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Counting Symmetries with Burnside's Lemma and Pólya's Theorem

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Abstract. Counting concerns a large part of combinational analysis. Burnside's lemma, sometimes also called Burnside's counting theorem, the Cauchy-Frobenius lemma or the orbit-counting theorem [5], is often useful in taking account of symmetry when counting mathematical objects. The Pólya's theorem is also known as the Redfield-Pólya Theorem which both follows and ultimately generalizes Burnside's lemma on the number of orbits of a group action on a set. Pólya's Theory is a spectacular tool that allows us to count the number of distinct items given a certain number of colors or other characteristics. Sometimes it is interesting to know more information about the characteristics of these distinct objects. Pólya's Theory is a unique and useful theory which acts as a picture function by producing a polynomial that demonstrates what the different configurations are, and how many of each exist.The dynamics of counting symmetries are the most interesting part. This subject has been a fascination for mathematicians and scientist for ages. Here 16 Bead Necklace, patterns of n tetrahedron with 2 colors, patterns of n cubes with 3 and 4 colorings and so on have been solved.

2010 Mathematics Subject Classifications: 14L,14P,14Q

Key Words and Phrases: Burnside's Lemma, Pólya's Theorem, Counting Symmetries,Group

1. Introduction and Preliminaries

1.1. Group Action

Let G be a group and A is a set which is nonempty. Suppose that G is a permutation group acting on a set A, and consider the set B^A of all functions

$$f : A \to B.$$

Then G acts naturally on B^A as follows: if $g \in G$ and $f \in B^A$ then define the function fg by [8]

$$fg(a) = f(ag^{-1}).$$

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1.2. The Cycle Index of a Permutation Group

Let us consider G is a permutation group acting on a set A which contains p number of elements. Suppose $X \in G$ which can be written as the product of t_1 disjoint cycles of length 1, t_2 disjoint cycles of length $2, \ldots, t_p$ disjoint cycles of length p then X is called of cycle type $(t_1, t_2, \ldots, t_p)$.

Example 1. *Let $G = \{X_1, X_2, X_3\}$ be a cyclic group of order 3 and the set $A = \{1, 2, 3\}$. Then we have*

$$X_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$$
$$X_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$
$$X_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

Here X_1 has 3 cycle type of length 1. It is of type $(3, 0, 0)$, known as identity mapping. X_2 and X_3 each has 1 cycle type of length 3 and are of type (0,0,1) [6].

Note that if we split A into k_1 cycles of length 1, k_2 cycles of length $2, \ldots, k_p$ cycles of length p then we have $k_1 + 2k_2 + 3k_3 + \ldots + pk_p = p$, the sum of lengths of the cycles are the total number of elements in A [2].

Definition 1. *Let G be a group and its elements are the permutations of the set A. Consider the group operation be multiplication and let us define a polynomial in p variables $t_1, t_2, t_3, \ldots, t_p$ where the coefficients are non negative. Then for every $g \in G$ we can form the product $t_1^{k_1}, t_2^{k_2}, t_3^{k_3}, \ldots, t_p^{k_p}$. So the cycle index G is the polynomial defined by*

$$Z(G, t_1, t_2, t_3, \ldots, t_p) = \frac{1}{|G|}\{\sum_{g \in G} t_1^{k_1} t_2^{k_2} t_3^{k_3} \ldots t_p^{k_p}\}$$

Example 2. *Consider $G = S_3 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ where*

$$X_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$$
$$X_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$
$$X_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$
$$X_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (2)(13)$$
$$X_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(23)$$

$$X_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3)$$

We observe that the permutations of each X_4, X_5 and X_6 has one cycle of length 2 and one cycle of length 1 and are of type (1,1,0). X_1,X_2 and X_3 are known to us by the Example 1. The order of this group is $3! = 6$. Hence by the definition the cycle index of S_3 is [6]

$$Z(S_3, t_1, t_2, t_3) = \frac{1}{6}\{t_1^3 + 3t_1t_2 + 2t_3\}.$$

Theorem 1. *The cycle index of the symmetric group $Z(S_p)$ can be defined by*

$$Z(S_p) = \frac{1}{p!}\{\sum t(t_1^{k_1} t_2^{k_2} t_3^{k_3} \dots t_p^{k_p})\}$$

where

$$t = \frac{p!}{(1^{k_1}!)(2^{k_2}!)(3^{k_3}!)\dots(p^{k_p}!)}$$

and the summation is over p-tuple $(k_1, k_2, k_3, \dots, k_p)$ which are nonnegative integers k_i satisfying $1k_1 + 2k_2 + 3k_3 + \dots + pk_p = p$ [6].

