



## GENERALIZED THERMOELASTICITY PROBLEM OF A HOLLOW SPHERE UNDER THERMAL SHOCK

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**Abstract.** This problem deals with the thermo-elastic interaction due to step input of temperature on the boundaries of a homogeneous isotropic spherical shell in the context of generalized theories of thermo-elasticity. Using the Laplace transformation the fundamental equations have been expressed in the form of vector-matrix differential equation which is then solved by eigen value approach. The inverse of the transform solution is carried out by applying a method of Bellman et al. Stresses, displacements and temperature distribution have been computed numerically and presented graphically in a number of figures for copper material. A comparison of the results for different theories (CTE, CCTE, TRDTE(GL), TEWOED(GN-II), TEWED(GN-III)) is presented. When the outer radius of the shell tends to infinity, the corresponding results agree with that of existing literature.

**Key words:** Generalized thermo-elasticity, energy dissipation, Laplace transform, step input temperature, vector-matrix differential equation.

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### 1. Introduction

The classical theories of thermo-elasticity involving infinite speed of propagation of thermal signals, contradict physical facts. During the last three decades, non-classical theories involving finite speed of heat transportation in elastic solids have been developed to remove this paradox. In contrast with the conventional coupled thermo-elasticity theory [1],

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which involves a parabolic-type heat transport equation, these generalized theories involving a hyperbolic-type heat transport equation are supported by experiments exhibiting the actual occurrence of wave-type heat transport in solids, called second sound effect. The extended thermo-elasticity theory (ETE) proposed by Lord and Shulman [1], incorporates a flux-rate term into Fourier's law of heat conduction, and formulates a generalized form that involves a hyperbolic-type heat transport equation admitting finite speed of thermal signals. Green and Lindsay [2] developed temperature-rate-dependent thermo-elasticity (TRDTE(GL)) theory by introducing relaxation time factors that does not violate the classical Fourier law of heat conduction and this theory also predicts a finite speed for heat propagation. The closed-form solutions for thermo-elastic problems in generalized theory of thermo-elasticity have been obtained by Hetnarski and Ignaczak [3]. Hetnarski and Ignaczak [4] studied the response of semi-space to a short laser pulse in the context of generalized thermo-elasticity. Sinha and Sinha [5] and Sinha and Elsibai [6], [7] studied the reflections of thermo-elastic waves from the free surface of a solid half-space and at the interface of two semi-infinite media in welded contact in the context of generalized thermo-elasticity. Sharma [8] investigated the reflection of thermo-elastic waves from the stress-free insulated boundary of a transversely isotropic half-space in the light of the theory developed by Dhaliwal and Singh [9] in which the theory developed in [1] is extended to anisotropic (transversely isotropic) thermo-elastic materials. Kar and Kanoria [10] investigated thermo-elastic stress wave propagation in an unbounded body with a spherical hole following the theories developed in [2], [15]. Roychoudhury and Bhatta [12] and Roychoudhury and Chatterjee [13] studied thermo-elastic wave propagation in an infinitely extended thin plate following the theory developed in [2] for short-time approximation using the Laplace transform technique.

Most engineering materials such as metals possess a relatively high rate of thermal damping and thus are not suitable for use in experiments concerning second sound propagation. But, given the state of recent advances in material science, it may be possible in the foreseeable future to identify (or even manufacture for laboratory purposes) an idealized material for the purpose of studying the propagation of thermal waves at finite speed. The relevant theoretical developments on the subject are due to Green and Naghdi[14-16] and provide

sufficient basic modifications in the constitutive equations that permit treatment of a much wider class of heat flow problems, labelled as types I, II, III. The natures of these three types of constitutive equations are such that when the respective theories are linearized, type-I is the same as the classical heat equation (based on Fourier's law) whereas the linearized versions of type-II and type-III theories permit propagation of thermal waves at finite speed. The entropy flux vector in type II and III ( i.e. thermo-elasticity without energy dissipation (TEWOED) and thermo-elasticity with energy dissipation (TEWED)) models are determined in terms of the potential that also determines stresses. When Fourier conductivity is dominant the temperature equation reduces to classical Fourier law of heat conduction and when the effect of conductivity is negligible the equation has undamped thermal wave solutions without energy dissipation. Several investigations relating to thermo-elasticity without energy dissipation theory (TEWOED) have been presented by Roychoudhuri and Datta [17], Sharma and Chouhan [18], Roychoudhury and Bandyopadhyay [19], Chandrasekharaiah [20], [21]. The main object of the present paper is to study the thermo-elastic stresses, displacement and temperature distribution in an isotropic homogeneous spherical shell due to step input of temperature on the boundaries of the shell in the context of TRDTE [2] and TEWED [16] of type-III of generalized thermo-elasticity. Using the Laplace transformation the fundamental equations have been expressed in the form of vector-matrix differential equation which is then solved by eigen value approach. The inversion of the Laplace transform is done following Bellman et al. [22]. The results obtained theoretically have been computed numerically and are presented graphically for Copper material. A complete and comprehensive analysis and comparison of results of the above theories are presented.

## 2. Basic equations and constitutive relations

We consider a homogeneous isotropic spherical shell of inner radius  $a$  and outer radius  $b$  in an undisturbed state and initially at uniform temperature  $T_0$ . We introduce spherical polar co-ordinate  $(r, \theta, \phi)$  with the centre of the cavity as the origin.

