Some Remarks on Finitely Quasi-injective Modules

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Abstract. Let $R$ be a ring. A right $R$-module $M$ is called finitely quasi-injective if each $R$-homomorphism from a finitely generated submodule of $M$ to $M$ can be extended to an endomorphism of $M$. Some conditions under which finitely generated finitely quasi-injective modules are of finite Goldie Dimensions are given, and finitely generated finitely quasi-injective Kasch modules are studied.

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1. Introduction

Throughout the paper, $R$ is an associative ring with identity and all modules are unitary. If $M_R$ is a right $R$-module with $S = \text{End}(M_R)$, and $A \subseteq S, X \subseteq M, B \subseteq R$, then we denote the Jacobson radical of $S$ by $J(S)$, and we write $l_S(X) = \{s \in S \mid sx = 0, \forall x \in X\}$, $r_M(A) = \{m \in M \mid am = 0, \forall a \in A\}$, $l_M(B) = \{m \in M \mid mb = 0, \forall b \in B\}$. Following [5], we write $W(S) = \{s \in S \mid \text{Ker}(s) \subseteq \text{ess} M\}$.

At first let we recall some concepts. A module $M_R$ is called finitely quasi-injective (or FQ-injective for short) [7] if each $R$-homomorphism from a finitely generated submodule of $M$ to $M$ can be extended to an endomorphism of $M$; a ring $R$ is said to be right F-injective if $R_R$ is finitely quasi-injective. F-injective rings have been studied by many authors such as [2, 3, 8]. A module $M_R$ is called a $C_1$ module if every submodule of $M$ is essential in a direct summand of $M$, $C_1$ modules are also called CS modules. A module $M_R$ is called a $C_2$ module if every submodule of $M$ that is isomorphic to a direct summand of $M$ is itself a direct summand of $M$. A module $M_R$ is called a $C_3$ module if, whenever $N$ and $K$ are submodules of $M$ with $N \subseteq \oplus M, K \subseteq \oplus M$, and $N \cap K = 0$, then $N \oplus K \subseteq \oplus M$. A module $M_R$ is called continuous if it is both $C_1$ and $C_2$. A module $M_R$ is called quasi-continuous if it is both $C_1$ and $C_3$. It is well-known that $C_2$ modules are $C_3$ modules, and so continuous modules are quasi-continuous.

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A module $M_R$ is said to be Kasch [1] provided that every simple module in $\sigma[M]$ embeds in $M$, where $\sigma[M]$ is the category consisting of all $M$-subgenerated right $R$-modules. In this note we shall mainly study finitely generated finitely quasi-injective modules with finite Goldie Dimensions, and finitely generated finitely quasi-injective Kasch modules, respectively.

2. Main Results

We begin with some Lemmas.

Lemma 1 ([10, Theorem 1.2]). For a module $M_R$ with $S = \text{End}(M_R)$, the following statements are equivalent:

(1) $M_R$ is FQ-injective;

(2) (a) $l_S(A \cap B) = l_S(A) + l_S(B)$ for any finitely generated submodules $A$, $B$ of $M$, and

(b) $l_M r_R(m) = S m$ for any $m \in M$. where $l_M r_R(m)$ consists of all elements $z \in M$ such that $m x = 0$ implies $x z = 0$ for any $x \in R$.

Lemma 2 ([10, Theorem 2.1, Theorem 2.2]). Let $M_R$ be a finitely generated FQ-injective module with $S = \text{End}(M_R)$. Then

(1) $l_S(Ker \alpha) = S \alpha$ for any $\alpha \in S$.

(2) $W(S) = J(S)$.

Lemma 3 ([10, Theorem 2.3]). Let $M_R$ be a finitely generated finite dimensional FQ-injective module with $S = \text{End}(M_R)$. Then $S$ is semilocal.

Lemma 4. Let $M_R$ be a finitely generated FQ-injective module. Then it is a $C_2$ module.

Proof. Write $S = \text{End}(M_R)$. Let $N$ be a submodule of $M$ and $N \cong eM$ for some $e^2 = e \in S$. Then there exists some $s \in S$ such that $N = seM$ and $\text{Ker}(se) = \text{Ker}(e)$. By Lemma 2(1), we have $Se e = Se$, and hence $e = tse$ for some $t \in S$ with $t = et$. Thus $(set)^2 = set$ and $N = (se)M = (set)M$. Therefore $N$ is a direct summand of $M$. □

Lemma 5. Let $M_R$ be a quasi-continuous module with $S = \text{End}(M_R)$. Then idempotents of $S/W(S)$ can be lifted.

