Low Dimensional Homology Groups of the Orthosymplectic Lie Superalgebra $\mathfrak{osp}(1, 2)$

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Abstract. We realize the Lie superalgebra $\mathfrak{osp}(1, 2)$ in terms of first order differential operators and endow it with the Lie superbracket of vector fields to determine the basis (co)cycles of low dimensional (co)homology groups of $\mathfrak{osp}(1, 2)$ with trivial coefficients, using the complex introduced by Tanaka [7]. Our calculations agree with the result obtained by Fuks and Leites in [2].

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1. Introduction and Generalities

Given a Lie superalgebra $\mathfrak{g}$ over a field $k$ of characteristic 0, D. Fuks [1] introduced a Koszul complex associated to $\mathfrak{g}$. Using this complex, Fuks and Leites [2] calculated the cohomology groups with trivial coefficients of the classical Lie superalgebras. In particular, they found that

$$H^*(\mathfrak{osp}(1, 2)) \cong H^*(\mathfrak{sp}(2)).$$

In [7], J. Tanaka introduced another Koszul complex for $\mathfrak{g}$. In this work, we use this complex and take advantage of the small basis of the superalgebra $\mathfrak{osp}(1, 2)$ to calculate low dimensional (co)homology groups of $\mathfrak{osp}(1, 2)$ with coefficients in $\mathbb{R}$. The result obtained agrees with (1). In particular, our calculations provide explicitly three generators of the group $H^3(\mathfrak{osp}(1, 2); \mathbb{R})$ in terms of the basis of $\mathfrak{osp}(1, 2)$. Note that in the non trivial case where these (co)homology groups are non zero, results providing the generators have been limited to the second degree [3, 4, 6, 8, 9].

Let us recall a few definitions. A Lie superalgebra [5] $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded algebra over a commutative ring or field such as $\mathbb{R}$ or $\mathbb{C}$ with a direct sum decomposition...
\[ \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \]
together with a bilinear operation \([-, -]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) such that \([\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}\), and satisfying

i) \([X, Y] = (-1)^{|X||Y|}[Y, X] = 0\) (super antisymmetry)

ii) \([X, [Y, Z]] = [[X, Y], Z] + (-1)^{|Y||Z|}Y[X, Z]\) (super Jacobi identity).

The elements \(X\) and \(Y\) are said to be homogeneous and the parity \(|X|\) of an homogeneous element \(X\) is 0 or 1 according to whether it is in \(\mathfrak{g}_0\) or \(\mathfrak{g}_1\), in which case \(X\) is said to be even or odd respectively. The Grassmann algebra \(\wedge^*(\mathfrak{g})\) is defined as the quotient of the tensor algebra \(\mathfrak{g} \otimes \mathfrak{g}\) by the two-sided ideal of \(\mathfrak{g} \otimes \mathfrak{g}\) generated by

\[ \{X \otimes Y + (-1)^{|X||Y|}Y \otimes X, X, Y \in \mathfrak{g}\}. \]

Let \(\text{Vect}(\mathfrak{g})\) be the superspace of vector fields on \(\mathfrak{g}\). The superbracket of two vector fields \(X\) and \(Y\) is bilinear and defined for two homogeneous vector fields by:

\[ [X, Y] = X \circ Y - (-1)^{|X||Y|}Y \circ X. \quad (2) \]

2. Lie Superalgebra Homology

For any Lie superalgebra \(\mathfrak{g}\) over a ring \(k\) and \(V\) any \(\mathfrak{g}\)-module, J. Tanaka [7] defined the Lie algebra homology of \(\mathfrak{g}\) with coefficients in \(V\), written \(H_\ast(\mathfrak{g}; V)\), as the homology of the complex \(\wedge^\ast(\mathfrak{g}) \otimes V\), namely

\[
0 \leftarrow V \xleftarrow{d} \mathfrak{g}^1 \otimes V \xleftarrow{d} \mathfrak{g}^2 \otimes V \xleftarrow{d} \cdots \xleftarrow{d} \mathfrak{g}^{n-1} \otimes V \xleftarrow{d} \mathfrak{g}^n \otimes V \xrightarrow{d} \cdots
\]