There are some formulas for cycle indices. In this study we will use indices for symmetric groups, cyclic groups, and dihedral groups.

Definition 2. *The totient function $\phi(n)$, also called Euler's totient function or the Euler-phi function, is defined as the number of positive integers $\leq n$ that are relatively prime to n, where 1 is counted as being relatively prime to all numbers*.*
For example, there are four numbers relatively prime to 12 that are less than 12 (1,5,7, and 11), so $\phi(12) = 4$.

For cyclic groups and dihedral groups we have the following formulas:

$$Z(C_n) = \frac{1}{n}\sum_{k/n} \phi(k) t_k^{n/k}, \tag{1}$$

and

$$Z(D_n) = \frac{1}{2}Z(C_n) + \begin{cases} \frac{1}{2}t_1 t_2^{(n-1)/2} & \text{if } n \text{ odd}, \\ \frac{1}{4}(t_2^{n/2} + t_1^2 t_2^{(n-2)/2}) & \text{if } n \text{ even}. \end{cases}$$

$Z(D_n)$ is divided into two cases by imagining a necklace of four beads versus a necklace of five beads. With four beads, there are two types of flips: those over the line through the centers of opposite edges, and those over the line connecting opposite vertices. However, with five beads, there is only one type of flip: those over lines connecting a vertex with the center of an opposite edge. This can be extended to any even or odd number of beads, respectively [4].

*K. R. Rumery, Twelve-Tone Composition, Part One, `http://jan.ucc.nau.edu/~krr2/12tone/12tone1.html`

Example 3. *Let us use this formula to compute the cycle index of C_4,*

$$\begin{aligned} Z_{C_4} =& \frac{1}{4}\sum_{l/4}\phi(l)t_l^{4/l} \\ =& \frac{1}{4}(\phi(1)t_1^{4/1}+\phi(2)t_2^{4/2}+\phi(4)t_4^{4/4}) \\ =& \frac{1}{4}(t_1^4+t_2^2+2t_4) \end{aligned}$$

Example 4. *Here we will illustrate the formula by computing the cycle index of D_4,*

$$\begin{aligned} Z_{D_4} =& \frac{1}{2}Z_{C_4}+\frac{1}{4}(t_2^{\frac{4}{2}}+t_1^2t_2^{\frac{4-2}{2}}) \\ =& \frac{1}{2}(\frac{1}{4}(t_1^4+t_2^2+2t_4))+\frac{1}{4}(t_2^{\frac{4}{2}}+t_1^2t_2^{\frac{4-2}{2}}) \\ =& \frac{1}{8}(t_1^4+t_2^2+2t_4)+\frac{1}{4}(t_2^2+t_1^2t_2) \\ =& \frac{1}{8}(t_1^4+2t_1^2t_2+3t_2^2+2t_4) \end{aligned}$$

Now we have already got the cycle index of the symmetric groups from Theorem 1. The following example will find the coefficient on n-th term.

Example 5. *Let A be a set of n elements and let G be the group of all permutations of A. That is G is a symmetric group of degree n. Its cycle index turns out to be equal to the coefficient of x^n in the development of*

$$exp(xt_1+\frac{x^2t_2}{2}+\frac{x^3t_3}{3}+\ldots)$$

as a power series of in x. This expression can be written as

$$\sum_{g_1=0}^{\infty}\frac{x^{g_1}t_1^{g_1}}{g_1!}\sum_{g_1=0}^{\infty}\frac{x^{2g_2}t_2^{g_2}}{g_2!2^{g_2}}\ldots$$

The coefficient of x^n is obtained by summing the expression

$$t_1^{g_1}t_2^{g_2}\ldots(g_1!g_2!2^{g_2}\ldots)^{-1}$$

over all possible $g_1, g_2, \ldots$ satisfying $g_1+2g_2+\ldots = n$ [2].

Definition 3. *The set of elements g of a group G such that $g^{-1}Hg = H$, is said to be the* ***normalizer*** *N_G with respect to a subset of group elements of H. If H is a subgroup of G, $N_G(H)$ is also a subgroup containing H.*

1.3. Equivalence Relations

Let A be a set and $\sim$ be a binary relation of A. Then $\sim$ is an **equivalence relation** if and only if for all $p, q, r \in A$ the following conditions are satisfied:

Reflexivity $p \sim p$

Symmetry If $p \sim q$ then $q \sim p$

Transitivity If $p \sim q$ and $q \sim r$ then $p \sim r$

The **equivalence class** of p under $\sim$ is the set $\{p \in A | p \sim q\}$ [1]. These equivalence classes are called **patterns**.