Proof. Let $s^2 - s \in W(S)$, then $\text{Ker}(s^2 - s) \triangleleft M$. If $x \in \text{Ker}(s^2 - s)$, then $(1 - s)x \in \text{Ker}(s)$, $sx \in \text{Ker}(1 - s)$, and hence $x = (1 - s)x + sx \in \text{Ker}(s) \oplus \text{Ker}(1 - s)$. It shows that $\text{Ker}(s^2 - s) \subseteq \text{Ker}(s) \oplus \text{Ker}(1 - s)$, and thus $\text{Ker}(s) \oplus \text{Ker}(1 - s) \triangleleft M$. Now let $N_1$ and $N_2$ be maximal essential extensions of $\text{Ker}(s)$ and $\text{Ker}(1 - s)$ in $M$, respectively. Then it is clear that $N_1 \cap N_2 = 0$ and $N_1 \oplus N_2 \triangleleft M$. Since $M$ is a $C_1$ module and $N_1$ and $N_2$ are closed submodules of $M$, $N_1$ and $N_2$ are direct summand of $M$. But $M$ is a $C_3$ module, $N_1 \oplus N_2$ is a direct summand of $M$, so that $N_1 \oplus N_2 = M$. This implies that there exists an $e^2 = e \in S$ such that $N_1 = (1 - e)M$ and $N_2 = eM$. Let $y \in \text{Ker}(s), z \in \text{Ker}(1 - s)$, then noting that $y \in (1 - e)M$ and $z \in eM$, we have $(e - s)(y + z) = z - sz = (1 - s)z = 0$, so that $\text{Ker}(s) \oplus \text{Ker}(1 - s) \subseteq \text{Ker}(e - s)$. And hence $e - s \in W(S)$, that is, idempotents modulo $W(S)$ lift. □
Corollary 1. Let $M_R$ be a finitely generated $FQ$-injective $C_1$ module with $S = \text{End}(M_R)$. Then $S$ is semiperfect if and only if $S$ is semilocal.

Proof. Since $M_R$ is a finitely generated $FQ$-injective module, by Lemma 4, it is a $C_2$ module and hence a $C_3$ module. Thus $M_R$ is quasi-continuous by the condition that $M_R$ is a $C_1$ module. And so the result follows from Lemma 5 and Lemma 2(2). $\square$

Recall that a ring $R$ is called right MP-injective [11] if every monomorphism from a principal right ideal of $R$ to $R$ extends to an endomorphism of $R$; a ring $R$ is called right MGP-injective [11] if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and any $R$-monomorphism from $a^n R$ to $R$ extends to an endomorphism of $R$; a ring $R$ is said to be right AP-injective [6] if, for any $a \in R$, there exists a left ideal $X_a$ such that $\text{lr}(a) = Ra \oplus X_a$; a ring $R$ is called right AGP-injective if, for any $0 \neq a \in R$, there exists a positive integer $n$ and a left ideal $X_a^n$ such that $a^n \neq 0$ and $\text{lr}(a^n) = Ra^n \oplus X_a^n$. Clearly, right MP-injective rings are right MGP-injective, and right AP-injective rings are right AGP-injective. If $R$ is a right MP-injective rings, then $R$ is right $C_2$ by [11, Theorem 2.7] and $J(R) = Z(R_R)$ by [11, Theorem 3.4]. If $R$ is a right AP-injective rings, then $R$ is right $C_2$ by [9, Corollary 3.4] and $J(R) = Z(R_R)$ by [6, Corollary 2.3]. So by Lemma 5, we have immediately the following corollary.

Corollary 2. Let $R$ be a right CS ring. If $R$ is right MP-injective or right AP-injective, then $R$ is semiperfect if and only if $R$ is semilocal.

Let $M$ be a right $R$-module. A finite set $A_1, \ldots, A_n$ of proper submodules of $M$ is said to be coindependent if for each $i, 1 \leq i \leq n, A_i \cap \bigcap_{j \neq i} A_j = M$, and a family of submodules of $M$ is said to be coindependent if each of its finite subfamily is coindependent. The module $M$ is said to have finite dual Goldie dimension if every coindependent family of submodules of $M$ is finite. Refer to [4] for details concerning the dual Goldie dimension.

