\alpha) \cap P_\alpha = 0_\alpha$ for each $\alpha \in I$ and $r_{S_\alpha}(x_\alpha) \cap P_\alpha \neq 0_\alpha$, for each $\alpha \in \Lambda - I$. Now $\prod_{\alpha\in\Lambda-I}(0_\alpha) \times \prod_{\alpha\in I} P_\alpha$ is a nonzero right ideal of S but

$$r_S((x_\alpha)_{\alpha\in\Lambda}) \cap (\prod_{\alpha\in\Lambda-I}(0_\alpha) \times \prod_{\alpha\in I} P_\alpha) = [\prod_{\alpha\in\Lambda-I} r_{S_\alpha}(x_\alpha) \cap (0_\alpha)] \times [\prod_{\alpha\in I} r_{S_\alpha}(x_\alpha) \cap P_\alpha] = (0_\alpha)_{\alpha\in\Lambda},$$

a contradiction since $(x_\alpha)_{\alpha\in\Lambda} \in Z(S)$. Thus, $x_\alpha \in Z(S_\alpha)$ for each $\alpha \in \Lambda \Rightarrow (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} Z(S_\alpha)$.

Conversely, let $(x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} Z(S_\alpha) \Rightarrow x_\alpha \in Z(S_\alpha)$ for each $\alpha \in \Lambda$. Let P be a nonzero right ideal of S.

Now, let $\pi_\alpha : S \to S_\alpha$ be the projection map. Then, $P_\alpha = \pi_\alpha(P)$ is a nonzero right ideal of S_α for at least one α. This shows that $P = \prod_{\alpha\in\Lambda} P_\alpha$ by Lemma 3. Now , $r_{S_\alpha}(x_\alpha) \cap P_\alpha \neq 0_\alpha$ for at least one

$$\alpha \Rightarrow \prod_{\alpha\in\Lambda} r_{S_\alpha}(x_\alpha) \cap \prod_{\alpha\in\Lambda} P_\alpha \neq (0_\alpha)_{\alpha\in\Lambda} \Rightarrow r_S((x_\alpha)_{\alpha\in\Lambda}) \cap P \neq (0_\alpha)_{\alpha\in\Lambda} \Rightarrow (x_\alpha)_{\alpha\in\Lambda} \in Z(S).$$

Thus, $Z(S) = \prod_{\alpha\in\Lambda} Z(S_\alpha)$.

Similarly, we can prove that $Z(\bigoplus_{\alpha\in\Lambda} S_\alpha) = \bigoplus_{\alpha\in\Lambda} Z(S_\alpha)$.

The following theorem is a main theorem of this paper.

Theorem 1. *The class of singular ternary semirings with identity as well as the class of non-singular ternary semirings with identity is closed under product and direct sum.*

Definition 26. *A surjective homomorphism of ternary semirings $\gamma : R \to S$ is called a ternary semi-isomorphism if $ker\gamma = 0$.*

Lemma 4. *If $\gamma : S \to S'$ is a ternary semi-isomorphism and $Z(S) = S$ then $Z(S') = S'$.*

Proof. If possible, let $Z(S') \subset S'$. Then there exists $s' \in S'$ but $s' \notin Z(S')$. Let $s \in S$ such that $\gamma(s) = s'$. Since $s' \notin Z(S')$, there exists a nonzero right ideal H' of S' such that $r_{S'}(s') \cap H' = 0$. Let $H = \{b \in S : \gamma(b) \in H'\}$. Then H is a nonzero right ideal of S. So $r_S(s) \cap H \neq 0$. Therefore there exists a nonzero element h in H such that $sht = 0$ for all $t \in S$. Consequently $\gamma(sht) = \gamma(0)$ for all $t \in S \Rightarrow \gamma(s)\gamma(h)\gamma(t) = 0$ for all $t \in S \Rightarrow s'\gamma(h)t' = 0$ for all $t' \in S'$ as γ is surjective. But $\gamma(h) \in H'$. Therefore $\gamma(h) \in r_{S'}(s') \cap H'$ which implies that $\gamma(h) = 0$. Thus $h \in ker\gamma = 0$. So $h = 0$, a contradiction. Hence $Z(S') = S'$.

