On the Number of Pairs of Points in a Quadratic Equation with Rational Distance

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Abstract. In this paper is shown a solution to the number of pair of points in a quadratic equation with rational distance, this result have an important impact to solve the open problem “Points on a parabola” [3] proposed in The Center for Discrete Mathematics and Theoretical Computer Science (DIMACS), because it’s an approach to set down basis in the problem.

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1. Introduction

Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by the function $f(x) = ax^2 + bx + c$; where $x > 0$ and $a, b, c \in \mathbb{R}$ with $a \neq 0$, then the question is: how many pairs of points, so that the distance between them is a rational number?, although exist some references about quadratic equations and distances[1–3, 6, 7, 9, 10], there is no information about this specifically question and the solution of this problem allows to start to solve the still open problem “Points on a parabola” [3], that it’s about to find the maximum number of points that satisfies the condition to have a rational distance between any of them.

2. Main Result

Theorem 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = ax^2 + bx + c$; where $x \in \mathbb{Z}^+$ and $a, b, c \in \mathbb{R}$ with $a \neq 0$ ⇒ exist infinite pairs of points within the polynomial, where the distance between them is a rational number.

By reductio ad absurdum, we suppose that the quadratic equation with form $f(x) = ax^2 + bx + c$ have finite pairs of points that satisfies the condition that its distance is a rational number.

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Select two points in the polynomial: \((r, ar^2 + br + c)\) and \((s, as^2 + bs + c)\); where \(r, s \in \mathbb{Z}^+\)

Define the distance function for this case [5]

\[
d = \sqrt{(s - r)^2 + (as^2 + bs + c - ar^2 - br - c)^2}
\]

Cancel the constants: \(c - c = 0\)

\[
d = \sqrt{(s - r)^2 + (as^2 + bs - ar^2 - br)^2}
\]

Factorize by common factor:

\[
d = \sqrt{(s - r)^2 + (a(s^2 - r^2) + b(s - r))^2}
\]

Factorize by difference of squares:

\[
d = \sqrt{(s - r)^2 + (a(s - r)(s + r) + b(s - r))^2}
\]

Again, factorize by common factor:

\[
d = \sqrt{(s - r)^2 + ((s - r)(a(s + r) + b))^2}  \tag{1}
\]

Now define \(d = \frac{p}{q}\); where \(p, q \in \mathbb{Z}\) and \(q \neq 0\), to find all points where the distance between them is a rational number. Replace \(d\) in the equation by (1).

\[
\frac{p}{q} = \sqrt{(s - r)^2 + ((s - r)(a(s + r) + b))^2}
\]

Squaring both sides:

\[
\frac{p^2}{q^2} = (s - r)^2 + ((s - r)(a(s + r) + b))^2
\]

Reorganizing the equation:

\[
\left(\frac{p}{q}\right)^2 = (s - r)^2 + ((s - r)(a(s + r) + b))^2  \tag{2}
\]

Without loss of generality, we will use the equation [5]

\[
(5n)^2 = (-3n)^2 + (4n)^2  \tag{3}
\]

Where \(n \in \mathbb{Q}\), to represent that this family of pairs of points is infinite even if it’s a subset of all points that satisfies the condition to have a rational distance between them inside the quadratic equation.

Matching the equations (2) and (3)

\[
\frac{p}{q} = 5n  \tag{4}
\]
\[ s - r = -3n \]  \hspace{1cm} (5)  
\[ (s - r)(a(s + r) + b) = 4n \]  \hspace{1cm} (6)  

Replace (5) in (6):  
\[ -3n(a(s + r) + b) = 4n \]  

Divide by \( 3n \) in both sides:  
\[ a(s + r) + b = -\frac{4}{3} \]  

Deduct \( b \) in both sides:  
\[ a(s + r) = -\frac{4}{3} - b \]  

Divide by \( a \) in both sides:  
\[ s + r = -\frac{4}{3a} + \frac{b}{a} \]  

Reorganizing the equation:  
\[ s + r = -\frac{\frac{4}{3} + \frac{3b}{3}}{a} \]  
\[ = -\frac{\frac{4 - 3b}{3}}{a} \]  
\[ = -\frac{4 - 3b}{3a} \]  \hspace{1cm} (7)  

Define \( j = -\frac{4 - 3b}{3a} \); where \( j \in \mathbb{R} \) and replace in the equation (7).  
\[ s + r = j \]  \hspace{1cm} (8)  

Do (5)+(8)  
\[ 2s = j - 3n \]  

Divide by 2 in both sides:  
\[ s = \frac{j - 3n}{2} \]  \hspace{1cm} (9)  

Now, do (8)-(5)  
\[ 2r = j + 3n \]  

Divide by 2 in both sides:  
\[ r = \frac{j + 3n}{2} \]  \hspace{1cm} (10)  

The equations (9) and (10) present some restriction:  
\[ j > 0 \]  \hspace{1cm} (11)  
\[ r > 0 \]  \hspace{1cm} (12)
Replace (10) in (12)
\[ \frac{j + 3n}{2} > 0 \]
Multiply 2 in both sides:
\[ j + 3n > 0 \]
Deduct \( j \) in both sides:
\[ 3n > -j \]
Divide by 3 in both sides
\[ n > \frac{-j}{3} \] (14)

Now, replace (9) in (13)
\[ \frac{j - 3n}{2} > 0 \]
Multiply 2 in both sides:
\[ j - 3n > 0 \]
Add 3n in both sides:
\[ j > 3n \]
Divide by 3 in both sides
\[ \frac{j}{3} > n \] (15)

From (14) and (15)
\[ -\frac{j}{3} < n < \frac{j}{3} \] (16)

**Lemma 1.** The set \( Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \) is countably infinite.

**Proof.** Let \( s \in Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \), then each \( s \) will be written in the (unique) form \( \frac{p}{q} \), where \( p, q \in Z^+ \) and have no common divisor other than 1 [8, 11]. Now, define \( f : Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \rightarrow Z^+ \times Z^+ \) by \( f \left( \frac{p}{q} \right) = (p, q) \), and let \( K = \text{range } f \). For \( \frac{p}{q} \in Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \), we find that \( f \left( \frac{p}{q} \right) = f \left( \frac{u}{v} \right) \Rightarrow (p, q) = (u, v) \Rightarrow p = u \) and \( q = v \Rightarrow \left( \frac{p}{q} \right) = \left( \frac{u}{v} \right) \), so \( f \) is a one-to-one function. Therefore \( |Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right)| = |K| \), a subset of the countable set \( Z^+ \times Z^+ \) (by Theorem A3.5 in [4] we know that \( Z^+ \times Z^+ \) is countable). From [4, Theorem A3.5] it now follows that the set \( Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \) is countable.

Define \( g : Z^+ \rightarrow Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \) by \( g(x) = \frac{-j(x-1)}{3(x+1)} \), where \( x \in Z^+ \) and let \( L = \text{range } g \). For \( c, d \in Z^+ \), we find that if \( g(c) = g(d) \Rightarrow c = d \), so \( g \) is a one-to-one function. Consequently \( |Z^+| = |L| \), then we can notice that \( L \) is countably infinite, but \( L \in Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \), therefore \( Q \cap \left( -\frac{j}{3}, \frac{j}{3} \right) \) is also infinite. \( \square \)
Now, it’s known that $n \in \mathbb{Q} \cap \left(\frac{-1}{3}, \frac{1}{3}\right)$, then $r, s$ could take infinite values, where the distance between them is a rational number, because they depend of $n$.

References