Lemma 6 ([4, Propositions 2.43]). A ring $R$ is semilocal if and only if $R_R$ has finite dual Goldie dimension, if and only if $R_R$ has finite dual Goldie dimension.

Theorem 1. Let $M_R$ be a finitely generated $FQ$-injective module with $S = \text{End}(M_R)$. Then the following conditions are equivalent:

(1) $S$ is semilocal.

(2) $M_R$ is finite dimensional.

Furthermore, if $M$ is a $C_1$ module, then these conditions are equivalent to:

(3) $S$ is semiperfect.

Proof. (1) $\Rightarrow$ (2). If $M_R$ is not finite dimensional, then there exists $0 \neq x_i \in M, i = 1, 2, 3, \cdots$, such that $\sum_{i=1}^{\infty} x_i R$ is a direct sum. Since $M_R$ is $FQ$-injective, by Lemma 1, for any positive integer $m$ and any finite subset

$I \subset \mathbb{N}, S = l_S(0) = l_S(x_m R \cap \sum_{i \in I} x_i R) = l_S(x_m) + l_S(\sum_{i \in I} x_i R) = l_S(x_m) + \cap_{i \in I} l_S(x_i).$
Thus $l_S(x_i), i = 1, 2, 3, \ldots$ is an infinite coindexdependent family of submodules of $S$. By Lemma 6, $S$ is not semilocal, a contradiction.

(2) $\Rightarrow$ (1). By Lemma 3.

Furthermore, if $M$ is a $C_1$ module, then since it is a $C_2$ module by Lemma 4 and $W(S) = J(S)$ by Lemma 2(2), we have $(1) \iff (3)$ by Lemma 5.

The equivalence of (1) and (2) in the next Corollary 3 appeared in [8, Corollary 4.5].

**Corollary 3.** Let $R$ be a right $F$-injective ring. Then the following conditions are equivalent:

(1) $R$ is semilocal.

(2) $R$ is right finite dimensional.

Furthermore, if $R$ is a right $CS$ ring, then these conditions are equivalent to:

(3) $R$ is semiperfect.

**Theorem 2.** Let $M_R$ be a $FQ$-injective Kasch module with $S = \text{End}(M_R)$, then

(1) $r_M l_S(K) = K$ for every finitely generated submodule $K$ of $M_R$.

(2) $Sm$ is simple if and only if $mR$ is simple. In particular, $\text{Soc}(M_R) = \text{Soc}(S)$.

(3) $l_M(J(R)) \triangleleft S M$.

Moreover, if $M_R$ is finitely generated, then

(4) $l_S(T)$ is a minimal left ideal of $S$ for any maximal submodule $T$ of $M$.

(5) $l_S(\text{Rad}(M)) \triangleleft S$.

**Proof.** (1). Always $K \subseteq r_M l_S(K)$. If $m \in r_M l_S(K) \setminus K$, let $K \subseteq T \subseteq \text{max} (mR + K)$. By the Kasch hypothesis, let $\sigma : (mR + K)/T \to M$ be monic, and define $\gamma : mR + K \to M$ by $\gamma(x) = \sigma(x + T)$. Since $M_R$ is $FQ$-injective, $\gamma = s \cdot$ for some $s \in S$, so $sK = \gamma(K) = 0$. This gives $sm = 0$ as $m \in r_M l_S(K)$. But $sm = \sigma(m + T) \neq 0$ because $m \notin T$, a contradiction. Therefore, $r_M l_S(K) = K$.

(2). If $mR$ is simple. Then if $0 \neq sm \in Sm$, define $\gamma : mR \to smR$ by $\gamma(x) = sx$. Then $\gamma$ is a right $R$-isomorphism, and hence $\gamma^{-1}$ extends to an endomorphism of $M$. Thus, $m = \gamma^{-1}(sm) = \alpha(sm)$ for some $\alpha \in S$, and so $Sm$ is simple. Conversely, if $Sm$ is simple. By (1), $r_M l_S(m) = mR$ for each $m \in M$, which implies that for any $m \in M$, every $S$-homomorphism from $Sm$ to $M$ is right multiplication by an element of $R$. Now for any $0 \neq ma \in mR$, the right multiplication $\cdot a : Sm \to Sm$ is a left $S$-isomorphism. So let $\theta : Sm \to Sm$ be its inverse, then $\theta$ is a right multiplication by an element $b$ of $R$. Thus, $m = \theta(ma) = mab \in (ma)R$. Hence $mR$ is simple.