In the present problem (due to spherical symmetry) the displacement and temperature are

assumed to be functions of  $r$  and time  $t$  only. The stress-strain-temperature relations in the present problem are

$$\tau_{ij} = \lambda \Delta \delta_{ij} + 2\mu e_{ij} - \gamma(T + \delta_{1k} \alpha \dot{T}) \delta_{ij} \quad (1)$$

and generalized heat conduction equation is

$$\begin{aligned} & \left( \delta_{1k} + \delta_{2k} \frac{\partial}{\partial t} \right) K \nabla^2 T + \delta_{2k} K^* \nabla^2 T \\ & = \rho C_e \left[ \delta_{1k} \dot{T} + (\delta_{1k} \alpha^* + \delta_{2k}) \ddot{T} \right] + \gamma T_0 \left( \zeta \delta_{1k} + \delta_{2k} \frac{\partial}{\partial t} \right) \dot{\Delta}, \end{aligned} \quad (2)$$

where  $\tau_{ij}$  ( $i = r, \theta$ ) is the stress tensor,  $\Delta$  is the dilatation,  $T$  is the temperature increase over the absolute reference temperature  $T_0$ ,  $\gamma = (3\lambda + 2\mu)\alpha_t$ ,  $\lambda$  and  $\mu$  are the Lamé's constants,  $\alpha_t$  is the coefficient of linear thermal expansion of the material,  $\alpha$  and  $\alpha^*$  are the relaxation times,  $K$  is the coefficient of thermal conductivity,  $K^*$  is the additional material constant,  $\rho$  is the mass density,  $C_e$  is the specific heat of the solid at constant strain,  $\delta_{ij}$  is the Kronecker delta.

Consider the following partial cases of equations (1) and (2):

(i) if  $\alpha = 0$ ,  $\alpha^* = 0$ ,  $k = 1$ , and  $\zeta = 0$ , then the equations (1) and (2) are reduced to the equations of classical theory of thermo-elasticity (CTE).

(ii) if  $\alpha = 0$ ,  $\alpha^* = 0$ ,  $k = 1$ , and  $\zeta = 1$ , then the equations (1) and (2) are reduced to the equations of classical coupled theory of thermo-elasticity (CCTE).

(iii) if  $k = 1$ , and  $\zeta = 1$ , then the equations (1) and (2) are reduced to the equations of temperature rate dependent thermo-elasticity theory (TRDTE(GL)).

(iv) if  $k = 2$  and  $K = 0$ , then the equations (1) and (2) are reduced to the equations of thermo-elasticity with energy dissipation (TEWOED(GN-II)).

(v) if  $k = 2$ , then the equations (1) and (2) are reduced to the equations of thermo-elasticity with energy dissipation (TEWED(GN-III)).

The thermal relaxation times satisfy the inequalities [2]

$$\alpha \geq \alpha^* > 0$$

in the case of TRDTE(GL) theory.

If  $\tilde{\mathbf{u}} = [u(r, t), 0, 0]$  be the displacement vector, then

$$e_{rr} = \frac{\partial u}{\partial r}, e_{\theta\theta} = e_{\phi\phi} = \frac{u}{r}. \tag{3}$$

The stress equation of motion in spherical polar co-ordinates is given by

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{2}{r}(\tau_{rr} - \tau_{\theta\theta}) = \rho \frac{\partial^2 u}{\partial t^2}. \tag{4}$$

Introducing the following dimensionless quantities

$$U = \frac{(\lambda + 2\mu)u}{a\gamma T_0}; (R, S) = \left(\frac{r}{a}, \frac{b}{a}\right); (\sigma_R, \sigma_\theta) = \frac{1}{\gamma T_0}(\tau_{rr}, \tau_{\theta\theta});$$

$$\Theta = \frac{T}{T_0}; \eta = \frac{Gt}{a}; G^2 = \frac{\lambda + 2\mu}{\rho}; \alpha' = \frac{\alpha G}{a}; \alpha^* = \frac{\alpha^* G}{a};$$

equations (1), (2) and (4) become

$$\sigma_R = \frac{\partial U}{\partial R} + 2\lambda_1 \frac{U}{R} - \left(\Theta + \delta_{1k} \alpha' \frac{\delta \Theta}{\delta \eta}\right), \tag{5}$$

$$\sigma_\theta = \lambda_1 \frac{\partial U}{\partial R} + (\lambda_1 + 1) \frac{U}{R} - \left(\Theta + \delta_{1k} \alpha' \frac{\delta \Theta}{\delta \eta}\right), \tag{6}$$

$$\left\{ \left( \delta_{1k} + \frac{G}{a} \frac{\partial}{\partial \eta} \delta_{2k} \right) c_K^2 + \frac{G}{a} c_T^2 \delta_{2k} \right\} \left( \frac{\partial^2 \Theta}{\partial R^2} + \frac{2}{R} \frac{\partial \Theta}{\partial R} \right) = \left\{ \delta_{1k} \frac{\partial \Theta}{\partial \eta} + \left( \alpha^* \delta_{1k} + \frac{G}{a} \delta_{2k} \right) \frac{\partial^2 \Theta}{\partial \eta^2} \right\} + \epsilon \left( \zeta \delta_{1k} \frac{\partial}{\partial \eta} + \frac{G}{a} \frac{\partial^2}{\partial \eta^2} \delta_{2k} \right) \left( \frac{\partial U}{\partial R} + \frac{2U}{R} \right) \tag{7}$$

and

$$\frac{\partial^2 U}{\partial R^2} + \frac{2}{R} \frac{\partial U}{\partial R} - \frac{2U}{R^2} = \left( 1 + \delta_{1k} \alpha' \frac{\partial}{\partial \eta} \right) \frac{\partial \Theta}{\partial R} + \frac{\partial^2 U}{\partial \eta^2}, \tag{8}$$

where  $\lambda_1 = \frac{\lambda}{\lambda + 2\mu}$ ,  $c_T^2 = \frac{K^*}{\rho C_e G^2}$ ,  $c_K^2 = \frac{K}{a \rho C_e G}$ , and  $\epsilon = \frac{\gamma^2 T_0}{\rho C_e (\lambda + 2\mu)}$  are dimensionless constants.  $\epsilon$  being the thermo-elastic coupling constant. Here  $c_T$  is the nondimensional

thermal wave velocity and  $c_K$  is the damping co-efficient. In the case  $c_K = 0$ , we arrive at the thermo-elasticity equations without energy dissipation theory (TEWOED(GN-II)) for Green-Naghdi model.