1.4. Weights and Inventory

Definition 4. *Given a set of colors C we want to associate a weight w_c for all $c \in C$. Then we will define the **weight** of a coloring $a \in A$ to be the product of the weights of the colored elements [1].*

$$w(a) = \prod_{i \in N} w_{a(i)}$$

*This is also called the **inventory** of A.*

Example 6. *Consider 4 bead necklace to illustrate with C={Red(r), Yellow(y)}. Let $a = rryy$ where*

$$a(1) = Red,\ a(2) = Red,\ a(3) = Yellow \text{ and } a(4) = Yellow$$

Let the colors Red and Yellow have weights R and Y respectively. Then the coloring of a as follows

$$\begin{aligned} w(a) =& \prod_{i \in N} w_{a(i)} \\ =& w_{a(1)} w_{a(2)} w_{a(3)} w_{a(4)} \\ =& RRYY \\ =& R^2 Y^2 \end{aligned}$$

Definition 5. *Let G be a permutation group on a set A. Suppose that $\sim$ defines an equivalence relation on A. Then the equivalence class containing the element a, denoted by $Orb(a)$ is called the G-**orbit** of a. It is defined as [6]*

$$Orb(a) = \{\pi(a) : \pi \in G\}$$

Example 7. *Consider 4 bead necklace where $A = \{1, 2, 3, 4\}$, $B = \{Red(R), Yellow(Y)\}$, C be the group of coloring with $f : A \to B$ and $G = \{e, (12), (34), (12)(34)\}$ be the permutation group [1]. Then*

$$Orb(RRRR) = \{RRRR\} Orb(RRRY) = Orb(RRYR) = \{RRRY, RRYR\}$$

$$\begin{aligned}
Orb(RRYY) =& \{RRYY\} Orb(RYRR) = Orb(YRRR) = \{RYRR, YRRR\} \\
Orb(YYYY) =& \{YYYY\} Orb(YYYR) = Orb(YYRY) = \{YYYR, YYRY\} \\
Orb(YYRR) =& \{YYRR\} Orb(YRYY) = Orb(RYYY) = \{YRYY, RYYY\} \\
Orb(RYRY) =& Orb(YRYR) = Orb(RYYR) = Orb(YRRY) = \{RYRY, YRYR, RYYR, YRRY\}
\end{aligned}$$

The problem of determining the equivalent objects in A is to count all G-orbits. However this method looks impractical and even more boring in complex situations. Fortunately, Burnside's Lemma gives an analytical formula for such counting of G-orbits. It is a powerful technique and particularly efficient when the order of the group is small. It is also considered one of the essential parts in development of the Pólya Theory.

Definition 6. *Let G be a permutation group on a set A. An element a in A is said to be invariant under a permutation $\pi \in G$ if and only if $\pi(a) = a$. To each $a \in A$, the* ***stabilizer*** *of a, denoted by $Stab(a)$ is defined to be the set*

$$Stab(a) = \{\pi \in G : \pi(a) = a\}$$

and $\forall a \in G$. Let $\eta(\pi)$ denotes the number of elements of A that are invariant under π that is [6]

$$\eta(\pi) = \{a \in A : \pi(a) = a\}$$

Example 8. *Following the Example 7 the stabilizer sets for each coloring in C are given below*

$$\begin{aligned}
Stab(RRRR) =& Stab(YYYY) = Stab(RRYY) = Stab(YYRR) = \{e, (12), (34), (12)(34)\} \\
Stab(RRRY) =& Stab(YYYR) = Stab(RRRY) = Stab(YYYR) = \{e, (12)\} \\
Stab(RYRR) =& Stab(YRYY) = Stab(RYYY) = Stab(YRRR) = \{e, (34)\} \\
Stab(YRRY) =& Stab(RYYR) = Stab(RYRY) = Stab(YRYR) = \{e\}
\end{aligned}$$

2. Burnside's Lemma and Pólya's Theorem

2.1. Orbit-Stabilizer Formula

Let G be a group, acting on a set A, then For all $a \in A$, $|G| = |Stab(a)||Orb(a)|$ [6].

Example 9. *By following the Example 7 and 8 we have*

$$\begin{aligned}
Stab(YYRR) =& \{e, (12), (34), (12)(34)\} Stab(YRRR) = \{e, (34)\} \\
Orb(YYRR) =& \{YYRR\} Orb(YRRR) = \{RYRR, YRRR\}
\end{aligned}$$

Therefore

$$\begin{array}{ll}
|Stab(YYRR)| = 4 & |Stab(YRRR)| = 2 \\
|Orb(YYRR)| = 1 & |Orb(YRRR)| = 2 \\
\Rightarrow |Stab(YYRR)||Orb(YYRR)| = 4.1 = 4 & \Rightarrow |Stab(YRRR)||Orb(YRRR)| = 2.2 = 4
\end{array}$$

2.2. Burnside's Lemma

Lemma 1. *Let G be a finite group that acts on a set A. For each $g \in G$, let A^g denote the set of elements in A that are fixed by g. Burnside's lemma asserts the following formula for the number of orbits, denoted $|A/G|$:*

$$|A/G| = \frac{1}{|G|}\sum_{g\in G}|A^g|$$

Thus the number of orbits (a natural number or $+\infty$) is equal to the average number of points fixed by an element of G [3].