(3). Let $0 \neq m \in M$. Suppose that $T$ is a maximal submodule of $mR$. By the Kasch hypothesis, let $\sigma : mR/T \to M$ be monic, and define $f : mR \to M$ by $f(x) = \sigma(x + T)$. Since $M_R$ is $FQ$-injective, $f = s \cdot$ for some $s \in S$, and then $sm = f(m) = \sigma(m + T) \neq 0$. But
Let $T$ be any maximal submodule of $M$. Since $M_R$ is Kasch, there exists a monomorphism $\varphi : M/T \to M$. Define $\alpha : M \to M$ by $x \mapsto \varphi(x + T)$. Then $0 \neq \alpha \in S$, $\alpha T = \varphi(0) = 0$, and so $I_S(T) \neq 0$. Let $\Omega = \{K \mid 0 \neq K = I_S(X) \text{ for some } X \subseteq M\}$, then $I_S(T)$ is minimal in $\Omega$ for any maximal submodule $T$ of $M$. In fact, if $I_S(T) \supseteq I_S(X) \neq 0$, where $X \subseteq M$, then $T \subseteq r_M I_S(X) \neq M$. So $T = r_M I_S(X)$, and hence $I_5(T) = I_S(X)$. Since $S$ is left finite dimensional, there exist some minimal members $I_1, I_2, \ldots, I_n$ in $\Omega$ such that $I = \oplus_{i=1}^n I_i$ is a maximal direct sum of minimal members in $\Omega$. Now we establish the following claims:

**Claim 1.** $r_M(I_i)$ is a maximal submodule of $M$ for each $i$.

Since $M$ is finitely generated and Kasch, $r_M(I_i) \subseteq T_i = r_M I_S(T_i)$ for some maximal submodule $T_i$. Thus $I_i \supseteq I_S r_M I_S(T_i) = I_S(T_i) \neq 0$, and so $I_i = I_S(T_i)$ by the minimality of $I_i$ in $\Omega$. Now we choose $0 \neq a_i \in I_S(T_i)$. Then $T_i = r_M(a_i)$, and hence $r_M(I_i) = r_M I_S(T_i) = r_M I_S r_M(a_i) = r_M(a_i) = T_i$.

**Claim 2.** $\text{Rad} M = \bigcap_{i=1}^n r_M(I_i)$.

Clearly, $\text{Rad} M \subseteq \bigcap_{i=1}^n r_M(I_i)$. If $T$ is a maximal submodule of $M$, then $I_S(T) \cap I \neq 0$. Taking some $0 \neq b \in I_S(T) \cap I$, we have $T = r_M(b) \supseteq \bigcap_{i=1}^n r_M(I_i)$. This gives that $\bigcap_{i=1}^n r_M(I_i) \subseteq \text{Rad} M$, and the claim follows.

Finally, observing that each $M/r_M(I_i)$ is simple by Claim 1, and the mapping

$$f : M/\text{Rad} M \to \bigoplus_{i=1}^n M/r_M(I_i); m + \text{Rad} M \mapsto (m + r_M(I_1), \ldots, m + r_M(I_n))$$

is a monomorphism by Claim 2, we have that $M/\text{Rad} M$ is semisimple. 

**Lemma 7.** Let $M_R$ be a finitely generated Kasch module with $S = \text{end}(M_R)$. If $S$ is left finite dimensional, then $M/\text{Rad} M$ is semisimple.

