Finiteness Conditions for Unions of Two Semigroups and Ranks of $B(G, n)$

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Abstract. In this paper we try to find the finiteness conditions for union of two finite semigroups with a specially defined binary equation. Moreover we find the ranks of the semigroup $B(G, n)$.

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1. Introduction

Finiteness conditions of semigroups (the properties of semigroups which all finite semigroups have) have been considered for certain classes of semigroup constructions. (for examples see [1, 2]). In this paper periodicity, residual finiteness and solvability of word problem of union of two finite semigroups are determined.

Let $S$ and $T$ be two finite semigroups with empty intersection. We define a binary equation on $S \cup T$ as follows:

If $s_1 \in S$ and $s_2 \in S$ then $s_1.s_2$ is considered as the same operation defined on $S$. If $t_1 \in T$ and $t_2 \in T$ then $t_1.t_2$ is considered as the same operation defined on $T$. If $s \in S$ and $t \in T$ then $st = ts = t$. In [3] it is shown that any finitely presented semigroup $S$ is embedded into an inefficient semigroup, namely, the semigroup $S \cup SL_n$ where $SL_n$ is the free semilattice of rank $n$.

Let $S$ be a finite semigroup. A subset $U$ of $S$ is called independent if, for every $u$ in $U$, the element $u$ does not belong to the semigroup $< U \setminus \{u\} >$ generated by the remaining elements of $U$ (see [4]). In [5] Howie and Ribeiro introduced $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$ defined as follows:

- $r_1(S) = \max \{ k : \text{every subset } U \text{ of } S \text{ of cardinality } k \text{ is independent} \}$
- $r_2(S) = \min \{ k : \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ which generates } S \}$

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Theorem 2. A congruence \( \phi \) is a homomorphism \( \rho \) with finite index (that is \( \rho \) has finitely many equivalence classes) such that \( \phi(s) \neq \phi(t) \). (Residual finiteness of completely (0)-simple semigroups, which are Rees matrix semigroups \( M[G; I, J, P] \) over groups was investigated in [2].)

Theorem 1. Let \( S \) and \( T \) be finite semigroups. Then \( S \) and \( T \) are periodic if and only if \( S \cup T \) is periodic.

Proof. (\( \Rightarrow \)) Let \( S \) and \( T \) be periodic. Let \( x \in S \cup T \). Then \( x \in S \) or \( x \in T \). If \( x \in S \), since \( S \) is periodic there exists \( m, n \in \mathbb{N} \) such that \( x^m = x^n \). If \( x \in T \), since \( T \) is periodic there exists \( k, l \in \mathbb{N} \) such that \( x^k = x^l \). So \( S \cup T \) is periodic.

(\( \Leftarrow \)) Let \( S \cup T \) be periodic. Let \( x \in S \). Since \( S \subseteq S \cup T \) we have \( x \in S \cup T \). Since \( S \cup T \) is periodic there exists \( k_1, k_2 \in \mathbb{N} \) such that \( x^{k_1} = x^{k_2} \). We obtain \( S \) is periodic. Let \( y \in T \). Since \( T \subseteq S \cup T \) we have \( y \in S \cup T \). Since \( S \cup T \) is periodic there exists \( k_3, k_4 \in \mathbb{N} \) such that \( y^{k_3} = y^{k_4} \). Thus \( T \) is also periodic.

3. Residual Finiteness

We call a semigroup residually finite if, for each pair \( s \neq t \in S \) there exists a homomorphism \( \phi \) from \( S \) onto a finite semigroup such that \( \phi(s) \neq \phi(t) \), or equivalently, there exists a congruence \( \rho \) with finite index (that is \( \rho \) has finitely many equivalence classes) such that \( (s, t) \notin \rho \). (Residual finiteness of completely (0)-simple semigroups, which are Rees matrix semigroups \( M[G; I, J, P] \) over groups was investigated in [2].)

Theorem 2. \( S \cup T \) is residually finite if and only if \( S \) and \( T \) are residually finite.

Proof. (\( \Rightarrow \)) Assume that \( S \cup T \) is residually finite. Since \( S \) and \( T \) are subsemigroups of \( S \cup T \) then \( S \) and \( T \) are residually finite.

(\( \Leftarrow \)) Assume that \( S \) and \( T \) are residually finite semigroups. We will show that \( S \cup T \) is residually finite. Let \( s_1, s_2 \in S \cup T \) and \( s_1 \neq s_2 \). Since \( S \) is residually finite there is a finite semigroup \( K \) and an onto homomorphism \( \phi : S \to K \) such that \( \phi(s_1) \neq \phi(s_2) \). Let \( \Psi : S \cup T \to K \cup \{0\} \). If \( x \in S \)
let $\Psi(x) = \phi(x)$ and if $x \in L$ let $\Psi(x) = 0$. Then $\Psi(s_1) = \phi(s_1) \neq \phi(s_2) = \Psi(s_2)$. If $s_1, s_2 \in S$ then $\Psi(s_1s_2) = \phi(s_1)\phi(s_2)$. If $t_1, t_2 \in T$ then $\Psi(t_1t_2) = 0 = \psi(t_1)\psi(t_2) = 0.0$. If $s \in S$ and $t \in T$ then $\Psi(st) = \Psi(t) = 0 = \Psi(s)\Psi(t)$. So $\Psi$ is an onto homomorphism.

