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On solving Partial Differential Equations of Fractional Order by Using the Variational Iteration Method and Multivariate Padé Approximations

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Abstract. In this article, multivariate Padé approximation and variational iteration method proposed by He is adopted for solving linear and nonlinear fractional partial differential equations. The fractional derivatives are described in the Caputo sense. Numerical illustrations that include nonlinear time-fractional hyperbolic equation and linear fractional Klein-Gordon equation are investigated to show efficiency of multivariate Padé approximation. Comparison of the results obtained by the variational iteration method with those obtained by multivariate Padé approximation reveals that the present methods are very effective and convenient.

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1. Introduction

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance, physical and biological processes and systems [4, 10, 11, 18, 19, 28–30, 43–45]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Recently, the Adomian decomposition method [2, 3, 31, 32, 34–36, 46, 48, 49] and variational iteration [14, 16, 17, 20–24, 33, 37, 40] method have been used for solving a wide range of problems.

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Many approximation and numerical techniques have been used to solve fractional differential equations. The variational iteration method is relatively new approach to provide an analytical approximation to linear and nonlinear problems and it is particularly valuable as tool for scientists and applied mathematicians, because it provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to fractional differential equations. In the literature, the unvariate Padé approximation has been used to obtain approximate solutions of fractional order [38, 39]. So the objective of the present paper is to show the application of the multivariate Padé approximation to provide approximate solutions for initial value problems of linear and nonlinear partial differential equations of fractional order and to make comparison with variational iteration method.

2. Basic Definitions

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 1. *A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu\in R$ if there exists a real number $p(>\mu)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)}\in C_\mu$, $m\in N$.*

Definition 2. *The Riemann–Liouville fractional integral operator of order $\alpha\geq 0$, of a function $f\in C_\mu$, $\mu\geq -1$ is defined as*

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt\ , \quad \alpha>0, \qquad x>0 \\ J^0 f(x) &= f(x). \end{aligned} \tag{1}$$

Properties of the operator J^α can be found in [28, 43, 44]. For $f\in C_\mu$, $\mu\geq -1$, $\alpha,\beta\geq 0$ and $\gamma>-1$:

1. $J^\alpha J^\beta f(x)=J^{\alpha+\beta}f(x)$,
2. $J^\alpha J^\beta f(x)=J^\alpha J^\beta f(x)$,
3. $J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity will be introduced [4].

Definition 3. *The fractional derivative off (x) in the Caputo sense is defined as*

$$D_*^\alpha f(x)=J^{m-\alpha}D^m f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^m(t)dt, \tag{2}$$

for $m-1<\alpha\le m$, $m\in\mathbb{N}$, $x>0$, $f\in C_{-1}^{m}$.

Definition 4. *For m to be the smallest integer that exceeds* α*, the Caputo time-fractional derivative operator of order* $\alpha>0$ *is defined as*

$$D_t^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}d\tau & m-1<\alpha<m\\ \frac{\partial^m u(x,t)}{\partial t^m} & \alpha=m\in\mathbb{N}\end{cases}\tag{3}$$

Lemma 1. *If* $m-1<\alpha\le m$, $m\in\mathbb{N}$, *and* $f\in C_{\mu}^{m}$, $\mu\ge-1$ *then*

$$D_*^{\alpha}J^{\alpha}f(x)=f(x)\tag{4}$$

$$J^{\alpha}D_*^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^k(0^+)\frac{x^k}{k!},\quad x>0\tag{5}$$