Proof. The proof uses orbit-stabilizer theorem and the fact that A is the disjoint union of the orbits:

$$\begin{aligned}\sum_{g\in G}|A^g| =& |\{(g,a)\in G\times A | g\cdot a = a\}| = \sum_{a\in A}|Stab(a)| = \sum_{a\in A}\frac{|G|}{|Orb(a)|}\\ =&|G|\sum_{a\in A}\frac{1}{|Orb(a)|} = |G|\sum_{P\in A/G}\sum_{a\in P}\frac{1}{|P|} = |G|\sum_{P\in A/G}1 = |G|\cdot|A/G|\end{aligned}$$

Therefore [6]

$$\frac{1}{|G|}\sum_{g\in G}|A^g| = |A/G|$$

This proves the lemma. □

Example 10. *Continuing With the Example 7 and 8, We have $A(e) = C$*

$$\begin{aligned}A(12) =& \{RRRR, RRYR, RRRY, RRYY, YYRR, YYYR, YYRY, YYYY\}\\ A(34) =& \{RRRR, YRRR, RYRR, RRYY, YYRR, RYYY, YRYY, YYYY\}\\ A((12)(34)) =& \{RRRR, RRYY, YYRR, YYYY\}\end{aligned}$$

By applying Burnside's Lemma, we have

$$\frac{1}{|G|}\sum_{g\in G}|A^g| = \frac{1}{4}(16+8+8+4) = 9$$

Example 11. *Let G be the group of rotations of a cube induced on the set of 6 faces (Fig. 1). The rotations of the cube which leaves it invariant are*

(i) 6 rotations of 90 degree(clockwise or anti-clockwise) about the axes joining the centers of the opposite faces;

(ii) 3 rotations of 180 degree each of the above axes;

(iii) 8 rotations of 120 degree(clockwise or anti-clockwise) about the axes joining the opposite vertices;

(iv) 6 rotations of 180 degree about the axes joining the midpoints of the opposite edges and

(v) the identity.

To determine the cyclic index of its action on the set of faces it is observed that

(i) has rotations of order 4 and cycle type $t_1^2 t_4$;

(ii) has rotations of order 2 and cycle type $t_1^2 t_2^2$;

(iii) has rotations of order 3 and cycle type t_3^2

(iv) has rotations of order 2 and cycle type t_2^3

(v) has rotations of order 1 and cycle type t_1^6

Therefore the cycle index of G is

$$Z(G, t_1, t_2, \ldots, t_6) = \frac{1}{24}(6t_1^2 t_4 + 3t_1^2 t_2^2 + 8t_3^2 + 6t_2^3 + t_1^6).$$

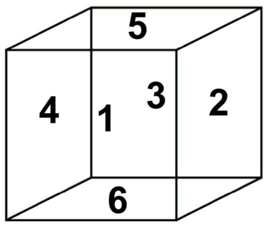
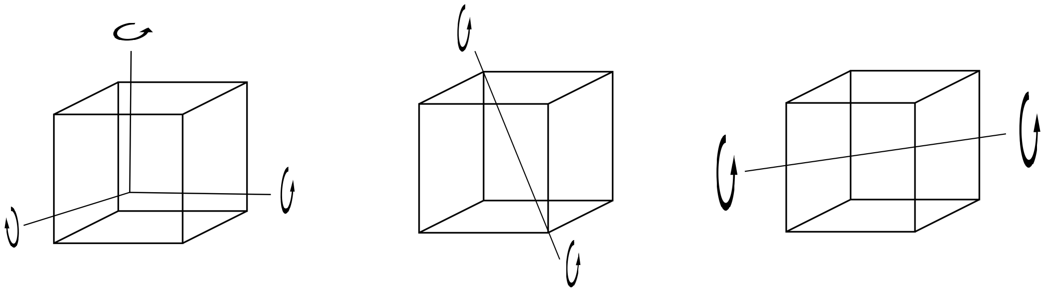