Let $t_1, t_2 \in S \cup T$ and $t_1 \neq t_2$. Since $T$ is residually finite there is a finite semigroup $L$ and an onto homomorphism $\theta: T \to L$ such that $\theta(t_1) \neq \theta(t_2)$. We define $\alpha : S \cup T \to L \cup \{1\}$ as follows. If $x \in S$ let $\alpha(x) = 1$ and if $x \in T$ let $\alpha(x) = \theta(x)$. It is clear that $\alpha(t_1) = \theta(t_1) \neq \theta(t_2) = \alpha(t_2)$. If $s_1, s_2 \in S$ then $\alpha(s_1s_2) = \alpha(s_1)\alpha(s_2) = 1.1 = 1$. If $t_1, t_2 \in T$ then $\alpha(t_1t_2) = \theta(t_1t_2) = \theta(t_1)\theta(t_2)$. If $s \in S$ and $t \in T$ then $\alpha(st) = \alpha(t) = \theta(t) = \alpha(s)\alpha(t) = 1.\theta(t)$. So $\alpha$ is an onto homomorphism.

Let $s, t \in S \cup T$ and $s \neq t$. We define $\mu : S \cup T \to R_2 = \{a, b\}$. Here $R_2 = \{a, b\}$ is the right zero semigroup with 2 elements and $ab = b, ba = a$. If $s \in S$ let $\mu(s) = a$ and if $t \in T$ let $\mu(t) = b$. We have $\mu(s) = a \neq \mu(t) = b$. If $s_1, s_2 \in S$ then $\mu(s_1s_2) = a = \mu(s_1)\mu(s_2) = a.a = a$. If $t_1, t_2 \in T$ then $\mu(t_1t_2) = b = \mu(t_1)\mu(t_2) = b.b = b$. If $s \in S$ and $t \in T$ then $\mu(st) = \mu(t) = b = \mu(s)\mu(t) = a.b = b$. Thus $\mu$ is an onto homomorphism.

\section{Solvable Word Problem}

A semigroup $S$ is said to have a solvable word problem with respect to a generating set $A$ if there exists a algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in $S$ or not. It is a well-known fact that, for a finitely generated semigroup $S$, the solvability of the word problem does not depend on the choice of the finite generating set for $S$. Thus we say that a semigroup $S$ has a solvable word problem with respect to any finite generating set.

\textbf{Theorem 3.} $S \cup T$ has solvable word problem if and only if $S$ and $T$ have solvable word problem.

\textbf{Proof.} ($\Rightarrow$) Let $S \cup T$ have solvable word problem. Since $S$ and $T$ are finitely generated, let $Y_1$ be generating set of $S$ and $Y_2$ be generating set of $T$. Then $Y_1 \cup Y_2$ is a generating set for $S \cup T$. Let $w_1, w_2 \in Y_1^+$. Since $w_1, w_2 \in Y_1^+ \subseteq (Y_1 \cup Y_2)^+$ and $S \cup T$ has solvable word problem there exists an algorithm which decides whether $w_1 = w_2$ holds in $S \cup T$. Since $w_1, w_2 \in Y_1^+$ and $Y_1$ is a generating set for $S$, the algorithm decides whether $w_1 = w_2$ holds in $S$. So $S$ has a solvable word problem. Similarly it is shown that $T$ has a solvable word problem.

($\Leftarrow$) Assume that $S$ and $T$ have solvable word problem. Let $X$ be a finite generating set for $S \cup T$. Then $X_1 = X \cap S$ and $X_2 = X \cap T$ are generating sets for $S$ and $T$. The set $Z = \{x_1x_2 = x_2, x_2x_1 = x_2 \mid x_1 \in X_1, x_2 \in X_2\}$ is finite. For $w_1, w_2 \in X^+$, if we apply some necessary relations from $Z$ we obtain $w_1, w_2 \in X^+$ such that $w_1 = w'_1$ and $w_2 = w'_2$ holds in $T$. $w'_1 \in X_1^+(i = 1, 2)$ or $w'_2 \in X_2^+(i = 1, 2)$. If $w'_1$ and $w'_2$ are not elements of the same free semigroup $X_1^+(i = 1, 2)$ then $w'_1 = w'_2$ does not hold in $S \cup T$. If $w'_1$ and $w'_2$ are in the same free semigroup $X_1^+(i = 1, 2)$ there exists an algorithm which decides whether the relation $w'_1 = w'_2$ holds in $S$ or $T$. Because $S$ and $T$ have solvable word problem. So $S \cup T$ has solvable word problem.