The boundary conditions are given by

$$\sigma_R = 0 \text{ on } R = 1, S \quad \eta \geq 0,$$

$$\Theta = \chi_1 H(\eta), \text{ on } R = 1, \quad \eta > 0,$$

$$= \chi_2 H(\eta) \text{ on } R = S, \quad \eta > 0,$$

where  $\chi_1$  and  $\chi_2$  are dimensionless constants and  $H(\eta)$  is the Heaviside unit step function. The above conditions indicate that for time  $\eta \leq 0$  there is no temperature ( $\Theta = 0$ ). Thermal shocks are given on the boundaries of the shell ( $R = 1, S$ ) immediately after time  $\eta = 0$ . Thermal stresses in the elastic medium due to the application of these thermal shocks are calculated. We assume that the medium is at rest and undisturbed initially. The initial and regularity conditions can be written as

$$U = \Theta = 0 \text{ at } \eta = 0, R \geq 1, \quad \frac{\partial U}{\partial \eta} = 0 \text{ at } \eta = 0,$$

$$U = \Theta = 0 \text{ when } R \rightarrow \infty.$$

### 3. Method of solution

Let

$$\{\bar{U}(R, p), \bar{\Theta}(R, p)\} = \int_0^\infty \{U(R, \eta), \Theta(R, \eta)\} e^{-p\eta} d\eta$$

with  $Re(p) > 0$  denote the Laplace transform of  $U$  and  $\Theta$  respectively.

Application of the Laplace transform, equations (7) and (8) yields

$$\frac{d^2 \bar{\Theta}}{dR^2} + \frac{2}{R} \frac{d\bar{\Theta}}{dR} = p \left\{ \frac{(1 + \alpha^* p)\delta_{1k} + p\delta_{2k}}{c_K^2(\delta_{1k} + p\delta_{2k}) + c_T^2\delta_{2k}} \right\} \bar{\Theta} +$$

$$\epsilon p \left\{ \frac{\zeta \delta_{1k} + p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right\} \left( \frac{d\bar{U}}{dR} + \frac{2\bar{U}}{R} \right) \tag{9}$$

and

$$\frac{d^2\bar{U}}{dR^2} + \frac{2}{R} \frac{d\bar{U}}{dR} - \frac{2\bar{U}}{R^2} = p^2\bar{U} + (1 + \alpha' p \delta_{1k}) \frac{d\bar{\Theta}}{dR}. \tag{10}$$

Differentiating equation (9) with respect to  $R$  and using equation (10) we get

$$\begin{aligned} \frac{d^2}{dR^2} \left( \frac{d\bar{\Theta}}{dR} \right) + \frac{2}{R} \frac{d}{dR} \left( \frac{d\bar{\Theta}}{dR} \right) - \frac{2}{R^2} \left( \frac{d\bar{\Theta}}{dR} \right) = \epsilon p^3 \left\{ \frac{\zeta \delta_{1k} + p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right\} \bar{U} + \\ p \left[ \frac{\{(1 + \alpha' p) + \epsilon \zeta(1 + \alpha' p)\} \delta_{1k} + (1 + \epsilon) p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right] \frac{d\bar{\Theta}}{dR}. \end{aligned} \tag{11}$$

Equations (10) and (11) can be written in the form

$$L(\bar{U}) = p^2\bar{U} + (1 + \alpha' p \delta_{1k}) \frac{d\bar{\Theta}}{dR} \tag{12}$$

and

$$\begin{aligned} L \left( \frac{d\bar{\Theta}}{dR} \right) = \epsilon p^3 \left\{ \frac{\zeta \delta_{1k} + p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right\} \bar{U} + \\ p \left[ \frac{\{(1 + \alpha' p) + \epsilon \zeta(1 + \alpha' p)\} \delta_{1k} + (1 + \epsilon) p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right] \frac{d\bar{\Theta}}{dR}, \end{aligned} \tag{13}$$

where  $\epsilon$  is the thermo-elastic coupling constant and

$$L \equiv \frac{d^2}{dR^2} + \frac{2}{R} \frac{d}{dR} - \frac{2}{R^2}. \tag{14}$$

From equations (12) and (13) we have the vector-matrix differential equation as follows

$$L\tilde{V} = \tilde{A}\tilde{V}, \tag{15}$$

where

$$\tilde{V} = \left[ \bar{U}, \frac{d\bar{\Theta}}{dR} \right]^T, \quad \tilde{A} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{16}$$

and  $C_{11} = p^2$ ,  $C_{12} = 1 + \alpha' p \delta_{1k}$ ,  $C_{21} = \epsilon p^3 \left\{ \frac{\zeta \delta_{1k} + p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right\}$ ,

$$C_{22} = p \left[ \frac{\{(1 + \alpha' p) + \epsilon \zeta(1 + \alpha' p)\} \delta_{1k} + (1 + \epsilon) p \delta_{2k}}{c_K^2(\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right].$$

#### 4. Solution of the vector-matrix differential equation

Let

$$\tilde{V} = \tilde{X}(m)\omega(R, m), \quad (17)$$

where  $m$  is a scalar,  $\tilde{X}$  is a vector independent of  $R$  and  $\omega(R, m)$  is a non-trivial solution of the scalar differential equation

$$L\omega = m^2\omega. \quad (18)$$

Let  $\omega = R^{-1/2}\omega_1$ . Therefore, from equation (18) we have

$$\frac{d^2\omega_1}{dR^2} + \frac{1}{R} \frac{d\omega_1}{dR} - \left( \frac{9}{4R^2} + m^2 \right) \omega_1 = 0. \quad (19)$$

Therefore, the solution of the equation (18) is

$$\omega = [A_1 I_{3/2}(mR) + A_2 K_{3/2}(mR)]/\sqrt{R} \quad (20)$$

Substituting equations (17) and (18) into equation (15) we get

$$\tilde{A}\tilde{X} = m^2\tilde{X}, \quad (21)$$

where  $\tilde{X}(m)$  is the eigenvector corresponding to the eigenvalue  $m^2$ .