where \(\mathfrak{g}^n\) is the \(n\)th exterior power (as defined above) of \(\mathfrak{g}\) over \(k\), and where

\[
d(g_1 \wedge \ldots \wedge g_n \otimes v) = \sum_{1 \leq j \leq n} (-1)^{i_1 + \ldots + i_{j-1}} g_1 \wedge \ldots \wedge \hat{g}_j \ldots \wedge g_n \otimes [g_j, v]
\]

\[
+ \sum_{1 \leq i < j \leq n} (-1)^{i+j+i_1+\ldots+i_{j-1}} [g_i, g_j] \wedge g_1 \wedge \ldots \wedge \hat{g}_i \ldots \wedge \hat{g}_j \ldots \wedge g_n \otimes v,
\]

where \(i_j = |X_j|\), \(j_i = i_j + \ldots + i_n\), \(i_1 = \sum_{1 \leq j \leq n} i_j\), \(i_1 = \sum_{1 \leq j \leq n} i_j\), and \(\hat{g}_i\) means that the variable \(g_i\) is deleted. In particular if \(V = k\) the trivial module, we identify \(g_1 \wedge \ldots \wedge g_n\) with \(g_1 \wedge \ldots \wedge g_n \otimes 1\) and have

\[
d(g_1 \wedge \ldots \wedge g_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j+i_1+\ldots+i_{j-1}} [g_i, g_j] \wedge g_1 \wedge \ldots \wedge \hat{g}_i \ldots \wedge \hat{g}_j \ldots \wedge g_n.
\]

Note that this complex is infinite since for \(g \in \mathfrak{g}_1\) (the odd part of \(\mathfrak{g}\), \(\mathfrak{g} \wedge \mathfrak{g}\) is not always 0 as in the Lie algebra case. The standard Koszul complex for homology of Lie superalgebras is the complex introduced by D. Fuks [1]. For trivial coefficients, it is defined as follows:
0 \leftarrow k \leftarrow C_1(g) \leftarrow C_2(g) \leftarrow \ldots \leftarrow C_{n-1}(g) \leftarrow C_n(g) \leftarrow \ldots

where \( C_n(g) = \bigoplus_{p+q=n} \wedge^p(g_0) \otimes S^q(g_1) \) and where the differentials \( d_n : C_n(g) \rightarrow C_{n-1}(g) \) are given by

\[
d_n((g_1 \wedge \ldots \wedge g_n) \otimes (h_1 \ldots h_q)) = \sum_{1 \leq i < j \leq P} (-1)^{i+j} ([g_i, g_j] \wedge g_1 \wedge \ldots \wedge \hat{g}_i \ldots \wedge \hat{g}_j \ldots \wedge g_p) \otimes (h_1 \ldots h_q)
\]

\[
+ \sum_{1 \leq i \leq n} (-1)^{i-1} (g_1 \wedge \ldots \wedge \hat{g}_i \ldots \wedge g_p) \otimes (g_i, (h_1 \ldots \ldots h_q))
\]

\[
+ \sum_{1 \leq i < j \leq q} (g_1 \wedge \ldots \wedge g_p) \otimes (h_1 \ldots \hat{h}_i \ldots \hat{h}_j \ldots h_q),
\]

for \( n \geq 2, x_i \in g_0, y_j \in g_1. \) In the following subsection, we calculate \( H_\bullet(\mathfrak{osp}(1,2); \mathbb{R}) \), using J. Tanaka’s definition.

### 3. Lie Superalgebra Homology of \( \mathfrak{osp}(1,2) \)

Throughout this section, we assume that \( k = \mathbb{R} \). Recall that \( \mathfrak{osp}(1,2n) \) consists of matrices of the form

\[
M = \begin{bmatrix}
0 & A_1 & A_2 \\
A_1^t & B & C \\
-A_2^t & -B & -D
\end{bmatrix}
\]

where \( A_1 \) and \( A_2 \) are \((1 \times n)\)-matrices, \( B \) is a \((n \times n)\)-matrix, \( C \) and \( D \) are symmetric \((n \times n)\)-matrices. Let \( e_{i,j} \) be matrices whose entries are 1 for \( i = j \) and 0 else. Then for \( n \geq 2 \), the following forms a basis of \( \mathfrak{osp}(1,2n) \):

\[
\mathfrak{B} = \{ e_{i,i} - e_{i+n,i+n}, \ e_{i,i+n}, \ e_{i+n,i}, \ e_{i,j} - e_{j+n,i+n}, \ e_{i,j+n} + e_{i,j+n}, \\
\phantom{=} e_{n+i,j} + e_{n+j,i}, \ e_{i,j} - e_{n+j,1}, \ e_{1,j+n} + e_{j,1}; \ 2 \leq i, j \leq n, \ i < j \}. 
\]