Figure 1: Faces of the cube



(a) Axis (b) Corner (c) Midpoint

Figure 2: Types of Rotations of the Cube

Consider $G = C_n$ *be cyclic group of order n with regarded as the group of permutations of the vertices of a regular n-gon. However it is the subgroup of* S_n *generated by an n-cycle(1,2,...,n). For a generator g of* S_n*, the element* g^i *has the same cyclic structure as that of* $gcd\{i,n\}$ *and cycle of length* $d = \frac{n}{gcd(i,n)}$ *and therefore has of order* d*. The number of elements of order* d *is equal to* *the number of integers* $1 \leq p \leq d$ *with* $gcd(p,d) = 1$*, which is already given by Euler's function* ϕ *from the elementary theory (Definition 4). Thus from* $(Eq1)$ *we obtain*

$$Z(G, t_1, t_2, \ldots, t_n) = \frac{1}{n} \sum_{d/n} \phi(d) t_d^{n/d}$$

where $\phi(p^n) = (p-1)p^{n-1}$ *for prime powers and* $\phi(m) = \Pi_{i=1}^{k} \phi(p_i^{n_i})$ *for* $m = \Pi_{i=1}^{k} p_i^{n_i}$ *[3].*

Example 12. *Again let G be the group of rotations of a cube induced on the set of 8 vertices. The rotations of the cube which leaves it invariant are the same as Example 11. It is now necessary to see what the rotations do with the vertices. To determine the cycle index of its action on the set of faces it is observed that*

(i) has rotations of order 4 and cycle type t_4^2*;*

(ii) has rotations of order 2 and cycle type t_2^4*;*

(iii) has rotations of order 4 and cycle type $t_1^2 t_3^2$

(iv) has rotations of order 2 and cycle type t_2^4

(v) has rotations of order 1 and cycle type t_1^8

Therefore the cycle index of G is [2]

$$Z(G, t_1, t_2, \ldots, t_8) = \frac{1}{24}(6t_4^2 + 9t_2^4 + 8t_1^2 t_3^2 + 6t_2^3 + t_1^8).$$

2.3. Pólya's Substitution

Let $\sigma(c_1, c_2, \ldots, c_k)$ be the polynomial obtained from the cycle index $Z(G, t_1, t_2, \ldots, t_n)$ by substituting

$$\begin{aligned}
t_1 &\longmapsto c_1 + c_2 + \ldots + c_k, \\
t_2 &\longmapsto c_1^2 + c_2^2 + \ldots + c_k^2, \\
&\ldots \\
t_n &\longmapsto c_1^n + c_2^n + \ldots + c_k^n,
\end{aligned}$$

Then the number of non-equivalent k colorings of A pattern $[p_1, p_2, \ldots, p_k]$ is equal to the coefficient of the term $c_1^{p_1} c_2^{p_2} \ldots c_k^{p_k}$ in $\sigma(c_1, c_2, \ldots, c_k)$ [3]. Then we find the generating function

$$Z_G\left(\sum_{i=1}^{k} c_i, \sum_{i=1}^{k} c_i^2, \ldots, \sum_{i=1}^{k} c_i^n\right),$$

where n stands for the largest cycle length. This is known as **Pólya's Enumeration Formula**.

Problem 1 (Edge Coloring of a Tetrahedron)**.** A tetrahedron has 6 edges. How many inequivalent ways are there to color it with

(a) 2 colors?

(b) 3 black and 3 yellow edges?

Solution: First we need to find out the symmetries of the tetrahedron with permutations of the edges. Let us consider the set A of the edges of a tetrahedron and G be the set of permutations of A which will be produced by rotating the tetrahedron. Their cyclic structures are given as follows

(i) The identity leaves all 6 edges fixed with structure t_1^6.

(ii) There are four 120° rotations about a corner and the middle of the opposite face (Fig. 3). They are two cycles of length three. They cyclically permute the edges incident to that corner and also the edges bounding the opposite face, so the cycle structure representation is t_3^2.

(iii) There are four 240° rotations as like the 120° rotations (Fig. 3). So the cycle structure is also same.

(iv) There are three 180° rotations about opposite edges leave the two edges fixed. The other four edges are left in cycles of length 2. Thus we have the cycle structure $t_1^2 t_2^2$ [7].