The characteristic equation corresponding to  $\tilde{A}$  can be written as

$$m^4 - (C_{11} + C_{22})m^2 + (C_{11}C_{22} - C_{12}C_{21}) = 0. \quad (22)$$

The roots of the characteristic equation (22) are of the form  $m^2 = m_1^2$  and  $m^2 = m_2^2$ , where

$$m_1^2 + m_2^2 = C_{11} + C_{22}, \quad m_1^2 m_2^2 = C_{11}C_{22} - C_{12}C_{21}. \quad (23)$$

The eigenvectors  $X(m_j)$ ,  $j=1,2$  corresponding to eigenvalues  $m_j^2$ ,  $j=1,2$  can be calculated as

$$\tilde{X}(m_j) = \begin{bmatrix} X_1(m_j) \\ X_2(m_j) \end{bmatrix} = \begin{bmatrix} C_{12} \\ -(C_{11} - m_j^2) \end{bmatrix}, \quad j = 1, 2. \quad (24)$$

Therefore, the equation (17) can be written as

$$\tilde{V} = \tilde{X}(m_j)[A_1 I_{3/2}(m_1 R) + B_1 K_{3/2}(m_1 R)]/\sqrt{R} +$$



$$\tilde{X}(m_j)[A_1 I_{3/2}(m_2 R) + B_1 K_{3/2}(m_2 R)]/\sqrt{R}, \quad \alpha = 1, 2. \tag{25}$$

Therefore, from equations in (15) we can write

$$\bar{U} = \sum_{i=1,2} C_{12} [A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)]/\sqrt{R} \tag{26}$$

and

$$\frac{d\bar{\Theta}}{dR} = - \sum_{i=1,2} (C_{11} - m_i^2) [A_i I_{3/2}(m_i R) + B_i K_{3/2}(m_i R)]/\sqrt{R}, \tag{27}$$

where  $I_{3/2}(m_i R)$  and  $K_{3/2}(m_i R)$  are the modified Bessel functions of order 3/2 of first and second kind respectively.  $A_i$  and  $B_i$ 's  $i = 1, 2$  are independent of  $R$  but depend on  $p$  and are to be determined from the boundary conditions.

Using the recurrence relations of modified Bessel functions [23] we obtain from the equation (27)

$$\bar{\Theta} = \sum_{i=1,2} \frac{(C_{11} - m_i^2)}{m_i} [A_i I_{1/2}(m_i R) + B_i K_{1/2}(m_i R)]/\sqrt{R}, \tag{28}$$

since

$$\frac{1}{R^{1/2}} P_{3/2}(mR) = - \frac{d}{dR} \left[ \frac{P_{1/2}(mR)}{mR^{1/2}} \right], \tag{29}$$

where  $P = I, K$ . Taking the Laplace transform of the equations (5) and (6) and using equations (26) and (28) we get

$$\begin{aligned} \bar{\sigma}_R = a_1 \sum_{i=1,2} A_i \left[ -\frac{4\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \frac{p^2}{m_i} R I_{1/2}(m_i R) \right] / R^{3/2} + \\ a_1 \sum_{i=1,2} B_i \left[ -\frac{4\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{p^2}{m_i} R K_{1/2}(m_i R) \right] / R^{3/2} \end{aligned} \tag{30}$$

and

$$\begin{aligned} \bar{\sigma}_\Theta = a_1 \sum_{i=1,2} A_i \left[ \frac{2\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \right. \\ \left. \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R I_{1/2}(m_i R) \right] / R^{3/2} + \\ a_1 \sum_{i=1,2} B_i \left[ \frac{2\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \right. \\ \left. \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R K_{1/2}(m_i R) \right] / R^{3/2}, \end{aligned} \tag{31}$$

where  $a_1 = 1 + \alpha' p \delta_{1k}$ . Using the boundary conditions  $\bar{\sigma}_R = 0$  on  $R = 1, R = S$  and  $\bar{\Theta} = \frac{\chi_1}{p}$  on  $R = 1, \bar{\Theta} = \frac{\chi_2}{p}$  on  $R = S$ . and using the recurrence relations [23] we obtain from equations (28) and (29)

$$\begin{aligned}
 A_1 W_{11} + A_2 W_{12} + B_1 W_{13} + B_2 W_{14} &= 0, \\
 A_1 W_{21} + A_2 W_{22} + B_1 W_{23} + B_2 W_{24} &= 0, \\
 A_1 W_{31} + A_2 W_{32} + B_1 W_{33} + B_2 W_{34} &= \frac{\chi_1}{p}, \\
 A_1 W_{41} + A_2 W_{42} + B_1 W_{43} + B_2 W_{44} &= \frac{\chi_2}{p},
 \end{aligned}
 \tag{32}$$

where

$$\begin{aligned}
 W_{1i} &= \frac{4\mu}{\lambda + 2\mu} P_{3/2}(m_j) + \frac{p^2}{m_j} P_{1/2}(m_j), \\
 W_{2i} &= \frac{4\mu}{\lambda + 2\mu} P_{3/2}(m_j S) + \frac{p^2}{m_j} S P_{1/2}(m_j S), \\
 W_{3i} &= \frac{p^2 - m_j^2}{m_j} P_{1/2}(m_j), \\
 W_{4i} &= \frac{p^2 - m_j^2}{m_j S^{1/2}} P_{1/2}(m_j S),
 \end{aligned}
 \tag{33}$$

where  $P = I$  for  $i = j = 1, 2$ ;  $P = K$  for  $i = 3, j = 1$  and  $i = 4, j = 2$ . From (32) the values of  $A_1, A_2, B_1$  and  $B_2$  are given as