Assume that \( \mathbb{R}^n \) is given the coordinates \((x_1, x_2, \ldots, x_n)\), and let \( \frac{\partial}{\partial x_i} \) be the unit vector fields parallel to the \( x_i \) axes respectively. It is easy to show in the case \( \mathfrak{osp}(1,2) \) that the Lie superalgebra generated by the family \( \mathfrak{B} \) below of vector fields (endowed with the superbracket of vector fields) is isomorphic to the orthosymplectic Lie superalgebra \( \mathfrak{osp}(1,2) \):

\[
\mathfrak{B} = \{ E_{23}, e_{23}, e_{32}, o_{23}, o_{32} \}
\]

where

\[
E_{23} := x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3},
\]

\[
e_{23} := x_2 \frac{\partial}{\partial x_3},
\]

\[
e_{32} := x_3 \frac{\partial}{\partial x_2}.
\]
Now using the identity (2) and the basis of superbrackets:

\[ o_{23} := x_1 \frac{\partial}{\partial x^2} - x_3 \frac{\partial}{\partial x_1}, \]
\[ o_{32} := x_1 \frac{\partial}{\partial x^3} + x_2 \frac{\partial}{\partial x_1}. \]

The remaining of this section details the proof that there are isomorphisms of super vector space

\[ H_r(osp(1, 2); \mathbb{R}) \cong \begin{cases} \mathbb{R}, & \text{for } r = 0 \\ 0, & \text{for } r = 1, 2 \\ \{E_{23} \wedge e_{23} \wedge e_{32}\} = \{E_{23} \wedge o_{23} \wedge o_{32}\} = \\ \{e_{23} \wedge o_{23} \wedge o_{32} - e_{32} \wedge o_{32} \wedge o_{32}\}, & \text{for } r = 3 \\ 0, & \text{for } r = 4. \end{cases} \]

3.1. Zero and First Homology Groups

Notice that in the Tanaka complex, we have the boundary maps \( d_0 : \mathbb{R} \to 0 \) and \( d_1 : osp(1, 2) \to \mathbb{R} \) with \( d_1(b) = 0 \) for all \( b \in osp(1, 2) \). So \( \ker d_0 = \mathbb{R} \), \( \text{Im} d_1 = 0 \). So

\[ H_0(osp(1, 2); \mathbb{R}) = \frac{\ker d_0}{\text{Im} d_1} = \frac{\mathbb{R}}{0} = \mathbb{R}. \]

Now using the identity (2) and the basis of \( osp(1, 2) \) provided above, we obtain the following superbrackets:

\[
\begin{align*}
[E_{23}, e_{23}] &= 2e_{23} & [E_{23}, o_{32}] &= o_{32} & [e_{32}, o_{32}] &= -o_{23} \\
[E_{23}, e_{32}] &= -2e_{32} & [e_{23}, o_{23}] &= -o_{32} & [o_{23}, o_{32}] &= E_{23} \\
[e_{23}, e_{32}] &= E_{23} & [e_{23}, o_{32}] &= 0 & [o_{23}, o_{23}] &= -2e_{32} \\
[E_{23}, o_{23}] &= -o_{23} & [e_{32}, o_{32}] &= 0 & [o_{32}, o_{32}] &= 2e_{23}
\end{align*}
\]

Remark 1. The set \( \{E_{23}, e_{23}, e_{32}\} \) constitutes the even part of \( osp(1, 2) \) and generates the Lie algebra \( sl(2) \) i.e., \( osp_0(1, 2) \cong sl(2) \). The set \( \{o_{23}, o_{32}\} \) constitutes the odd part of \( osp(1, 2) \) and is isomorphic to a 2-dimensional standard representation of \( sl(2) \).