Lemma 5. *If $\gamma : S \to S'$ is a semi-isomorphism and $Z(S') = S'$ then $Z(S) = S$.*

Proof. If possible, let $Z(S) \subset S$. So there exists $s(\neq 0) \in S$ but $s \notin Z(S)$. Therefore there exists a nonzero right ideal H of S such that $r_S(s) \cap H = 0$. Now $\gamma(H)$ is a nonzero right ideal of S'. Now $\gamma(s) \neq 0$ because $\gamma(s) = 0 \Rightarrow s \in ker\gamma = 0 \Rightarrow s = 0$. Let $\gamma(s) = s'$. We now prove that, $r_{S'}(s') \cap \gamma(H) = 0$. If possible, let $r_{S'}(s') \cap \gamma(H) \neq 0$ and $h'(\neq 0) \in r_{S'}(s') \cap \gamma(H)$. Let $\gamma(h) = h'$, then $h \neq 0$. Now $s'h't' = 0$ for all $t' \in S'$ and $h' \in \gamma(H)$. So $\gamma(sht) = \gamma(s)\gamma(h)\gamma(t) = s'h't' = 0$ for all $t \in S$. Therefore $sht \in ker\gamma = 0$ for all $t \in S$. So $sht = 0$ for all $t \in S$. Thus $h \in r_S(s) \cap H$, a contradiction. Therefore $r_{S'}(s') \cap \gamma(H) = 0$. So $s' \notin Z(S')$, a contradiction. Hence $Z(S) = S$.

Lemma 6. *If $\gamma : S \to S'$ is a semi-isomorphism and $Z(S') = 0$ then $Z(S) = 0$.*

Proof. If possible, let $Z(S) \neq 0$. So there exists $s(\neq 0) \in S$ such that $s \in Z(S)$. Now $\gamma(s) \neq 0$ because $\gamma(s) = 0 \Rightarrow s \in ker\gamma = 0 \Rightarrow s = 0$. Since $Z(S') = 0$, $\gamma(s) \notin Z(S')$. Hence there exists a nonzero right ideal H' of S' such that $r_{S'}(\gamma(s)) \cap H' = 0$. Let $H = \{s \in S : \gamma(s) \in H'\}$. Then H is a nonzero right ideal of S. Therefore $r_S(s) \cap H \neq 0$. Therefore there exists a nonzero element h in H such that $sht = 0$ for all $t \in S$. Consequently $\gamma(sht) = \gamma(0)$ for all $t \in S \Rightarrow \gamma(s)\gamma(h)\gamma(t) = 0$ for all $t \in S \Rightarrow \gamma(s)\gamma(h)t' = 0$ for all $t' \in S'$ as γ is surjective $\Rightarrow \gamma(h) \in r_{S'}(\gamma(s)) \cap H' = (0)$, which implies that $\gamma(h) = 0$. Thus $h \in ker\gamma = 0$. So $h = 0$, a contradiction. Hence $Z(S) = 0$.

Lemma 7. *If $\gamma : S \to S'$ is a ternary semi-isomorphism and $Z(S) = 0$ then $Z(S') = 0$.*

Proof. If possible, let $Z(S') \neq 0$ and $s'(\neq 0) \in Z(S')$. Since γ is surjective, there exists a nonzero element $s \in S$ such that $\gamma(s) = s'$. Since $Z(S) = 0$, so $s \notin Z(S)$. Therefore there exists a nonzero right ideal H of S such that $r_S(s) \cap H = 0$. Now $\gamma(H)$ is a nonzero right ideal of S'. We now prove that, $r_{S'}(s') \cap \gamma(H) = 0$. If possible, let $r_{S'}(s') \cap \gamma(H) \neq 0$ and $h'(\neq 0) \in r_{S'}(s') \cap \gamma(H)$. Let $\gamma(h) = h'$, then $h \neq 0$. Now $s'h't' = 0$ for all $t' \in S'$ and $h' \in \gamma(H)$. So $\gamma(sht) = \gamma(s)\gamma(h)\gamma(t) = s'h't' = 0$ for all $t \in S$. Therefore $sht \in ker\gamma = 0$ for all $t \in S$. So $sht = 0$ for all $t \in S$. Thus $h \in r_S(s) \cap H$, a contradiction. Therefore $r_{S'}(s') \cap \gamma(H) = 0$. So $s' \notin Z(S')$, a contradiction. Hence $Z(S') = 0$.