3. Multivariate Padé Approximation

The principles and theory of the multivariate Padé approximation and its applicability for various of differential equations are given in [1, 5–9, 12, 13, 47, 50, 51]. Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y)=\sum_{i,j=0}^{\infty}c_{ij}x^iy^j\tag{6}$$

around the origin. We know that a solution of unvariate Padé approximation problem for

$$f(x)=\sum_{i=0}^{\infty}c_ix^i\tag{7}$$

is given by

$$p(x)=\begin{vmatrix}\sum_{i=0}^{m}c_ix^i & x\sum_{i=0}^{m-1}c_ix^i & \cdots & x^n\sum_{i=0}^{m-n}c_ix^i\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m\end{vmatrix}\tag{8}$$

and

$$q(x)=\begin{vmatrix}1 & x & \cdots & x^n\\ c_{m+1} & c_m & \cdots & c_{m+1-n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m+n} & c_{m+n-1} & \cdots & c_m\end{vmatrix}\tag{9}$$

Let us now multiply jth row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j=2,\dots,n+1)$ and afterwards divide jth column in $p(x)$ and $q(x)$ by $x^{j-1}(j=2,\dots,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{10}$$

if $(D = \det D_{m,n} \neq 0)$.

This quotient of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced kth partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in (x,y). Here a bivariate term $c_{ij}x^i y^j$ is said to be of degree $i+j$. If we define

$$p(x,y) = \begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i y^j & \sum_{i+j=0}^{m-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{11}$$

and

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix} \tag{12}$$

Then it is easy to see that $p(x,y)$ and $q(x,y)$ are of the form

$$\begin{aligned} p(x,y) &= \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j \\ q(x,y) &= \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j \end{aligned} \tag{13}$$

We know that $p(x,y)$ and $q(x,y)$ are called Padé equations [8]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{14}$$

4. Variational Iteration Method

The principles of the variational iteration method are given in [14–26]. Ji-Huan He applied the variational iteration method to obtain analytical solution for the fractional differential equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = f(x,t), \quad u(a) = b, 1 < \alpha < 2. \tag{15}$$

The application of the variational iteration method has been extended in [42] to solve the time fractional differential equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} u(x,t) = R[x]\, u(x,t) + q(x,t), \quad t > 0, x \in R, \tag{16}$$

where $R[x]$ is a differential operator in x, subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) =& f(x), \quad 0 < \alpha \leq 1, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, \quad t > 0, \end{aligned} \tag{17}$$

and

$$\begin{aligned} u(x,0) =& f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), 1 < \alpha \leq 2, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, t > 0, \end{aligned} \tag{18}$$

where $f(x)$, $g(x)$ and $q(x,t)$ all are continuous functions and α, $m-1 < \alpha \leq m$ is a parameter describing the order of the time-fractional derivative in the Caputo sense. According to the variational iteration method, the correction functional for Eq. (16) has been constructed in [42] as:

$$\begin{aligned} u_{k+1}(x,t) =& u_k(x,t) + J_t^{\beta}\left[\lambda\left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} u_k(x,t) - R[x]\, \tilde{u}_k(x,t) - q(x,t)\right)\right] \\ =& u_k(x,t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \lambda(\tau) \left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} u_k(x,\tau) - R[x]\, \tilde{u}_k(x,\tau) - q(x,\tau)\right) d\tau. \end{aligned} \tag{19}$$

where J_t^{β} is the Riemann-Liouville fractional integral operator of order $\beta = \alpha - floor(\alpha)$ that is $\beta = \alpha + 1 - m$, with respect to the variable t and λ is a general Lagrange multiplier, which can be identified optimally via variational theory [27]. To identify approximately Lagrange multiplier, some approximation has been made in [42]. The correction functional (19) can be approximately expressed as follows

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\tau)\left(\frac{\partial^{m}}{\partial \tau^{m}} u_k(x,\tau) - R[x]\, \tilde{u}_k(x,\tau) - q(x,\tau)\right)\right] d\tau. \tag{20}$$