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{\chi_1}{p} \\ \frac{\chi_2}{p} \end{pmatrix}.
 \tag{34}$$

Equation (22) can be written as

$$\begin{aligned}
 m^4 - \left[ p^2 + \left\{ \frac{p}{c_K^2} (1 + \alpha^* p + \epsilon \zeta (1 + \alpha' p)) \delta_{1k} + \frac{(1 + \epsilon) p^2}{c_T^2 + c_K^2 p} \delta_{2k} \right\} \right] m^2 + \\
 p^3 \left\{ \frac{(1 + \alpha^* p) \delta_{1k} + p \delta_{2k}}{c_K^2 (\delta_{1k} + p \delta_{2k}) + c_T^2 \delta_{2k}} \right\} = 0.
 \end{aligned}
 \tag{35}$$

Therefore,

$$m_1, m_2 = \frac{\sqrt{p}(\delta_{1k} + \sqrt{p}\delta_{2k})}{2\sqrt{c_K^2(\delta_{1k} + p\delta_{2k}) + c_T^2\delta_{2k}}}(\sqrt{\alpha} \pm \sqrt{\beta}), \tag{36}$$

where

$$\alpha, \beta = [\sqrt{1 + \alpha' p\delta_{1k}} \pm \sqrt{c_K^2 p + c_T^2 \delta_{2k}}]^2 + [(1 + \alpha')\zeta\delta_{1k} + \delta_{2k}]\epsilon. \tag{37}$$

Therefore,  $m_1$  and  $m_2$  are real and positive quantities.

### 5. Special case (Case of infinite body)

From (30) and (31) we write

$$\bar{\sigma}_R = \bar{\sigma}_R(I) + \bar{\sigma}_R(K), \tag{38}$$

$$\bar{\sigma}_\theta = \bar{\sigma}_\theta(I) + \bar{\sigma}_\theta(K), \tag{39}$$

where

$$\bar{\sigma}_R(I) = a_1 \sum_{i=1,2} A_i \left[ -\frac{4\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \frac{p^2}{m_i} R I_{1/2}(m_i R) \right] / R^{3/2}, \tag{40}$$

$$\bar{\sigma}_R(K) = a_1 \sum_{i=1,2} B_i \left[ -\frac{4\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{p^2}{m_i} R K_{1/2}(m_i R) \right] / R^{3/2}, \tag{41}$$

$$\bar{\sigma}_\theta(I) = a_1 \sum_{i=1,2} A_i \left[ \frac{2\mu}{\lambda + 2\mu} I_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R I_{1/2}(m_i R) \right] / R^{3/2}, \tag{42}$$

and

$$\bar{\sigma}_\theta(K) = a_1 \sum_{i=1,2} B_i \left[ \frac{2\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R K_{1/2}(m_i R) \right] / R^{3/2}. \tag{43}$$

For large value of  $b$  i.e. for large value of  $S$ ,  $K_{1/2}(m_i S)$  and  $K_{3/2}(m_i S)$  tend to zero. Thus for large values of  $b$  the asymptotic expressions of  $\bar{\sigma}_R(I)$  and  $\bar{\sigma}_\theta(I)$  are given as

$$\bar{\sigma}_R(I) = a_1 \frac{\chi_2 \sqrt{S}}{p R^2} \frac{e^{-m_2(S-R)} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 S} \right) + \frac{p^2}{m_1} S \right] \times \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 R} \right) + \frac{p^2}{m_2} R \right] - e^{-m_1(S-R)} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 S} \right) + \frac{p^2}{m_2} S \right] \times \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 R} \right) + \frac{p^2}{m_1} R \right]}{\frac{p^2 - m_1^2}{m_1} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 S} \right) + \frac{p^2}{m_2} S \right] - \frac{p^2 - m_2^2}{m_2} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 S} \right) + \frac{p^2}{m_1} S \right]}, \tag{44}$$

$\rightarrow 0$  as  $S \rightarrow \infty$

and

$$\bar{\sigma}_\theta(I) = a_1 \frac{\chi_2 \sqrt{S}}{p R^2} \frac{e^{-m_1(S-R)} \left[ \frac{2\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 R} \right) - \frac{\lambda m_1^2 + (\lambda + 2\mu)(p^2 - m_1^2)}{(\lambda + 2\mu)m_1} R \right] \times \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 S} \right) + \frac{p^2}{m_2} S \right] - e^{-m_2(S-R)} \left[ \frac{2\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 R} \right) - \frac{\lambda m_2^2 + (\lambda + 2\mu)(p^2 - m_2^2)}{(\lambda + 2\mu)m_2} R \right] \times \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 S} \right) + \frac{p^2}{m_1} S \right]}{\frac{p^2 - m_1^2}{m_1} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_2 S} \right) + \frac{p^2}{m_2} S \right] - \frac{p^2 - m_2^2}{m_2} \left[ \frac{4\mu}{\lambda + 2\mu} \left( 1 - \frac{1}{m_1 S} \right) + \frac{p^2}{m_1} S \right]}, \tag{45}$$

$\rightarrow 0$  as  $S \rightarrow \infty$ .

