Local Solvability for the 2-Coupled System of Nonlinear Schrödinger Equations in a Banach Algebra $E_{2,1}^0$

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Abstract. This paper is concerned with initial value problem of the nonlinear coupled Schrödinger equations. We study local well posedness in the Banach algebra $E_{2,1}^0$($\mathbb{R}^n$) which is the extension of $H^s$($\mathbb{R}^n$) when $s \geq \frac{n}{2}$. The method we use is similar to the method of semigroup.

2000 Mathematics Subject Classifications: 35Q53, 35Q35.

Key Words and Phrases: Nonlinear, coupled Schrödinger equations, Banach algebra, local well-posedness

1. Introduction

It is well-known that $H^s$($\mathbb{R}^n$) is an algebra when $s > \frac{n}{2}$ and the Schrödinger operator generate an unitary group in $H^s$($\mathbb{R}^n$). The well-posedness in $H^s$($s \geq \frac{n}{2}$) for the Cauchy problem of the cubic semi-linear Schrödinger equation

$$iu_t + \Delta u = a|u|^2u, \quad x \in \mathbb{R}^n, t \in \mathbb{R}.$$  

were treated by using the method of semigroup [we refer to 6]. Recently, Wang et al in [10] introduce a new Banach algebra $E_{2,1}^0$($\mathbb{R}^n$) which is the extension of $H^s$($\mathbb{R}^n$) when $s \geq \frac{n}{2}$ and investigated the Cauchy problem of semi-linear Schrödinger equation with nonlinear term $|u|^{2k}u, k \in \mathbb{N}$. We shall study a coupled system by Wang’s approach in this paper.

As a natural extension of the single cubic nonlinear Schrödinger equation, the 2-coupled nonlinear Schrödinger equations:

$$\begin{cases}
iu_t + \Delta u = a|u|^2u + |v|^2u, & x \in \mathbb{R}, t \in \mathbb{R}, \\
i\nu_t + \Delta v = |u|^2v + a|v|^2v, & x \in \mathbb{R}, t \in \mathbb{R},
\end{cases}$$  

(1)

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have many applications including, for example, nonlinear optics [cf. 2, 4, 5, 9] and geophysical fluid dynamics [cf. 7, 8]. In the above equations, \( a \in \mathbb{R} \), the unknowns \( u(x, t), v(x, t) \) are the envelopes of wave packets in two different degrees of freedom of the underlying physical systems which we shall call 'modes'. The system is derived as an approximation to a more complex set of equations by singular perturbation theory.

Instead of studying (1), this paper is concerned with the following general Schrödinger system:

\[
\begin{aligned}
    iu_t + \Delta u &= a|u|^\alpha u + |v|^\alpha u, & x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\
    iv_t + \Delta v &= |u|^\alpha v + a|v|^\alpha v, & x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\
    u(0, x) &= \phi(x), & x \in \mathbb{R}^n, \\
    v(0, x) &= \psi(x), & x \in \mathbb{R}^n.
\end{aligned}
\]

As in [10], for technical reason, the restriction \( \alpha = 2k, \ k \in \mathbb{N} \) or \( |u|^\alpha = u^\alpha \) and \( |v|^\alpha = v^\alpha \) or \( \tilde{v}^\alpha \), \( \alpha \in \mathbb{N} \) is required in the nonlinear coupled terms. By denoting \( U = \begin{pmatrix} u \\ v \end{pmatrix} \), \( F(U) = \begin{pmatrix} a|u|^\alpha u + |v|^\alpha u \\ |u|^\alpha v + a|v|^\alpha v \end{pmatrix} \) and \( \Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \), we see readily that (2) take the following form:

\[
\begin{aligned}
    \partial_t U + \Delta U &= F(U) & x \in \mathbb{R}^n, \ t \in \mathbb{R}, \\
    U(0, x) &= \Phi(x) & x \in \mathbb{R}^n.
\end{aligned}
\]

Let

\[
\Lambda(t) = \begin{pmatrix} S(t) & 0 \\ 0 & S(t) \end{pmatrix},
\]

where \( S(t) = e^{it\Delta} \) is the fundamental solution operator of the Schrödinger equation and is given by

\[
S(t)\phi = \int_{\mathbb{R}^n} e^{-it|\xi|^2 + i\xi \cdot \phi} d\xi, \ \forall \ \phi \in S(\mathbb{R}^n).
\]

Then by the Duhanmel principle we see that the Cauchy problem (3) is equivalent to the following integral equation:

\[
U(t) = \Lambda(t)\Phi - i \int_0^t \Lambda(t - \tau)F(U(\tau))d\tau.
\]

Thus, in the sequel we shall solve this integral equation.