Here restricted variations are applied to the nonlinear term $R[x]u$, in this case the multiplier can be easily determined. Making the above functional stationary, noticing that $\delta\tilde{u}_k = 0$,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \left[\lambda(\tau)\left(\frac{\partial^m}{\partial\tau^m}u_k(x,\tau) - q(x,\tau)\right)\right] d\tau, \tag{21}$$

yields the following multipliers

$$\lambda = -1, \text{ for } m = 1 \tag{22}$$

$$\lambda = \tau - t, \text{ for } m = 2 \tag{23}$$

Therefore, for $m = 1$ $(0 < \alpha \leq 1)$, $\lambda = -1$ is substituted into the functional (19) to obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) - J_t^\alpha\left[\frac{\partial^\alpha}{\partial t^\alpha}u_k(x,t) - R[x]u_k(x,t) - q(x,t)\right]. \tag{24}$$

For $m = 2$, $(1 < \alpha \leq 2)$, $\lambda = \tau - t$ is substituted into the functional (19) to get

$$\begin{aligned} u_{k+1}(x,t) =& u_k(x,t) + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}(\tau-t) \\ &\times\left(\frac{\partial^\alpha}{\partial t^\alpha}u_k(x,\tau) - R[x]u_k(x,\tau) - q(x,\tau)\right)d\tau \\ =& u_k(x,t) - \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\tau-t) \\ &\times\left(\frac{\partial^\alpha}{\partial t^\alpha}u_k(x,\tau) - R[x]u_k(x,\tau) - q(x,\tau)\right)d\tau. \end{aligned} \tag{25}$$

So, the following iteration formula is obtained in [42]

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha-1)J_t^\alpha\left[\frac{\partial^\alpha}{\partial t^\alpha}u_k(x,t) - R[x]u_k(x,t) - q(x,t)\right]. \tag{26}$$

The initial approximation (trial function) u_0 can be freely chosen if it satisfies the initial and boundary conditions of the problem. However the success of the method depends on the proper selection of the initial approximation u_0. Finally, the solution $u(x,t) = \lim_{k\to\infty} u_k(x,t)$ is approximated by the Nth term $u_N(x,t)$.

4.1. Nonlinear Time-fractional Partial Differential Equation

The following nonlinear time-fractional partial differential equation is considered in [41]

$$D_{*t}^\alpha u(x,t) = f(u,u_x,u_{xx}) + g(x,t), \quad m-1 < \alpha \leq m, \tag{27}$$

where $D^{\alpha}_{*t}u(x,t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$, is the Caputo fractional derivative of order α, $m \in \mathbb{N}$, f is a nonlinear function and $g$is the source function. The initial and boundary conditions associated with (27) are of the from

$$\begin{aligned} u(x,0) =& h(x), \quad 0 < \alpha \leq 1, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, \quad t > 0, \end{aligned} \tag{28}$$

and

$$\begin{aligned} u(x,0) =& h(x), \quad \frac{\partial u(x,0)}{\partial t} = k(x), 1 < \alpha \leq 2, \\ u(x,t) \to& 0 \text{ as } |x| \to \infty, \quad t > 0. \end{aligned} \tag{29}$$

The correction functional for Eq. (27) has been approximately expressed in [41] as follows:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi) - f(\tilde{u}_k,(\tilde{u}_k)_x,(\tilde{u}_k)_{xx}) - g(x,\xi)\right)d\xi, \tag{30}$$

where λ is a general Lagrange multiplier [27], which can be identified optimally via variational theory [14, 22–24, 27], here $\tilde{u}_k,(\tilde{u}_k)_x,(\tilde{u}_k)_{xx}$ are considered as restricted variations, i.e.,$\delta\tilde{u}_n = 0$. Making the above functional stationary,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \lambda(\xi)\left(\frac{\partial^m}{\partial \xi^m}u_k(x,\xi) - g(x,\xi)\right)d\xi, \tag{31}$$

yields the following Lagrange multipliers

$$\begin{gathered} \lambda = -1 \text{ for } m = 1, \\ \lambda = \xi - t, \text{ for } m = 2. \end{gathered}$$

Therefore, for $m = 1$, the following iteration formula has been obtained in [41]:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) - f(u_k,(u_k)_x,(u_k)_{xx}) - g(x,\xi)\right)d\xi. \tag{32}$$