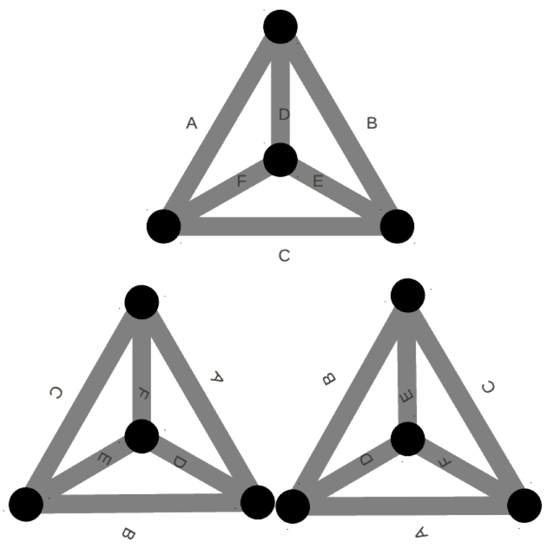


Figure 3: Rotation of tetrahedron by 120° and 240°

Then by Burnside's Lemma we have

$$Z(G,t_1,t_2,\ldots,t_6)=\frac{1}{12}(t_1^6+8t_3^2+3t_1^2t_2^2)$$

(a) The number of distinct 2 colorings on tetrahedron edge will be

$$Z_G=\frac{1}{12}(2^6+8(2^2)+3(2^2)(2^2))=12$$

(b) Now we will apply Pólya's Substitution method. Let the colors black and yellow be denoted by x and y respectively. Then substituting $t_i\to(x^i+y^i)$ for each $i=\{1,2,3,4,5,6\}$ in the previous equation, we get

$$Z_G=\frac{1}{12}((x+y)^6+8(x^3+y^3)^2+3(x+y)^2(x^2+y^2)^2)$$

By Maple we get,

$$Z_G=x^6+x^5y+2x^4y^2+4x^3y^3+2x^2y^4+xy^5+y^6$$

The term $4x^3y^3$ contains tetrahedron with 3 black and 3 yellow edges. The coefficient of this term is 4. Hence there are 4 ways to color the edge of tetrahedron.

2.4. Pólya's Fundamental Theorem

Theorem 2. *The Pattern inventory is*

$$PI=Z_G(\sum_{c\in C}w_c,\sum_{c\in C}w_c^2,\sum_{c\in C}w_c^3,\ldots,\sum_{c\in C}w_c^p)$$

where Z_G is the cycle index and $p=|D|$ [2].

Proof. Let $X=X_1\cup X_2\cup\ldots\cup X_m$ be the equivalence classes of X under the relation

$$x\sim y \text{ if and only if } W(x)=W(y)$$

By Orbit-Stabilizer Formula, $g*x\sim x$ for all $x\in X$, $g\in G$ and so we can think of G acting on each X_i individually. We use the fact that $x\in X_i$ implies $g*x\in X_i$ for all $i\in[m]$, $g\in G$. We use the notation $g^{(i)}\in G^{(i)}$ when we restrict attention to X_i.

Let m_i denote the number of orbits. Then

$$PI=\sum_{i=1}^{m}m_iW_i=\sum_{i=1}^{m}W_i(\frac{1}{|G|}\sum_{g\in G}|Fix(g^{(i)})|),$$

By Burnside's Lemma, we have

$$PI=\frac{1}{|G|}\sum_{g\in G}\sum_{i=1}^{m}|Fix(g^{(i)})|W_i=\frac{1}{|G|}\sum_{g\in G}W(Fix(g)) \tag{2}$$

Note that (1) follows $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ if and only if $x \in W_i$ and $g * x = x$.

Suppose that $Z_G = x_1^{k_1} x_2^{k_2} \dots x_p^{k_p}$. Then we can claim that

$$W(Fix(g)) = (\sum_{c \in C} w_c)^{k_1} (\sum_{c \in C} w_c)^{k_2} \dots (\sum_{c \in C} w_c)^{k_p} \tag{3}$$

Substituting (2) and (3) yields the theorem.

To verify (2) we use the fact that if $x \in Fix(g))$, then the elements of a cycle of g must be given the same color. A cycle of length i will then contribute a factor $\sum_{c \in C} w_c^i$ where the term w_c^i comes from the choice of c for every element of the cycle [8]. □

Example 13. *Let* $A = \{1, 2, 3, 4, 5, 6\}$ *and* $B = \{white, black, gray\}$. *Consider a function* $f \in B^A$ *which is a 6-tuple of colors and can be viewed as necklace representation.*

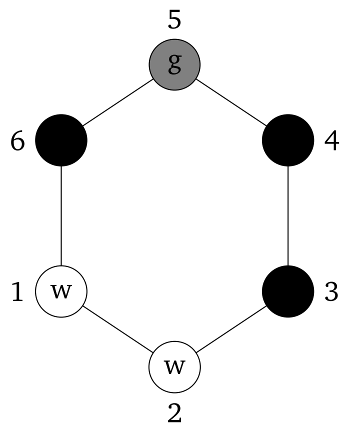


Figure 4: Colored Necklace

Now we are interested to assign different weights to the beads. Suppose that $black(b) = 1$, $white(w) = 10$ *and* $gray(g) = 100$. *Then the weight of the necklace is*

$$W(f) = 10 \times 10 \times 1 \times 1 \times 100 = 10^4$$

Note that rotation does not change the weight.