We shall use the notation \( || \cdot || \) to denote the norm of 2-dimensional vector functions, and use \( \| \cdot \| \) to denote the norm of scale functions, so that \( \|U\| = \|u\| + \|v\| \) if \( U = \begin{pmatrix} u \\ v \end{pmatrix} \).

The main result of this paper is

**Theorem 1.** Let \( \Phi \in E_{2,1}^0(\mathbb{R}^n) \). Then there exists \( T^* \equiv T^*(\|\Phi\|_{E_{2,1}^0(\mathbb{R}^n)}) > 0 \) such that the Cauchy problem (3) has a unique solution

\[
U \in C([0, T^*), E_{2,1}^0(\mathbb{R}^n)).
\]
Moreover, if \( T^* < \infty \) then
\[
\lim_{t \to T^*} \sup \|U(t)\|_{E_{2,1}^0(R^n)} = \infty.
\]

\( E_{2,1}^0(R^n) \) will be introduced in the next section.

In the sequel, \( C \) will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. For \( p \geq 1 \) we set \( p' = \frac{p}{p-1} \).

### 2. Preliminaries

#### 2.1. The Banach Algebra \( E_{2,1}^0 \)

We denote by \( S(R^n) \) and \( S'(R^n) \) the Schwartz space and its dual space, respectively. Let \( \rho \in S(R^n) \) and \( \rho : R^n \to [0,1] \) be a smooth radial bump function adapted to the ball \( B(0, \sqrt{2n}) \), say \( \rho(\xi) = 1 \) as \( 0 \leq |\xi| \leq \sqrt{2n} \), and \( \rho(\xi) = 0 \) as \( |\xi| \geq \sqrt{2n} \). Let \( \rho_k \) be a translation of \( \rho : \)
\[
\rho_k(\xi) = \rho(\xi - k), k \in Z^n,
\]
where \( k \in Z^n \) means that \( k = (k_1, k_2, \cdots, k_n) \), and \( k_1, k_2, \cdots, k_n \) are all integers. Since \( \rho(\xi) = 1 \) in the unit closed cube \( Q_k \) with center \( k \) and \( \{Q_k\}_{k \in Z^n} \) is a covering of \( R^n \), one has that \( \sum_{k \in Z^n} \rho_k(\xi) \geq 1 \) for all \( \xi \in R^n \). We write
\[
\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{k \in Z^n} \rho_k(\xi) \right)^{-1}, \quad k \in Z^n.
\]

It is easy to see that
\[
\left\{ \begin{array}{l}
|\sigma_k(\xi)| \geq C, \quad \forall \xi \in Q_k; \\
\sup \sigma_k(\xi) \subset \{ \xi : |\xi - k| \leq \sqrt{2n} \}; \\
\sum_{k \in Z^n} \sigma_k(\xi) = 1, \quad \forall \xi \in R^n; \\
|\sigma_k^{(m)}(\xi)| \leq C_m, \quad \forall \xi \in R^n.
\end{array} \right.
\]

Hence, the set
\[
\Upsilon = \{ \{\sigma_k\}_{k \in Z^n} : \{\sigma_k\}_{k \in Z^n} \text{ satisfies } (4) \}
\]
is non-void. Let \( \{\sigma_k\}_{k \in Z^n} \in \Upsilon \) be a function sequence. Define operator:
\[
\Box_k \equiv \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad k \in Z^n,
\]
where the operator \( \mathcal{F} \) means Fourier transformation.