**Proof.** Let $T$ be any maximal submodule of $M$. Since $M_R$ is Kasch, there exists a monomorphism $\varphi : M/T \to M$. Define $\alpha : M \to M$ by $x \mapsto \varphi(x + T)$. Then $0 \neq \alpha \in S$, $\alpha T = \varphi(0) = 0$, and so $I_S(T) \neq 0$. Let $\Omega = \{K \mid 0 \neq K = I_S(X) \text{ for some } X \subseteq M\}$, then $I_S(T)$ is minimal in $\Omega$ for any maximal submodule $T$ of $M$. In fact, if $I_S(T) \supseteq I_S(X) \neq 0$, where $X \subseteq M$, then $T \subseteq r_M I_S(X) \neq M$. So $T = r_M I_S(X)$, and hence $I_5(T) = I_S(X)$. Since $S$ is left finite dimensional, there exist some minimal members $I_1, I_2, \ldots, I_n$ in $\Omega$ such that $I = \oplus_{i=1}^n I_i$ is a maximal direct sum of minimal members in $\Omega$. Now we establish the following claims:

**Claim 1.** $r_M(I_i)$ is a maximal submodule of $M$ for each $i$.

Since $M$ is finitely generated and Kasch, $r_M(I_i) \subseteq T_i = r_M I_S(T_i)$ for some maximal submodule $T_i$. Thus $I_i \supseteq I_S r_M I_S(T_i) = I_S(T_i) \neq 0$, and so $I_i = I_S(T_i)$ by the minimality of $I_i$ in $\Omega$. Now we choose $0 \neq a_i \in I_S(T_i)$. Then $T_i = r_M(a_i)$, and hence $r_M(I_i) = r_M I_S(T_i) = r_M I_S r_M(a_i) = r_M(a_i) = T_i$.

**Claim 2.** $\text{Rad} M = \bigcap_{i=1}^n r_M(I_i)$.

Clearly, $\text{Rad} M \subseteq \bigcap_{i=1}^n r_M(I_i)$. If $T$ is a maximal submodule of $M$, then $I_S(T) \cap I \neq 0$. Taking some $0 \neq b \in I_S(T) \cap I$, we have $T = r_M(b) \supseteq \bigcap_{i=1}^n r_M(I_i)$. This gives that $\bigcap_{i=1}^n r_M(I_i) \subseteq \text{Rad} M$, and the claim follows.

Finally, observing that each $M/r_M(I_i)$ is simple by Claim 1, and the mapping

$$f : M/\text{Rad} M \to \bigoplus_{i=1}^n M/r_M(I_i); m + \text{Rad} M \mapsto (m + r_M(I_1), \ldots, m + r_M(I_n))$$

is a monomorphism by Claim 2, we have that $M/\text{Rad} M$ is semisimple. 

**Theorem 3.** Let $M_R$ be a finitely generated and FQ-injective Kasch module with $S = \text{End}(M_R)$. Then the following conditions are equivalent:
(1) $M / \text{Rad}(M)$ is semisimple.

(2) $S$ is left finitely cogenerated.

(3) $S$ is left finite dimensional.

In this case, $\text{Soc}(S) = l_S(\text{Rad}(M))$, and $G(S) = c(S\text{Soc}(S)) = c(M / \text{Rad}(M))$.

Proof. (1) $\Rightarrow$ (2). It is trivial in case $M = 0$. If $M \neq 0$, then $M / \text{Rad}M \neq 0$ because $M$ is finitely generated. As $M / \text{Rad}M$ is semisimple, there exist maximal submodules $T_1, T_2, \ldots, T_n$ such that $M / \text{Rad}M \cong \bigoplus_{i=1}^n M / T_i$. Hence, by Theorem 2(4),

$$l_S(\text{Rad}M) \cong s\text{Hom}_R(M / \text{Rad}M, sM_R) \cong s\text{Hom}_R(\bigoplus_{i=1}^n M / T_i, sM_R) \cong \bigoplus_{i=1}^n l_S(T_i)$$

is an $n$-generated semisimple left ideal of $S$. This implies that $l_S(\text{Rad}M) = \text{Soc}(S) \triangleleft_S S$ by Theorem 2(5), and therefore $S$ is left finitely cogenerated, and $G(S) = n = c(S\text{Soc}(S))$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). See Lemma 7. □

**Corollary 4.** Let $R$ be a right $F$-injective right Kasch ring. Then the following conditions are equivalent:

(1) $R$ is semilocal.

(2) $R$ is left finitely cogenerated.

(3) $R$ is left finite dimensional.

In this case, $\text{Soc}(R) = l_R(J(R))$, and $G(R) = c(R\text{Soc}(R)) = c(R / J(R))$.

**References**