\hfill $\square$
5. Ranks of $B(G, n)$

The semigroup $B(G, n) = \{1, 2, \ldots, n\} \times G \times \{1, 2, \ldots, n\} \cup \{0\}$ is the Brandt semigroup. The binary operation on $B(G, n)$ is defined as follows

$$(i, a, j)(k, b, l) = (i, ab, l)$$ if $j = k$
$$0$$ if $j \neq k$
$$0(i, a, j) = (i, a, j).0 = 0 = 0$$

In [5] $r_5(B(G, n))$ is given. Now we define other ranks of $B(G, n)$.

**Lemma 1.** Let $B(G, n)$ be the Brandt semigroup. Let $A$ be the minimum generating set of $G$. Then $r_1(B(G, n)) = 1$, $r_2(B(G, n)) = 2n.|A|$ and $r_3(B(G, n)) = 2n.|A|$.

**Proof.** Let $A$ be the minimum generating set of $G$. We show the set
$$B = \{(1, a, j),(i, a, 1)|a \in A, 1 \leq i \leq n, 1 \leq j \leq n\}$$
is the minimum generating set for $B(G, n)$. For $(i, g, j) \in B(G, n)$ we have
$$(i, g, j) = (i, a_1, 1).(1, a_2, 1).(1, a_3, 1)\ldots(1, a_m, j)$$
$(a_i \in A, i = 1, 2, \ldots m)$.

So $B$ is a generating set for $B(G, n)$. Let $C$ be a generating set for $B(G, n)$. Since $(i, a, 1) = (i, a, 1).(1, 1, 1)(a \in A)$ and $(1, 1, j) = (1, 1, 1).(1, 1, j)$. So we have $B \subseteq C$. Thus $B$ is the minimum generating set for $B(G, n)$. We have $r_2(B(G, n)) = 2n.|A|$.

Let $D$ be a generating set for $B(G, n)$ and assume that $D$ is independent. Since $D$ is a generating set and $B$ is the minimum generating set then $B \subseteq D$. Let $(i', g', j') \in D B$. Let $g = a_1'a_2'\ldots a'_i'(a_i' \in A)$. Then $(i', g, j') = (i', a_1', 1).(1, a_2', 1)(1, a_3', 1)\ldots(1, a_i', 1')$. This contradicts with the assumption of $D$ to be independent. So $B$ is the unique independent generating subset of $B(G, n)$. Thus $r_3(B(G, n)) = 2n.|A|$.

Let $(i, g, j) \in B(G, n)$. If $i = j$ then $(i, g, j). (i, g, j) = (i, g^2, j) \neq (i, g, j)$ unless $g^2 = g$. If $i \neq j$ then $(i, g, j). (i, g, j) = 0$. So $B(G, n)$ is not a band. Since $r_2(B(G, n)) \neq |B(G, n)|$ then $B(G, n)$ is not royal. (see [5]) So $r_1(B(G, n)) = 1$.

In the following theorem we determine $r_4(B(G, n))$.

**Theorem 4.** Let $B(G, n)$ be a Brandt semigroup. Let $r_4(G) = k$. Then $r_4(B(G, n)) = k + 1$.

**Proof.** Let $U$ be the maximum independent subset of $G$. Since $r_4(G) = k$ then $|U| = k$. We will show that $U' = \{(1, u, 1)|u \in U\} \cup \{0\}$ is the maximum independent subset of $B(G, n)$. Let $(1, u, 1) = (1, u_1, 1).(1, u_2, 1)(u_1, u_2 \in U)$. Then $u = u_1.u_2$. Since $U$ is independent then $u = u_1$ or $u = u_2$. We obtain $U'$ is independent. Let $U'' \subseteq B(G, n)$ be another independent set. We have $U' \cup (S \setminus U') = S = B(G, n)$. Let $s \in S \setminus U'$. Let

$$s = (i, g, j)(1 \leq i \leq n, g \in G\setminus U, 1 \leq j \leq n).$$

Since $g \in G\setminus U$ then $g = g_1.g_2(g_1, g_2 \in G, g_1 \neq g, g_2 \neq g)$. We have $(i, g, j) = (i, g_1, 1).(j, g_2, j)$ and $U'' = ((U' \cup \{0\}) \cap U'') \cup (U'' \cap (S \setminus U'))$. Since the elements of $S \setminus (U' \cup \{0\})$ can be written
as a product of two elements. So $U'' \cap (S\setminus(U' \cup \{0\})) = \emptyset$. Then $U'' = (U' \cup \{0\}) \cap U''$. So $U'' \subseteq (U' \cup \{0\})$. We obtain $U' \cup \{0\}$ is the maximum independent set. So $r_4(B(G, n)) = k + 1$.

The studies on finiteness conditions of semigroups and ranks of semigroups may be expanded to different classes of semigroups as future work.

References