For any \( k \in Z^n \), we write \( |k| = |k_1| + |k_2| + \cdots + |k_n| \). Let \( 0 \leq \lambda < \infty \), \( 0 < p, q \leq \infty \), we introduce the following function space
\[
E_{p,q}^\lambda(R^n) = \left\{ f \in S'(R^n) : \|f\|_{E_{p,q}^\lambda(R^n)} \equiv \left( \sum_{k \in Z^n} [2^{\lambda|k|}\|\Box_k f\|_{L^p(R^n)}]^q \right)^{\frac{1}{q}} < \infty \right\}.
\]
Obviously, the function space $E_{p,q}^{\lambda}(\mathbb{R}^n)$ is modified from the Besov space $B_{p,q}^{s}(\mathbb{R}^n)$ [see 1]. Since the relation between $E_{p,q}^{\lambda}(\mathbb{R}^n)$ and the Besov space $B_{p,q}^{s}(\mathbb{R}^n)$ have nothing to do with our result, we omit it here [for the details, we refer to 10].

The algebra property of $E_{2,1}^{0}$ may deduce from the following embedding property and bilinear estimate.

**Lemma 1.** Let $0 \leq \lambda < \infty$, $0 < p_1 \leq p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$. Then we have

$$E_{p_1,q_1}^{\lambda}(\mathbb{R}^n) \subset E_{p_2,q_2}^{\lambda}(\mathbb{R}^n).$$

**Proof.** See the proof of Proposition 3.5 in [10].

**Lemma 2.** Let $0 \leq \lambda < \infty$, $0 < p \leq p_1$, $p_2 \leq \infty$, $0 < q \leq \infty$. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then we have

$$\|uv\|_{E_{p,q}^{\lambda}} \leq C 2^{Cq\lambda} \|u\|_{E_{p_1,q_1}^{\lambda}} \|v\|_{E_{p_2,q_1}^{\lambda}},$$

where $a \wedge b = \min\{a, b\}$. $C$ is independent of $\lambda$, $q$ and if $p$ is fixed, then $C$ is also independent of $p_1$, $p_2$.

**Proof.** See the proof of Lemma 4.1 in [10].

As a matter of fact, by Lemma 1 and Lemma 2, we have

$$\|uv\|_{E_{2,1}^{0}} \leq C \|uv\|_{E_{2,1}^{0}} \leq C \|u\|_{E_{2,1}^{0}} \|v\|_{E_{2,1}^{0}}. \quad (5)$$

Which suggest that $E_{2,1}^{0}$ is a Banach algebra.

From the comparison between $E_{2,q}^{0}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$, we find that $E_{2,1}^{0}(\mathbb{R}^n)$ is the extension of $H^s(\mathbb{R}^n)$:

$$H^s(\mathbb{R}^n) \subset E_{2,1}^{0}(\mathbb{R}^n) \quad \text{for} \quad s > \frac{n}{2}, \quad \text{and} \quad H^s(\mathbb{R}^n) \subset E_{2,1}^{0}(\mathbb{R}^n) \quad \text{fails, for} \quad s \leq \frac{n}{2}.$$  

Indeed, we have

**Lemma 3.** We have

$$H^s(\mathbb{R}^n) \subset E_{2,q}^{0}(\mathbb{R}^n), \quad s > n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 0 < q < 2,$$

$$L^2(\mathbb{R}^n) = E_{2,2}^{0}(\mathbb{R}^n) \quad \text{(equivalent norm)},$$

$$E_{2,q}^{0}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n), \quad s < n\left(\frac{1}{q} - \frac{1}{2}\right), \quad 2 < q \leq \infty.$$  

Furthermore, $E_{2,1}^{0}$ is the intermediate space of $H^s(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, that is

$$H^s(\mathbb{R}^n) \subset E_{2,1}^{0} \subset L^\infty(\mathbb{R}^n), \quad s > n/2.$$  

[(3.43) in 10].