To calculate the first homology group, notice that from the boundary map \( d_1 \) above, \( \ker d_1 = osp(1, 2) \). Now by definition of Tanaka’s complex, the boundary map \( d_2 \) is given by:

\[
\begin{align*}
d(E_{23} \wedge e_{23}) &= -[E_{23}, e_{23}] = -2e_{23} & d(e_{23} \wedge o_{32}) &= -[e_{23}, o_{32}] = 0 \\
d(E_{23} \wedge e_{32}) &= -[E_{23}, e_{32}] = 2e_{32} & d(e_{32} \wedge o_{23}) &= -[e_{32}, o_{23}] = 0 \\
d(E_{23} \wedge o_{23}) &= -[E_{23}, o_{23}] = 0 & d(e_{32} \wedge o_{32}) &= -[e_{32}, o_{32}] = 0 \\
d(E_{23} \wedge o_{32}) &= -[E_{23}, o_{32}] = o_{32} & d(o_{23} \wedge o_{23}) &= -[o_{23}, o_{23}] = E_{23} \\
d(E_{23} \wedge o_{23}) &= -[E_{23}, o_{23}] = -o_{32} & d(o_{23} \wedge o_{32}) &= -[o_{23}, o_{32}] = -E_{23} \\
d(e_{23} \wedge o_{23}) &= -[e_{23}, o_{23}] = o_{32} & d(o_{32} \wedge o_{32}) &= -[o_{32}, o_{32}] = -2e_{23}.
\end{align*}
\]

From these formulas, it is clear that \( \text{Im} d_2 = osp(1, 2) \). So

\[ H_1(osp(1, 2); \mathbb{R}) = \frac{\ker d_1}{\text{Im} d_2} = \frac{osp(1, 2)}{osp(1, 2)} = 0 \]
3.2. Second Homology Group

From the boundary map $d_2$ above, we have

$$\ker d_2 = \langle E_{23} \land o_{32} + e_{23} \land o_{23}, e_{23} \land o_{32}, e_{32} \land o_{23}, E_{23} \land e_{23} - o_{32} \land o_{32}, e_{23} \land e_{32} - o_{23} \land o_{32} - E_{23} \land e_{23} - o_{32} \land o_{32} \rangle.$$  

Now by definition of Tanaka’s complex, the boundary map $d_3$ is given by:

$$d(E_{23} \land e_{23} \land e_{32}) = 0$$
$$d(E_{23} \land e_{23} \land o_{23}) = -e_{23} \land o_{23} - E_{23} \land o_{32}$$
$$d(E_{23} \land e_{23} \land o_{32}) = -3e_{23} \land o_{32}$$
$$d(E_{23} \land e_{32} \land o_{23}) = 3e_{32} \land o_{23}$$
$$d(E_{23} \land e_{32} \land o_{32}) = e_{32} \land o_{32} - E_{23} \land o_{23}$$
$$d(e_{23} \land e_{32} \land o_{23}) = -E_{23} \land o_{23} + e_{32} \land o_{32}$$
$$d(e_{23} \land e_{32} \land o_{32}) = -E_{23} \land o_{32} - e_{23} \land o_{23}$$
$$d(E_{23} \land o_{23} \land o_{32}) = 0$$
$$d(e_{23} \land o_{23} \land o_{32}) = o_{32} \land o_{32} - E_{23} \land e_{23}$$
$$d(e_{32} \land o_{23} \land o_{32}) = -E_{23} \land e_{32} + o_{23} \land o_{32}$$
$$d(E_{23} \land o_{23} \land o_{23}) = 2o_{23} \land o_{23} + 2e_{32} \land E_{23}$$
$$d(E_{23} \land o_{32} \land o_{32}) = -2o_{32} \land o_{32} + 2E_{23} \land e_{23}$$
$$d(e_{23} \land o_{23} \land o_{32}) = 2o_{23} \land o_{32} - 2e_{23} \land e_{32}$$
$$d(e_{23} \land o_{32} \land o_{32}) = 0$$
$$d(e_{32} \land o_{23} \land o_{32}) = 0$$
$$d(e_{32} \land o_{23} \land o_{23}) = 2o_{23} \land o_{32} - 2e_{23} \land e_{32}$$
$$d(o_{23} \land o_{23} \land o_{32}) = 6e_{32} \land o_{23}$$
$$d(o_{23} \land o_{23} \land o_{32}) = 2e_{32} \land o_{32} - 2E_{23} \land o_{23}$$
$$d(o_{23} \land o_{32} \land o_{32}) = -2e_{23} \land o_{23} - 2E_{23} \land o_{32}$$
$$d(o_{32} \land o_{32} \land o_{32}) = -6e_{32} \land o_{32}.$$  