Theorem 2. *If S and S' be two semi-isomorphic ternary semirings. Then S is singular (nonsingular) iff S' is singular (resp. nonsingular).*

Proposition 13. *If I is an ideal of a ternary semiring S and as a ternary semiring, I is semiprime, then $Z(I) = I \cap Z(S)$.*

Proof. Let $x \in Z(I)$ and P be a nonzero right ideal of S. If $PSI = 0$, then
$(IPS)^3 = (IPS)(IPS)(IPS) = I(PSI)(PSI)PS = 0$. Now, I is a semiprime ternary semiring and IPS is an ideal of I. and so, $IPS = 0$. Consequently $P \subseteq r_S(I) \subseteq r_S(x)$, since $x \in Z(I) \subseteq I$. So $P \cap r_S(x) = P \neq 0$. If $PSI \neq 0$, then PSI is a nonzero right ideal of I. So $r_I(x) \cap PSI \neq 0$. Let $0 \neq a \in r_I(x) \cap PSI$. Hence, $xai = 0$ for all $i \in I \Rightarrow xaii_1s = 0$ for all $s \in S$ and for all $i, i_1 \in I$. Now if $aii_1 = 0$ for all $i, i_1 \in I$, then $aII = 0, aIa = 0$ and $a^3 = 0$ as $a \in I$. Now $\langle a\rangle^3 = (aII + IIa + IaI + IIaII + na)^3 = (0) \Rightarrow a = 0$ as I is semiprime. This arrives a contradiction as $a \neq 0$. Thus, $aii_1 \neq 0$ for some $i, i_1 \in I \Rightarrow 0 \neq aii_1 \in r_S(x)$ for some $i, i_1 \in I$. Also $aii_1 \in PSI \subseteq P$ for all $i, i_1 \in I$. Hence, $0 \neq aii_1 \in r_S(x) \cap P$. This leads to $r_S(x) \cap P \neq (0)$. Thus, in any case, $r_S(x)$ is an essential right ideal of S. Hence, $x \in Z(S)$. Also, $x \in I$. This shows that $x \in I \cap Z(S)$. Thus, $Z(I) \subseteq I \cap Z(S)$.

Conversely, let $x \in I \cap Z(S)$ and P be a nonzero right ideal of I. Since the ternary semiring I is semiprime, $PII \neq 0$ as $PII = (0) \Rightarrow P^3 \subseteq PII = (0) \Rightarrow P = (0)$, a contradiction. Thus, PII is a nonzero right ideal of S. Hence, $r_S(x) \cap PII \neq (0)$. However, $PII \subseteq P$ and whence, $r_S(x) \cap P \neq (0)$. Consequently, we have $0 \neq r_S(x) \cap P \subseteq r_I(x) \cap P$. Thus, $x \in Z(I)$. Hence, $I \cap Z(S) \subseteq Z(I)$. Thus, $Z(I) = I \cap Z(S)$.

Theorem 3. *The class of semiprime non-singular(singular) ternary semirings is hereditary.*

Proof. The Proof follows from Proposition 13.

Corollary 1. *The class of prime non-singular (singular) ternary semirings is hereditary.*

Proof. The proof of this corollary is an immediate consequence of Theorem 3.

Finally, we state the following propositions:

Proposition 14. *[7] Let S be a ternary semiring. If Q is a semiprime ideal of S and I is an ideal of S then $Q \cap I$ is a semiprime ideal of I.*

In closing this paper, we give the following theorem of semiprime non-singular ternary semirings.

Theorem 4. *The class of homomorphic images of semiprime non-singular ternary semirings is hereditary.*

Proof. Let $\mathscr{A} = \{\phi(S) : S$ be a semiprime nonsingular ternary semiring $\}$. Let $\phi(S) \in \mathscr{A}$ and J a nonzero ideal of $\phi(S)$. Then, there exists a nonzero ideal I of S such that $\phi(I) = J$. Now by Proposition 14, I is semiprime as S is semiprime. Also by Proposition 13, I is nonsingular. Hence, $\phi(I) \in \mathscr{A}$, that is, $J \in \mathscr{A}$. Thus, the class of homomorphic images of semiprime non-singular ternary semirings is hereditary.

Remark 3. *We make the following observations.*

(i) The ternary subsemiring of a semiprime non-singular ternary semiring is not necessarily semiprime non-singular. If the ring is commutative then the subring of a semiprime ring is always semiprime. Similar result holds for ternary semirings.