Therefore, as  $b \rightarrow \infty$

$$\bar{\sigma}_R = a_1 \sum_{i=1,2} B_i \left[ -\frac{4\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{p^2}{m_i} R K_{1/2}(m_i R) \right] / R^{3/2}, \tag{46}$$

$$\bar{\sigma}_\theta = a_1 \sum_{i=1,2} B_i \left[ \frac{2\mu}{\lambda + 2\mu} K_{3/2}(m_i R) - \frac{\lambda m_i^2 + (\lambda + 2\mu)(p^2 - m_i^2)}{(\lambda + 2\mu)m_i} R K_{1/2}(m_i R) \right] / R^{3/2}, \tag{47}$$

where  $B_i$ 's ( $i = 1, 2$ ) are given as

$$B_1 = -\frac{\chi_1}{P} \frac{m_1 \left[ 4\mu m_2 K_{3/2}(m_2) + (\lambda + 2\mu)p^2 K_{1/2}(m_2) \right]}{4\mu \left[ (p^2 - m_2^2)m_1 K_{3/2}(m_1) K_{1/2}(m_2) - (p^2 - m_1^2)m_2 K_{1/2}(m_1) K_{3/2}(m_2) \right] + (\lambda + 2\mu)p^2(m_1^2 - m_2^2) K_{1/2}(m_1) K_{1/2}(m_2)}, \tag{48}$$

and

$$B_2 = \frac{\chi_1}{P} \frac{m_2 \left[ 4\mu m_1 K_{3/2}(m_1) + (\lambda + 2\mu)p^2 K_{1/2}(m_1) \right]}{4\mu \left[ (p^2 - m_2^2)m_1 K_{3/2}(m_1) K_{1/2}(m_2) - (p^2 - m_1^2)m_2 K_{1/2}(m_1) K_{3/2}(m_2) \right] + (\lambda + 2\mu)p^2(m_1^2 - m_2^2) K_{1/2}(m_1) K_{1/2}(m_2)}. \tag{49}$$

The values of  $m_1$  and  $m_2$  take the same values for this problem and those in [10] though the dimensionless forms are different. Therefore the above results are equivalent to those in [10].

### 6. Numerical results and discussions

The solution in the space-time domain is obtained numerically by the method of Bellman et al. [22] for fixed value of the space variable and for dimensionless time  $\eta = \eta_i, i = 1(1)7$ , where  $\eta_i$ 's are computed from roots of the shifted Legendre polynomial of degree 7 (see Appendix) with  $S = 4$ . The computations for the state variables are carried out for different values of  $\eta_i, \eta_i = 0.0257750, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612, 3.67119$ . The material chosen for numerical evaluation is copper. The physical data for copper are taken as [11]

$$\rho = 8.96 \text{ gm/cm}^3, \quad \epsilon = 0.0168, \quad \lambda = 1.387 \times 10^{12} \text{ dy/cm}^2, \quad \mu = .448 \times 10^{12} \text{ dy/cm}^2,$$

and the hypothetical values of relaxation time parameters and material parameters  $c_T$  and  $c_K$  are taken as

$$\alpha = 2.0 \times 10^{-13} \text{ sec}, \quad \alpha^* = 1.0 \times 10^{-13} \text{ sec}, \quad c_T = 2$$

and  $c_K = 1.2$ .

The results of the numerical evaluation of the thermo-elastic stress variations, temperature distribution and displacement are illustrated in figs. 1 to 6. The magnitudes of the variation of stresses, temperature and displacement are observed when the step input of temperatures  $\chi_1 = 5$  and  $\chi_2 = 3$  are applied on the inner boundary  $R=1$  and outer boundary  $S=4$  of the shell respectively in all theories (CTE, CCTE, TRDTE(GL), TEWOED(GN-II) and TEWED(GN-III)). Figs.1(a) and 1(b) show the radial stress  $\sigma_R$  against radial distance  $R$  for time  $\eta=.69$  and  $.026$  respectively. In fig.1(a) it is observed that the qualitative behavior of CTE, CCTE, GL and GN-III models are almost the same. But in GN-II model oscillatory nature is observed. This is due to the rapid reflection of stress wave from outer boundary to inner boundary, since there is no energy dissipation in this theory.

In fig.1(b), when time is small ( $\eta = .026$ ) i.e, at early stage of wave propagation all the models except GN-II model give close results, whereas for advanced time (Fig.1(a)) the wave propagates with different speeds. The boundary conditions on the boundaries of the shell are found to be satisfied numerically. This is consistent with our theoretical results.

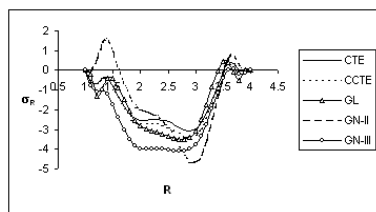
Figs.2(a) and 2(b) are plotted for hoop stress  $\sigma_\theta$  against radial distance  $R$  for time  $\eta= .69$  and  $.026$  respectively for all the models. It is clear from fig.2(a) that the maximum stress occurs near the inner boundary and almost oscillatory nature are observed for all the models for time  $\eta = .69$ , whereas for small time  $\eta = .026$  i.e, at the early stage the maximum stress occurs near the boundaries and almost zero in the interior of the shell for CTE, CCTE, GL and GN-III except in the case of GN-II model, where the oscillatory nature becomes more prominent near the boundaries.

Figs.3 and 4 depict  $\sigma_R$  and  $\sigma_\theta$  versus time  $\eta$  for  $R = 1.5$  respectively. It is seen that the stress waves propagate with time and magnitude of both the stresses in GN-II model are large in comparison with other models.