From the formulas (3)–(9) above, it is clear that $\text{Im} d_3 = \ker d_2$. Therefore

$$H_2(\mathfrak{osp}(1, 2) ; \mathbb{R}) = 0.$$  

3.3. Third Homology Group

From the boundary map $d_3$ above, we have

$$\ker d_3 = \langle E_{23} \land e_{23} \land e_{32}, E_{23} \land o_{23} \land o_{32}, e_{23} \land o_{32} \land o_{32}, e_{32} \land o_{23} \land o_{23} $$,
\[2E_{23} \land e_{23} \land o_{32} - o_{32} \land o_{32} \land o_{32}, \quad 2E_{23} \land e_{32} \land o_{23} - o_{23} \land o_{23} \land o_{23},
\]
\[2E_{23} \land e_{23} \land o_{23} - o_{23} \land o_{32} \land o_{32}, \quad 2E_{23} \land e_{32} \land o_{32} - o_{23} \land o_{23} \land o_{32},
\]
\[E_{23} \land e_{23} \land o_{23} - e_{23} \land e_{32} \land o_{32}, \quad E_{23} \land e_{32} \land o_{32} - e_{23} \land e_{32} \land o_{23},
\]
\[E_{23} \land o_{23} \land o_{23} - 2e_{32} \land o_{23} \land o_{32}, \quad E_{23} \land o_{32} \land o_{32} + 2e_{23} \land o_{23} \land o_{32},
\]
\[e_{23} \land o_{23} \land o_{23} - e_{32} \land o_{32} \land o_{32} > .
\]

Now by definition of Tanaka’s complex, the boundary map \( d_4 \) is given by:

\[d(E_{23} \land e_{23} \land e_{32} \land o_{23}) = e_{23} \land e_{32} \land o_{23} - E_{23} \land e_{32} \land o_{32} \quad (10)
\]
\[d(E_{23} \land e_{23} \land e_{32} \land o_{32}) = -e_{23} \land e_{32} \land o_{32} + E_{23} \land e_{23} \land o_{32} \quad (11)
\]
\[d(E_{23} \land e_{23} \land o_{23} \land o_{32}) = -E_{23} \land o_{32} \land o_{32} - 2e_{23} \land o_{23} \land o_{32} \quad (12)
\]
\[d(E_{23} \land e_{32} \land o_{23} \land o_{32}) = 2e_{32} \land o_{23} \land o_{32} - E_{23} \land o_{23} \land o_{32} \quad (13)
\]
\[d(e_{23} \land e_{32} \land o_{23} \land o_{32}) = -E_{23} \land o_{23} \land o_{32} + e_{32} \land o_{32} \land o_{32} \quad (14)
\]
\[-e_{23} \land o_{23} \land o_{32} - E_{23} \land e_{23} \land o_{32} \quad (15)
\]
\[d(E_{23} \land e_{23} \land o_{23} \land o_{32}) = -2E_{23} \land o_{23} \land o_{32} + 2E_{23} \land e_{23} \land o_{32} \quad (16)
\]
\[d(E_{23} \land e_{23} \land o_{32} \land o_{32}) = -4e_{23} \land o_{32} \land o_{32} \quad (17)
\]
\[d(E_{23} \land e_{23} \land o_{23} \land o_{32}) = 4e_{32} \land o_{23} \land o_{32} \quad (18)
\]
\[d(E_{23} \land e_{32} \land o_{32} \land o_{32}) = -2E_{23} \land o_{32} \land o_{32} + 2E_{23} \land e_{32} \land o_{32} \quad (19)
\]
\[d(e_{23} \land e_{32} \land o_{23} \land o_{32}) = -E_{23} \land o_{23} \land o_{32} + 2e_{32} \land o_{32} \land o_{32} \quad (20)
\]
\[d(e_{23} \land e_{32} \land o_{32} \land o_{32}) = -E_{23} \land o_{32} \land o_{32} - 2e_{23} \land o_{32} \land o_{32} \quad (21)
\]
\[d(E_{23} \land o_{23} \land o_{23} \land o_{32}) = o_{23} \land o_{32} \land o_{32} + 2E_{23} \land e_{23} \land o_{32} \quad (22)
\]
\[d(e_{23} \land o_{23} \land o_{23} \land o_{32}) = -3o_{32} \land o_{32} \land o_{32} + 6E_{23} \land e_{23} \land o_{32} \quad (23)
\]
\[d(e_{23} \land o_{23} \land o_{23} \land o_{32}) = 0.
\]
\[d(e_{32} \land o_{23} \land o_{23} \land o_{32}) = 0.
\]
\[d(e_{32} \land o_{23} \land o_{23} \land o_{32}) = 0.
\]
\[d(e_{32} \land o_{23} \land o_{32} \land o_{32}) = 2o_{23} \land o_{23} \land o_{32} - 2E_{23} \land e_{32} \land o_{32} - 2e_{23} \land e_{32} \land o_{23} \quad (24)
\]
\[d(e_{32} \land o_{32} \land o_{32} \land o_{32}) = 3o_{23} \land o_{32} \land o_{32} - 6e_{23} \land e_{32} \land o_{32} \quad (25)
\]
\[d(o_{23} \land o_{23} \land o_{23} \land o_{23}) = 12e_{32} \land o_{23} \land o_{23} \quad (26)
\]
\[d(o_{23} \land o_{23} \land o_{23} \land o_{32}) = 6e_{32} \land o_{23} \land o_{32} - 3E_{23} \land o_{23} \land o_{23} \quad (27)
\]
\[d(o_{23} \land o_{23} \land o_{32} \land o_{32}) = 2e_{32} \land o_{32} \land o_{32} - 4E_{23} \land o_{32} \land o_{32} - 2e_{23} \land o_{23} \land o_{32} \quad (28)
\]
\[d(o_{32} \land o_{32} \land o_{32} \land o_{32}) = -3E_{23} \land o_{32} \land o_{32} - 6e_{23} \land o_{23} \land o_{32} \quad (29)
\]
\[d(o_{32} \land o_{32} \land o_{32} \land o_{32}) = -12e_{23} \land o_{32} \land o_{32}.
\]
From the formulas (10)–(13), (17)–(25) above, it is clear that all the cycles but $E_{23} \wedge e_{23} \wedge e_{32}$, $E_{23} \wedge o_{23} \wedge o_{32}$, $e_{23} \wedge o_{23} \wedge o_{23} - e_{32} \wedge o_{32} \wedge o_{32}$ are boundaries, so are zero in homology. By (15) and (16) or (26), these remaining three cycles differ by a boundary, so they generate the same homology class. Therefore

$$H_3(osp(1, 2); \mathbb{R}) = \langle E_{23} \wedge e_{23} \wedge e_{32} \rangle = \langle E_{23} \wedge o_{23} \wedge o_{32} \rangle = \langle e_{23} \wedge o_{23} \wedge o_{23} - e_{32} \wedge o_{32} \wedge o_{32} \rangle.$$  

### 3.4. Fourth Homology Group

In this subsection, $e^{\wedge k}$ stands for $e \wedge e \wedge \ldots \wedge e$. From the boundary map $d_4$ above, we have

$$\ker d_4 = \left\{ e_{23} \wedge o_{32}^{\wedge 3}, e_{32} \wedge o_{23}^{\wedge 3}, E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 2}, -E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2}, \right.$$  

$$3E_{23} \wedge e_{23} \wedge o_{32} - o_{32} - o_{32} \wedge o_{32}, 3E_{23} \wedge e_{32} \wedge o_{32} - o_{32} \wedge o_{32},$$  

$$3E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 2} - o_{32} \wedge o_{32}^{\wedge 2}, 3E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2} - o_{32}^{\wedge 2},$$  

$$E_{23} \wedge o_{32}^{\wedge 3} - 3e_{32} \wedge o_{32}^{\wedge 2} \wedge o_{32}, 3E_{23} \wedge o_{32}^{\wedge 3} + 3e_{23} \wedge o_{32} \wedge o_{32}^{\wedge 2},$$  

$$E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2}, E_{23} \wedge e_{32} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32}.$$  