In this case, it can be begun with the initial approximation

$$u_0(x,t) = h(x). \tag{33}$$

For $m = 2$, the following iteration formula is obtained [41]:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^\alpha}{\partial \xi^\alpha}u_k(x,\xi) - f(u_k,(u_k)_x,(u_k)_{xx}) - g(x,\xi)\right)d\xi. \tag{34}$$

In this case, it can be begun with the initial approximation

$$u_0(x,t) = h(x) + tk(x). \tag{35}$$

The correction functional (30) will give several approximations, and therefore the exact solution is obtained as

$$u(x,t) = \lim_{k\to\infty} u_k(x,t) \tag{36}$$

5. Numerical Experiments

In this section two methods, VIM and MPA, shall be illustrated by two examples. All the numerical results are calculated by using the software Maple12.

Example 1. *Consider the one-dimensional linear inhomogeneous fractional Klein-Gordon equation [42]*

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, x \in R, 1 < \alpha \leq 2, \tag{37}$$

subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0. \tag{38}$$

According to the variational iteration method and to Eq. (26)*, the iteration formula for Eq.* (37) *is given by*

$$u_{k+1}(x,t) = u_k(x,t) - (\alpha - 1) J_t^{\alpha} \left[\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^2 u}{\partial x^2} + u - 6x^3 t - (x^3 - 6x)t^3 \right]. \tag{39}$$

By the above variational iteration formula, if it is begun with $u_0 = 0$, so following approximations has been obtained in [42]

$$u_1(x,t) = (\alpha - 1)\left[6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} \right],$$

$$\begin{aligned} u_2(x,t) =& 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 6(x^3 - 6x)\frac{t^{\alpha+3}}{\Gamma(\alpha+4)} \\ & - (\alpha-1)^2 \left[6(x^3 - 6x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 6(x^3 - 12x)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \right] + \ldots . \end{aligned} \tag{40}$$

the variational iteration method gives the solution for the classical Klein-Gordon Eq. (37) *(when $\alpha = 2$) which is given by*

$$u(x,t) = x^3 t^3 + (x^3 - 6x)\frac{6t^5}{\Gamma(6)} + 36x\frac{t^5}{\Gamma(6)} - 36x\frac{6t^7}{\Gamma(8)} - 6x^3 \frac{t^5}{\Gamma(6)} - (x^3 - 6x)\frac{6t^7}{\Gamma(8)} + \ldots \tag{41}$$

$$= x^3 t^3 - 0.001190476190 x^3 t^7 - 0.01428571428 x t^7 \tag{42}$$

the exact solution of (37)*, for the special case $\alpha = 2$ is given in [42]*

$$u(x,t) = x^3 t^3 \tag{43}$$

Now let us calculate the approximate solution of Eq. (42) **for $\mathbf{m} = 8$ and $\mathbf{n} = 2$ by using Multivariate Padé approximation. To obtain Multivariate Padé equations of Eq.** (42) **for $\mathbf{m} = 8$ and $\mathbf{n} = 2$, we use Eqs.** (11) **and** (12)**. By using Eqs.** (11) **and** (12) **we obtain,**

$$\mathbf{p}(\mathbf{x},\mathbf{t}) = \begin{array}{ccc} \mathbf{x}^3\mathbf{t}^3 \quad 0.01428571428\mathbf{x}\mathbf{t}^7 & \mathbf{x}^3\mathbf{t}^3 & \mathbf{x}^3\mathbf{t}^3 \\ 0 & 0.01428571428\mathbf{x}\mathbf{t}^7 & 0 \\ 0.001190476190\mathbf{x}^3\mathbf{t}^7 & 0 & 0.01428571428\mathbf{x}\mathbf{t}^7 \end{array} \tag{44}$$

$$= \quad 0.00001700680271(\mathbf{x}^4 \quad 12.00000000\mathbf{x}^2 + 0.1714285714\mathbf{t}^4)\mathbf{x}^3\mathbf{t}^{17} \tag{45}$$

and

$$\mathbf{q}(\mathbf{x},\mathbf{t}) = \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0.01428571428\mathbf{x}\mathbf{t}^7 & 0 \\ 0.001190476190\mathbf{x}^3\mathbf{t}^7 & 0 & 0.01428571428\mathbf{x}\mathbf{t}^7 \end{array} \tag{46}$$

$$= \quad 0.00001700680271(\quad 12.00000000 + \mathbf{x}^2)\mathbf{x}^2\mathbf{t}^{14} \tag{47}$$