To count the necklaces (without weight), we have to calculate the number of fixed points for each element of C_6.

Table 1: Fixed Points

Element g	$\|fix(g)\|$
e	3^6
$(1,2,3,4,5,6)$	3
$(1,3,5)(2,4,6)$	3^2
$(1,4)(2,5)(3,6)$	3^3
$(1,5,3)(2,6,4)$	3^2
$(1,6,5,4,3,2)$	3

Now we have to consider counting the sums of the weights of the fixed points of each element of C_6. Let us calculate Wfix(g) for the element $g = (1,4)(2,5)(3,6)$. For a function f to be fixed by g we must have

$$f(1) = f(4)\ f(2) = f(5)\ f(3) = f(6)$$

and hence

$$W(f) = W(f(1))^2 W(f(2))^2 W(f(3))^2$$

Now $f(1)$, $f(2)$ and $f(3)$ can be any white, black or gray and that is why

$$Wfix(g) = (W(white)^2 + W(black)^2 + W(gray)^2)^3$$

. By repeating this process we can find out the other elements of C_6. Then we get

Table 2: Weighted Fixed Points

Element g	Wfix(g)
e	$(W(white) + W(black) + W(gray))^6$
$(1,2,3,4,5,6)$	$W(white)^6 + W(black)^6 + W(gray)^6$
$(1,3,5)(2,4,6)$	$(W(white)^3 + W(black)^3 + W(gray)^3)^2$
$(1,4)(2,5)(3,6)$	$(W(white)^2 + W(black)^2 + W(gray)^2)^3$
$(1,5,3)(2,6,4)$	$(W(white)^3 + W(black)^3 + W(gray)^3)^2$
$(1,6,5,4,3,2)$	$W(white)^6 + W(black)^6 + W(gray)^6$

For any given values of $W(white), W(black)$ and $W(gray)$ we can calculate this sum and find the sum of the weights of the weighted $(6,3)-$ necklaces.

At this moment we would like to apply Pólya's Fundamental Theorem. First we need to assign the weights of the three colors. Let $W(white) = x$, $W(black) = y$ and $W(gray) = z$. Then the weight of a necklace representative is not a number, but a multivariate polynomial in the variables x, y and z [8].

$$W(f) = x \times x \times y \times y \times z \times y = x^2 y^3 z.$$

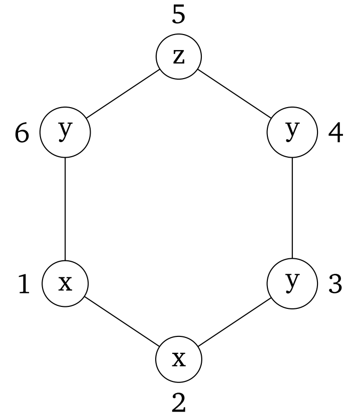


Figure 5: Weighted Necklace

By using Burnside's Lemma we get that

$$Z(C_6, t_1, t_2, \dots, t_6) = \frac{1}{6}(t_1^6 + t_2^3 + 2t_3^2 + 2t_6)$$

According to Pólya's Substitution we have

$$\begin{aligned}
t_1 &\longmapsto x + y + z,\\
t_2 &\longmapsto x^2 + y^2 + z^2,\\
t_3 &\longmapsto x^3 + y^3 + z^3,\\
t_4 &\longmapsto x^4 + y^4 + z^4,\\
t_5 &\longmapsto x^5 + y^5 + z^5,\\
t_6 &\longmapsto x^6 + y^6 + z^6,
\end{aligned}$$

Now

$$PI = \frac{1}{6}((x + y + z)^6 + (x^2 + y^2 + z^2)^3 + 2(x^3 + y^3 + z^3)^2 + 2(x^6 + y^6 + z^6))$$

Using Maple the result becomes

$$\begin{aligned}
PI =& 16x^2y^2z^2 + 10x^3y^2z + 5x^4yz + 10x^3yz^2 + 10x^2y^3z + 10xy^3z^2 + 10xy^2z^3\\
&+ 5xy^4z + x^5y + x^5z + 3x^4y^2 + 3x^4z^2 + 4x^3y^3 + 4x^3z^3 + \mathbf{3x^2y^4} + 3x^2z^4 + xy^5\\
&+ xz^5 + y^5z + 3y^4z^2 + 4y^3z^3 + 3y^2z^4 + yz^5 + 10x^2yz^3 + 5xyz^4 + x^6 + y^6 + z^6
\end{aligned}$$

Each term of this expression corresponds to necklaces with a fixed number of red, green and blue beads. For example, the term $3x^2y^4$ says that there are 3 orbits with weight xy^4z or in other words 3 necklaces with 2 white beads, 4 black beads and 0 gray. They are