Now by definition of Tanaka’s complex, the boundary map $d_5$ is given by:

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}) = E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - E_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2}) = 2e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} - 2E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2}) = 2E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32} - 2e_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2}) = e_{23} \wedge o_{32}^{\wedge 2} - 3E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} + 6E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32}^{\wedge 2}) = 2E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} + 6E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32}^{\wedge 2}) = -2E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} + 6E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}) = -E_{23} \wedge o_{23}^{\wedge 3} + 3e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}$$  

$$d(E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2}) = 5e_{32} \wedge o_{32}^{\wedge 3}$$  

$$d(E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 3}) = 5e_{32} \wedge o_{32}^{\wedge 3}$$  

$$d(E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}) = 2E_{23} \wedge e_{23} \wedge o_{32} \wedge o_{32} - 2E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}$$  

$$+ e_{32} \wedge o_{32} \wedge o_{32}^{\wedge 2}$$  

$$d(E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 3}) = 6E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32} - 3E_{23} \wedge o_{32} \wedge o_{32}^{\wedge 2} + e_{32} \wedge o_{32}^{\wedge 3}$$  

$$d(e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 3}) = -E_{23} \wedge o_{23}^{\wedge 3} + 3e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}$$  

$$d(e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2}) = -E_{23} \wedge o_{23}^{\wedge 2} - o_{32} - 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}$$  

$$+ 2e_{32} \wedge o_{32} \wedge o_{32}^{\wedge 2} - e_{23} \wedge o_{32}^{\wedge 3}$$  

$$d(e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}) = -E_{23} \wedge o_{23} \wedge e_{32}^{\wedge 2} - 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}$$  

$$- 2e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32} + e_{32} \wedge o_{32}^{\wedge 3}$$
\[ d(e_{23} \land o_{32} \land o_{32}^{3}) = -E_{23} \land o_{32}^{3} - 3e_{23} \land o_{23} \land o_{32}^{2} \]
\[ d(E_{23} \land o_{32}^{4}) = 6E_{23} \land e_{32} \land o_{32}^{2} + 4o_{23}^{4} \]
\[ d(E_{23} \land o_{23}^{3} \land o_{32}) = -6E_{23} \land e_{32} \land o_{32} \land o_{32} + 2o_{23}^{3} \land o_{32} \]
\[ d(E_{23} \land o_{23} \land o_{32}^{3}) = 6E_{23} \land e_{32} \land o_{32} \land o_{32} - 2o_{23} \land o_{32}^{3} \]
\[ d(E_{23} \land o_{23}^{3}) = 6E_{23} \land e_{32} \land o_{32}^{2} - 4o_{32}^{3} \]
\[ d(e_{23} \land o_{23}^{4}) = -6e_{23} \land e_{32} \land o_{23}^{2} + 4o_{23}^{3} \land o_{32} \]
\[ d(e_{23} \land o_{23}^{3} \land o_{32}) = 3o_{23}^{2} \land o_{32} - 6e_{23} \land e_{32} \land o_{32} \land o_{32} \]
\[ + 3E_{23} \land e_{23} \land o_{32}^{2} \]
\[ d(e_{23} \land o_{23}^{2} \land o_{32}^{3}) = -4E_{23} \land e_{23} \land o_{23} \land o_{32} - 2e_{23} \land e_{32} \land o_{32}^{2} \]
\[ + 2o_{23} \land o_{32}^{3} \]
\[ d(e_{23} \land o_{32}^{5}) = 3E_{23} \land e_{23} \land o_{23}^{2} + o_{32}^{4} \]
\[ d(e_{23} \land o_{32}^{4}) = 0 \]
\[ d(e_{32} \land o_{23}^{4}) = 0 \]
\[ d(e_{32} \land o_{23}^{3} \land o_{32}) = 3E_{23} \land e_{32} \land o_{23}^{2} + o_{23}^{4} \]
\[ d(e_{32} \land o_{23}^{2} \land o_{32}^{2}) = 4E_{23} \land e_{32} \land o_{23} \land o_{32} + 2e_{23} \land e_{32} \land o_{23}^{2} \]
\[ - 2o_{23}^{3} \land o_{32} \]
\[ d(e_{32} \land o_{23} \land o_{32}^{3}) = 3o_{23}^{2} \land o_{32}^{2} + 6e_{23} \land e_{32} \land o_{23} \land o_{32} + 3E_{23} \land e_{32} \land o_{32}^{2} \]
\[ d(e_{32} \land o_{32}^{4}) = 6e_{23} \land e_{32} \land o_{23}^{2} + 4o_{32} \land o_{32}^{3} \]
\[ d(o_{32}^{5}) = 20e_{32} \land o_{32}^{3} \]
\[ d(o_{32}^{4} \land o_{32}) = 12e_{32} \land o_{23}^{2} \land o_{32} - 4E_{23} \land o_{32}^{3} \]
\[ d(o_{32}^{3} \land o_{32}) = 6e_{32} \land o_{32} \land o_{32}^{2} - 6E_{23} \land o_{23}^{2} \land o_{32} - 2e_{23} \land o_{32}^{3} \]
\[ d(o_{32}^{2} \land o_{32}^{2}) = -6e_{23} \land o_{32} \land o_{32} + 4e_{32} \land o_{32}^{2} + 2e_{32} \land o_{32}^{3} \]
\[ d(o_{32} \land o_{32}^{2}) = -12e_{23} \land o_{23} \land o_{32} - 4E_{23} \land o_{32}^{3} \]
\[ d(o_{32}^{4}) = -20e_{23} \land o_{32}^{3} \].