**Proof.** See the proof of Proposition 3.8 in [10].
2.2. Some Preliminary Lemmas

Estimate for the Schrödinger group

**Lemma 4.** Let $0 < r ≤ 2 ≤ p ≤ ∞$, $0 < q ≤ ∞$. Then for the Schrödinger group $S(t) = e^{tΔ}$ we have the estimate

$$\|S(t)φ\|_{F^0_{p,q}} \leq C\|φ\|_{L^p_{r,q}}.$$

In particular,

$$\|S(t)φ\|_{L^2_{p,1}} \leq C\|φ\|_{L^2_{p,1}}.$$

**Proof.** See the proof of Proposition 5.5 in [10].

From Lemma 4, we deduce that

**Lemma 5.** Let $0 < r ≤ 2 ≤ p ≤ ∞$, $0 < q ≤ ∞$. Then for the group $Λ(t)$ we have the estimate

$$\|Λ(t)φ\|_{F^0_{p,q}} \leq C\|φ\|_{L^p_{r,q}}.$$

In particular,

$$\|Λ(t)φ\|_{L^2_{p,1}} \leq C\|φ\|_{L^2_{p,1}}.$$

With the algebra property, we have the estimates for the nonlinear coupled terms

**Lemma 6.**

$$\|F(U)\|_{E^0_{2,1}} \leq C\|U\|_{E^0_{2,1}}^{α+1},$$

and

$$\|F(U_1) - F(U_2)\|_{E^0_{2,1}} \leq C\|U_1 - U_2\|_{E^0_{2,1}} \left(\|U_1\|_{E^0_{2,1}}^{α} + \|U_2\|_{E^0_{2,1}}^{α}\right),$$

where $U = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, $U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$.

**Proof.** By (5), we have

$$\|F(U)\|_{E^0_{2,1}} = \|a|u|^a u + |v|^a u\|_{E^0_{2,1}} + \|a|u|^a v + a|v|^a v\|_{E^0_{2,1}} \leq \|a\|\|u|^a u\|_{E^0_{2,1}} + \|v|^a u\|_{E^0_{2,1}} + \|u|^a v\|_{E^0_{2,1}} + |a|\|v|^a v\|_{E^0_{2,1}} \leq |a|\|u\|_{E^0_{2,1}}^{α+1} + \|v\|_{E^0_{2,1}}^{α} \|u\|_{E^0_{2,1}} + |a|\|v\|_{E^0_{2,1}}^{α+1} \|u\|_{E^0_{2,1}} + |a|\|v\|_{E^0_{2,1}}^{α+1} \|v\|_{E^0_{2,1}} \leq \|u\|_{E^0_{2,1}}^{α} \left(\|u\|_{E^0_{2,1}} + \|v\|_{E^0_{2,1}}\right) + C\|v\|_{E^0_{2,1}}^{α} \left(\|u\|_{E^0_{2,1}} + \|v\|_{E^0_{2,1}}\right) \leq C\|U\|_{E^0_{2,1}}^{α+1}$$

By mean value theorem we obtain $x^α - y^α = α(x - y)(x - η y)^{α-1}$, ($0 ≤ η ≤ 1$). Using this fact and (5), Young’s inequality (since $\frac{α-1}{α} + \frac{1}{α} = 1$), we have

$$\|F(U_1) - F(U_2)\|_{E^0_{2,1}} = \|a|u_1|^a u_1 + |v_1|^a u_1 - (a|u_2|^a u_2 + |v_2|^a u_2)\|_{E^0_{2,1}}.$$
Let us consider the mapping $\mathcal{T}_{cal}$. By Lemma 5 and the first inequality of Lemma 6, we obtain

\begin{align*}
&+\|a|v_1|^a v_1 - (a|v_2|^a v_2 + a|v_2|^a v_2)\|_{E_{2,1}^0}^a \\
&= \|a|v_1|^a v_1 - (a|v_2|^a v_2 + a|v_2|^a v_2)\|_{E_{2,1}^0}^a \\
&+\|a|v_1|^a v_1 - (a|v_2|^a v_2 + a|v_2|^a v_2)\|_{E_{2,1}^0}^a \\
&\leq \|a|v_1|^a v_1 - (a|v_2|^a v_2 + a|v_2|^a v_2)\|_{E_{2,1}^0}^a \\
&+\|a|v_1|^a v_1 - (a|v_2|^a v_2 + a|v_2|^a v_2)\|_{E_{2,1}^0}^a \\
&\leq C \left[\|u_1\|^a_{E_{2,1}^0} + \|u_2\|^a_{E_{2,1}^0} + \|v_1\|^a_{E_{2,1}^0} + \|v_2\|^a_{E_{2,1}^0} \right] \|U_1 - U_2\|_{E_{2,1}^0}^a \\
&\leq C (\|U_1\|^a_{E_{2,1}^0} + \|U_2\|^a_{E_{2,1}^0}) \|U_1 - U_2\|_{E_{2,1}^0}^a.
\end{align*}