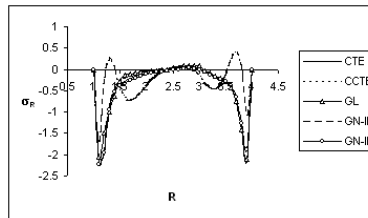
Fig.5 shows the graphs of temperature distribution ( $\Theta$ ) with the radial distance  $R$  for fixed time  $\eta = .69$ . All the profiles except GN-II model are found to be convex upwards with respect to the  $R$ -axis, and the magnitude of temperature distribution in GN-III model is observed to be greater than that of the other models (CTE, CCTE and GL). This is due to the fact that the term

involving thermal wave velocity ( $c_T$ ) is present in the GN-III model. The profile associated with GN-II model appears to be oscillatory in the region  $1 \leq R \leq 4$ , as in this case there is no damping term ( $c_K$ ) present in the expression of temperature distribution. It is also to be seen that the boundary conditions are satisfied on the boundaries ( $\chi_1 = 5$  on  $R=1$  and  $\chi_2 = 3$  on  $S = 4$ ). Which are in agreement with our theoretical results.

Fig.6 is plotted for radial variation of the displacement for fixed time  $\eta = .69$ . Almost similar behaviors are observed in all the theories. Now figs. 7-11 depicts  $\sigma_R$  versus  $R$  for fixed time  $\eta = .69$  with inner radius  $R = 1$  and different outer radius  $S = 3, 6, 8, 10$  and  $12$  for GN-III model. It is observed that as structural size increases the radial stress confined near the boundaries of the hollow sphere where the step input of temperatures are applied. Which is physically plausible. It is also observed that oscillations are accompanied by large  $R$ . This is due to the fact that stress waves propagate from the inner boundary to the outer boundary and from the outer boundary to the inner boundary at the same time. Similar behavior is observed in figs. 12-16 for hoop stress. For all above numerical calculations FORTRAN-77 programming Language has been used.

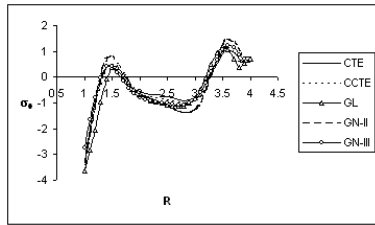


(a) Radial Stress vs.  $R$  for  $\eta = .69$ .

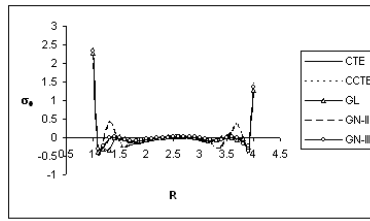


(b) Radial Stress vs.  $R$  for  $\eta = .026$ .

Figure 1: Radial Stress.



(a) Hoop Stress vs.  $R$  for  $\eta = .69$ .



(b) Hoop Stress vs.  $R$  for  $\eta = .026$ .

Figure 2: Hoop Stress.

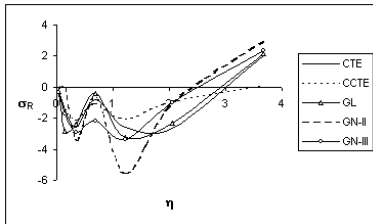


Figure 3: Radial Stress vs.  $\eta$  for  $R = 1.5$

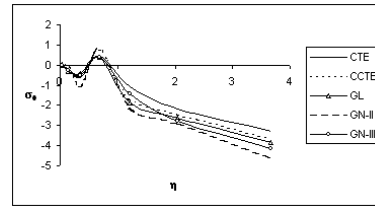


Figure 4: Hoop Stress vs.  $\eta$  for  $R = 1.5$

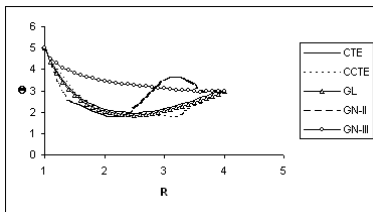


Figure 5: Variation of Temperature Distribution vs.  $R$  for  $\eta = .69$ .

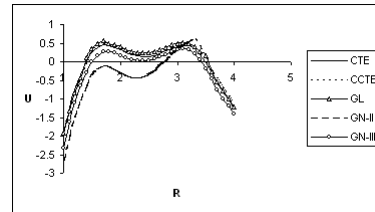


Figure 6: Variation of Displacement vs.  $R$  for  $\eta = .69$ .



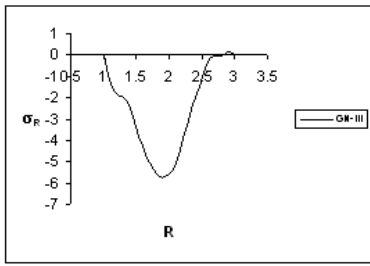


Figure 7: Radial Stress vs.  $R$  for  $\eta = .69$  and  $S = 3$ .

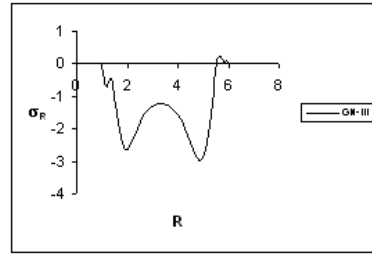


Figure 8: Radial Stress vs.  $R$  for  $\eta = .69$  and  $S = 6$ .

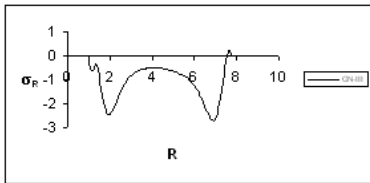


Figure 9: Radial Stress vs.  $R$  for  $\eta = .69$  and  $S = 9$ .

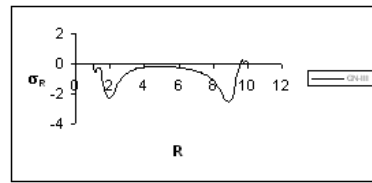


Figure 10: Radial Stress vs.  $R$  for  $\eta = .69$  and  $S = 10$ .