From the formulas (24)–(32), (34), (35) above, it is clear that all the cycles but

\[ E_{23} \land e_{23} \land o_{32} \land o_{32} - e_{23} \land e_{32} \land o_{32} \land o_{32}, \]
\[ E_{23} \land e_{32} \land o_{23} \land o_{32} - e_{23} \land e_{32} \land o_{23} \land o_{32} \]

are boundaries, so are zero in homology. For these remaining two cycles, combining (31) and (36) shows that the first is a boundary. Similarly, combining (32) and (33) shows that the second is also boundary. Therefore

\[ H_{4}(\text{osp}(1, 2); \mathbb{R}) = 0. \]
In the following section, we provide the low dimensional cohomology groups with trivial coefficients of $\text{osp}(1, 2)$.

4. Lie Superalgebra Cohomology of $\text{osp}(1, 2)$

**Theorem 1.** There are isomorphisms of super vector spaces

$$H^r(\text{osp}(1, 2); \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{for } r = 0, \\ 0 & \text{for } r = 1, 2, \\ \langle E_{23}^e \wedge e_{23}^e \wedge e_{32}^o \rangle = \langle E_{23}^e \wedge o_{23}^o \wedge o_{32}^o \rangle = \\ \langle e_{23}^e \wedge o_{23}^e \wedge o_{23}^o \rangle = \langle e_{23}^e \wedge o_{23}^o \wedge o_{32}^o \wedge o_{32}^o \rangle, & \text{for } r = 3 \\ 0, & \text{for } r = 4. \end{cases}$$

**Proof.** We use the super vector space isomorphism (see [8, lemma 1.7])

$$H^r(\text{osp}(1, 2); \mathbb{R}) \cong \text{Hom}(H_*(\text{osp}(1, 2); \mathbb{R}), \mathbb{R})$$

and the dual basis

- $E_{23}^e := x_2 d x^2 - x_3 d x^3$
- $e_{23}^e := x_3 d x^2$
- $o_{32}^o := x_4 d x^3 + x_2 d x^1$
- $e_{23}^o := x_2 d x^3$
- $o_{23}^o := x_1 d x^2 - x_3 d x^1$

where $d x^i$ is the dual of $\frac{\partial}{\partial x^i}$ with respect to the basis of $\text{osp}(1, 2)$ given in section 2.  

References