3. Proof of the Main Result

We shall make use of the fixed point Theorem to solve the integral equation

$$U = \mathcal{T}(U) = \Lambda(t)\Phi - i \int_0^t \Lambda(t-\tau)F(U(\tau))d\tau. \quad (6)$$

Define a metric space as follows:

$$D = \{ U : \|U\|_{C(0,T;E_{2,1}^0)} \leq M \},$$

$$d(U,V) = \|U - V\|_{C(0,T;E_{2,1}^0)}.$$

By Lemma 5, we have

$$\|\Lambda(t)\Phi\|_{C(0,T;E_{2,1}^0)} \leq C \Phi_{E_{2,1}^0}.$$

By Lemma 5 and the first inequality of Lemma 6, we obtain

$$\left\| \int_0^t \Lambda(t-\tau)F(U(\tau))d\tau \right\|_{C(0,T;E_{2,1}^0)} \leq CT \left\| U \right\|_{C(0,T;E_{2,1}^0)}.$$

Let us consider the mapping $\mathcal{T} : U \rightarrow \Lambda(t)\Phi - i \int_0^t \Lambda(t-\tau)F(U(\tau))d\tau$. We show that $\mathcal{T} : (D, d) \rightarrow (D, d)$ is a contraction mapping. Indeed, for any $U \in D$, by (7) and (8) we have

$$\|\mathcal{T}(U)\|_{C(0,T;E_{2,1}^0)} \leq C \Phi_{E_{2,1}^0} + CT \left\| U \right\|_{C(0,T;E_{2,1}^0)}.$$

Put $M = 2C \Phi_{E_{2,1}^0}$, we have

$$\|\mathcal{T}(U)\|_{C(0,T;E_{2,1}^0)} \leq \frac{M}{2 + CTM^a+1}. \quad (9)$$

Let $T$ be small enough to satisfies $CTM^a \leq \frac{1}{4}$. It follows from (9) that $\mathcal{T}(U) \in D$. 
Similarly, we have
\[ ||\mathcal{T}(U) - \mathcal{T}(V)||_{C(0,T;E_{2,1}^0)} \leq \frac{1}{2} ||U - V||_{C(0,T;E_{2,1}^0)}. \]

Indeed, \( \forall U, V \in (D,d) \), by Lemma 5 and the second inequality of Lemma 6, we obtain
\[
\begin{align*}
||\mathcal{T}(U) - \mathcal{T}(V)||_{C(0,T;E_{2,1}^0)} &\leq \left|\left| \int_0^t \Lambda(t-\tau)[F(U(\tau)) - F(V)d\tau \right|\right|_{C(0,T;E_{2,1}^0)} \\
&\leq CT||F(U) - F(V)||_{C(0,T;E_{2,1}^0)} \\
&\leq CT \left[ ||U||_{C(0,T;E_{2,1}^0)}^q + ||V||_{C(0,T;E_{2,1}^0)}^q \right] ||U - V||_{C(0,T;E_{2,1}^0)} \\
&\leq 2CTM^q||U - V||_{C(0,T;E_{2,1}^0)} \\
&\leq \frac{1}{2} ||U - V||_{C(0,T;E_{2,1}^0)}.
\end{align*}
\]

Hence, by Banach fixed point theorem, we see that \( \mathcal{T} \) has a fixed point \( U \in D \) which is a solution of integral equation (6). We can extend this solution step by step and finally find a maximal \( T^* > 0 \) such that \( U \in C([0,T^*),E_{2,1}^0(R^n)) \) and \( \lim_{t \to T^*} \sup \|U(t)\|_{E_{2,1}^0(R^n)} = \infty \). The uniqueness of such solutions can also be shown in a standard way. This finishes the proof of the Theorem.

**ACKNOWLEDGEMENTS** This work is financially supported by the research projects of Zhejiang Ocean University under the grant numbers X08M014 and 21065030608.

**References**