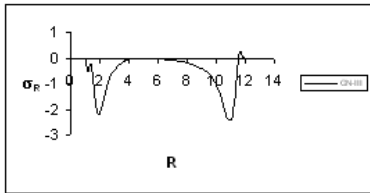


Figure 11: Radial Stress vs.  $R$  for  $\eta = .69$  and  $S = 12$ .

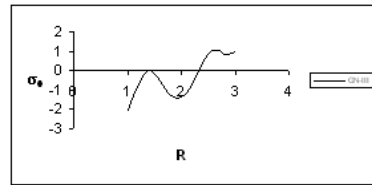


Figure 12: Hoop Stress vs.  $R$  for  $\eta = .69$  and  $S = 3$ .

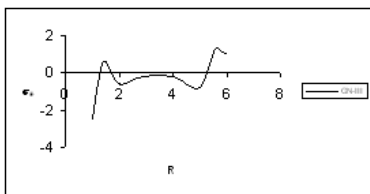


Figure 13: Hoop Stress vs.  $R$  for  $\eta = .69$  and  $S = 6$ .

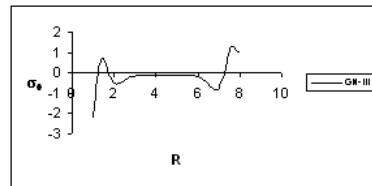


Figure 14: Hoop Stress vs.  $R$  for  $\eta = .69$  and  $S = 8$ .

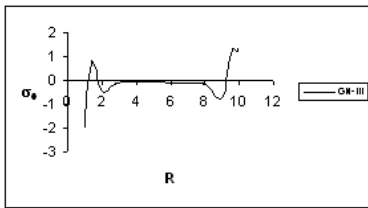


Figure 15: Hoop Stress vs.  $R$  for  $\eta = .69$  and  $S = 10$ .

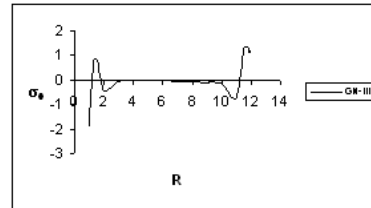


Figure 16: Hoop Stress vs.  $R$  for  $\eta = .69$  and  $S = 12$ .

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### Appendix

Let the Laplace transform of  $\sigma_i(R, \eta)$  be given by

$$\bar{\sigma}_j(R, p) = \int_0^{\infty} e^{-p\eta} \sigma_j(R, \eta) d\eta. \quad (A.1)$$

We assume that  $\sigma_j(R, \eta)$  is sufficiently smooth to permit the use of the approximate method we apply.

Putting  $x = e^{-\eta}$  in equation (A.1) we obtain

$$\bar{\sigma}_j(R, p) = \int_0^1 x^{p-1} g_j(R, x) dx, \quad (A.2)$$

where

$$g_j(R, x) = \sigma_j(R, -\log x). \quad (A.3)$$

Applying the Gaussian quadrature rule to the equation (A.2) we obtain the approximate relation

$$\sum_{i=1}^n W_i x_i^{p-1} g_j(R, x_i) = \bar{\sigma}_j(R, p), \quad (A.4)$$

where  $x_i$ 's ( $i = 1, 2, \dots, n$ ) are the roots of the shifted Legendre polynomial and  $W_i$ 's ( $i = 1, 2, \dots, n$ ) are the corresponding weights [22] and  $p = 1(1)n$ .

For  $p = 1(1)n$ , the equations (A.4) can be written as

$$W_1 g_j(R, x_1) + W_2 g_j(R, x_2) + \dots + W_n g_j(R, x_n) = \bar{\sigma}_j(R, 1)$$

$$W_1 x_1 g_j(R, x_1) + W_2 x_2 g_j(R, x_2) + \dots + W_n x_n g_j(R, x_n) = \bar{\sigma}_j(R, 2)$$

.....

.....

$$W_1x_1^{n-1}g_j(R, x_1) + W_2x_2^{n-1}g_j(R, x_2) + ..... + W_nx_n^{n-1}g_j(R, x_n) = \bar{\sigma}_j(R, n)$$

Therefore

$$\begin{pmatrix} g_j(R, x_1) \\ g_j(R, x_2) \\ \dots \\ \dots \\ g_j(R, x_n) \end{pmatrix} = \begin{pmatrix} W_1 & W_2 & \dots & W_n \\ W_1x_1 & W_2x_2 & \dots & W_nx_n \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ W_1x_1^{n-1} & W_2x_2^{n-1} & \dots & W_nx_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \bar{\sigma}_j(R, 1) \\ \bar{\sigma}_j(R, 2) \\ \cdot \\ \cdot \\ \bar{\sigma}_j(R, n) \end{pmatrix} \quad (A.5)$$

(As the matrix is the product of  $\text{diag}\{W_i\}$  multiplied by Vander Monde matrix, it can be shown that the matrix is non-singular.)

Hence  $g_j(R, x_1), g_j(R, x_2), \dots, g_j(R, x_n)$  are known.

For  $n = 7$  we have

Roots of the shifted

Corresponding Weights

Legendre Polynomial

2.5446043828620886E-2	6.4742483084434816E-2
1.2923440720030282E-1	1.3985269574463828E-1
2.9707742431130145E-1	1.9091502525255938E-1
5.0000000000000000E-1	2.0897959183673466E-1
7.0292257568869853E-1	1.9091502525255938E-1
8.7076559279969706E-1	1.3985269574463828E-1
9.7455395617137909E-1	6.4742483084434816E-2

From equations in (A.5) we can calculate the discrete values of  $g_j(R, x_i)$  i.e,  $\sigma_j(R, \eta_i); (i = 1, 2, \dots, 7)$  and finally using interpolation we obtain the stress components  $\sigma_i(R, \eta); (i = R, \theta)$ .