So the Multivariate Padé approximation of order $(8,2)$ **for eq.** (42)**, that is,**

$$[8,2]_{(\mathbf{x},\mathbf{t})} = \frac{(\mathbf{x}^4 \quad 12.00000000\mathbf{x}^2 + 0.1714285714\mathbf{t}^4)\mathbf{x}\mathbf{t}^3}{12.00000000 + \mathbf{x}^2} \tag{48}$$

the variational iteration method gives the solution for the classical Klein-Gordon Eq. (37) **(when** $= 1.5$**) which is given by**

$$\begin{aligned} \mathbf{u}(\mathbf{x},\mathbf{t}) &= 1.805406668\mathbf{x}^3\mathbf{t}^{2.5} + 0.1146289948(\mathbf{x}^3 \quad 6\mathbf{x})\mathbf{t}^{4.5} \\ &\quad 0.06250000000(\mathbf{x}^3 \quad 6\mathbf{x})\mathbf{t}^{4.0} \quad 0.002083333334(\mathbf{x}^3 \quad 12\mathbf{x})\mathbf{t}^{6.0} \\ &= 1.805406668\mathbf{x}^3\mathbf{t}^{2.5} + 0.1146289948\mathbf{x}^3\mathbf{t}^{4.5} \quad 0.6877739688\mathbf{x}\mathbf{t}^{4.5} \\ &\quad 0.06250000000\mathbf{x}^3\mathbf{t}^{4.0} + 0.3750000000\mathbf{x}\mathbf{t}^{4.0} \quad 0.002083333334\mathbf{x}^3\mathbf{t}^{6.0} \\ &\quad + 0.02500000001\mathbf{x}\mathbf{t}^{6.0} \end{aligned} \tag{49}$$

For simplicity, let $\mathbf{t}^{1=2} = \mathbf{a}$; then

$$\begin{aligned} \mathbf{u}(\mathbf{x},\mathbf{a}) &= 1.805406668\mathbf{x}^3\mathbf{a}^5 + 0.1146289948\mathbf{x}^3\mathbf{a}^9 \quad 0.6877739688\mathbf{x}\mathbf{a}^9 \quad 0.06250000000\mathbf{x}^3\mathbf{a}^8 \\ &\quad + 0.3750000000\mathbf{x}\mathbf{a}^8 \quad 0.002083333334\mathbf{x}^3\mathbf{a}^{12} + 0.02500000001\mathbf{x}\mathbf{a}^{12} \end{aligned} \tag{50}$$

and let

$$\begin{aligned} \mathbf{K} &= 1.805406668\mathbf{x}^3\mathbf{a}^5 + 0.1146289948\mathbf{x}^3\mathbf{a}^9 \quad 0.6877739688\mathbf{x}\mathbf{a}^9 \quad 0.06250000000\mathbf{x}^3\mathbf{a}^8 \\ &\quad + 0.3750000000\mathbf{x}\mathbf{a}^8 + 0.02500000001\mathbf{x}\mathbf{a}^{12} \end{aligned}$$

$$\begin{aligned} \mathbf{L} &= 1.805406668\mathbf{x}^3\mathbf{a}^5 + 0.1146289948\mathbf{x}^3\mathbf{a}^9 \quad 0.6877739688\mathbf{x}\mathbf{a}^9 \\ &\quad 0.06250000000\mathbf{x}^3\mathbf{a}^8 + 0.3750000000\mathbf{x}\mathbf{a}^8 \end{aligned}$$