(ii) It has been proved in Proposition 13 that if I is an ideal of a ternary semiring S and I a semiprime ideal as a ternary semiring, then $Z(I) = I \cap Z(S)$ .Thus, by the above result, we can easily deduce that if S is a non-singular ternary semiprime semiring, then I is non-singular. Similar situation has been investigated by T.K. Dutta and M. L. Das in [9] in the case of a semiring. In the proof of this result, the properties of ideals have been used. No result has been proved so far for subsemiring or subring and hence, in the case of ternary semirings, the above result may not be true.

The reader is invited to provide a counter example.

(iii) It is noted that the direct product or direct sum of semiprime non-singular ternary semirings with identity is semiprime and non-singular.

In the theory of finite groups and semigroups, the notion of the formation and the hereditary classes of groups and semigroups have been studied by L. A. Shemetkov in [16]. In view of the above observations and the terminology of formation given by L.A. Shemetkov and A. N. Skiba [17], we propose the following conjecture. The hereditary semiprime non-singular ternary semirings form a hereditary formation of ternary semirings.

ACKNOWLEDGEMENTS The third author is thankful to CSIR, India for financial assistance.

References

[1] T. K. Dutta and S. Kar. *On Regular Ternary Semirings*; Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, 343 - 355. 2003.

[2] T. K. Dutta and S. Kar. *On Prime Ideals and Prime Radical of Ternary Semirings*; Bull. Cal. Math. Soc., Vol. 97, No. 5. 445 - 454. 2005.

[3] T.K. Dutta and S. Mandal. A general type of regular and semiprime ideals in ternary semirings, Advances in algebraic structure,Proceedings of International conference in Algebra, World Scientific,INC., 219-232. 2010.

[4] T. K. Dutta and S. Kar. *On Semiprime Ideals and Irreducible Ideals of Ternary Semirings*; Bull. Cal. Math. Soc., Vol. 97, No. 5, 467 - 476. 2005.

[5] T. K. Dutta and S. Kar. *On Ternary Semifields*; Discussiones Mathematicae - General Algebra and Applications, Vol 24, No.2, 185 - 198. 2004.

[6] T. K. Dutta and S. Kar. *On the Jacobson Radical of a Ternary Semiring*; Southeast Asian Bull.of Math., Vol. 28, No. 1, 1 - 13. 2004.

[7] T.K. Dutta, K. P. Shum and S. Mandal. *Weakly special radical class and special radical class of Ternary Semirings*; Journal of International Mathematics and Mathematical Science. 2012 (To appear).

[8] T. K. Dutta and S. Kar. Two types of Jacobson radicals of ternary semirings; Southeast Asian Bull. Math. 29, no. 4, 677 - 687. 2005.

[9] T. K. Dutta and M. L. Das. *Singular Radicals in Semiring*; Southeast Asian Bull. of Math., Vol. 34, 405-416. 2010.

[10] M. Ferrero and E. R. Puczylowski. *The singular ideal and radicals*, J. Austral. Math. Soc.(Series A) 64, 195-209. 1998.

[11] S. Kar. On structure space of ternary semirings; Southeast Asian Bull. Math. 31, no. 3, 537 - 545. 2007.

[12] S. Kar, J. Sircar and S. Mandal. *On Right Strongly Prime Ternary Semirings*; East-West J. of Mathematics, Vol. 12, No. 1, pp. 59 - 68. 2010.

[13] D. H. Lehmer. *A ternary analogue of abelian groups, Amer. J. Math*, 54(2), 329-338. 1932.

[14] W. G. Lister. *Ternary Rings*; Trans. Amer. Math. Soc. 154, 37-55. 1971.

[15] S. J. Lö. *On the Extending Models I*; Fundamenta Mathematicae.,42, 38-54. 1955.

[16] L. A. Shemetkov. *New ideas and results in the theory of formations*(Russian), Problems in algebra, no.4 (Russian), (Gomel), 65-67, Problems in algebra, (Russian), in Universitet.skoe, Minsk, 1986,1989.

[17] A. N. Skiba and L. A. Shemetkov. *Hereditarily indecomposable formations of groups*(Russian), Dokl. Akad. Nauk BSSR 33, no. 7, 581-582. 1989.