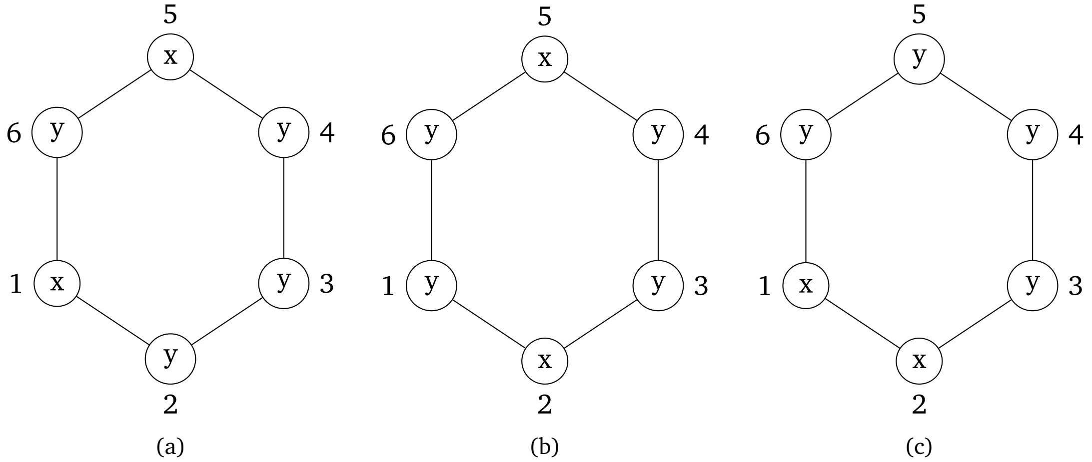


Figure 6: 3 Necklaces with 2 White Beads and 4 Black Beads

Problem 2 (Face Coloring of a Cube)**.** A cube has 6 faces and the faces are to be colored. How many inequivalent ways are there to color it with namely

(a) 4 red and 2 green faces?

(b) 3 red, 2 green and 1 blue faces?

(c) 2 red, 1 green, 1 blue and 2 yellow faces?

Solution: We have to color the faces of the cube. So recall Example 11. Then the cycle index is

$$Z(G, t_1, t_2, \ldots, t_6) = \frac{1}{24}(6t_1^2 t_4 + 3t_1^2 t_2^2 + 8t_3^2 + 6t_2^3 + t_1^6)$$

Now we will apply Pólya's Substitution method. Let the colors red, green, blue and yellow be denoted by x, y, z and p respectively. Then substituting $t_i \to (x^i + y^i + z^i + p^i)$ for each $i = \{1, 2, 3, 4, 5, 6\}$ in the previous equation, we get

$$\begin{aligned} Z_G =& \frac{1}{24}(6(x+y+z+p)^2(x^4+y^4+z^4+p^4) + 3(x+y+z+p)^2(x^2+y^2+z^2+p^2)^2 \\ &+ 8(x^3+y^3+z^3+p^3)^2 + 6(x^2+y^2+z^2+p^2)^3 + (x+y+z+p)^6) \end{aligned}$$

Using Maple we find that,

$$\begin{aligned} Z_G =& 3y^3px^2 + 3y^3zp^2 + 3yz^3p^2 + 3y^3pz^2 + 3yp^3z^2 + 3xp^3y^2 + 3x^3py^2 + 3x^3pz^2 \\ &+ 3xp^3z^2 + 6x^2z^2p^2 + 2ypx^4 + 2zpx^4 + 2yzp^4 + 2xpz^4 + 3zp^3y^2 + 2yzx^4 \\ &+ 8xyz^2p^2 + 8xzy^2p^2 + 8xpy^2z^2 + \mathbf{8yzx^2p^2} + 8ypx^2z^2 + 8zpx^2y^2 + 5x^3yzp \\ &+ 5xy^3zp + 5xyzp^3 + 5xyz^3p + 3yz^3x^2 + 3yp^3x^2 + 3xy^3z^2 + 2xyz^4 + 6x^2y^2z^2 \\ &+ 6x^2y^2p^2 + 2ypz^4 + 2xpy^4 + 2xyp^4 + 3xz^3p^2 + 3x^3zp^2 + 2xzp^4 + 3x^3yp^2 \end{aligned}$$

$$+3x^3yz^2+3xy^3p^2+2zpy^4+2xzy^4+3y^3zx^2+\mathbf{3x^3zy^2}+3z^3px^2+6y^2z^2p^2$$
$$+3z^3py^2+3zp^3x^2+3xz^3y^2+2y^2z^4+y^5z+yz^5+2z^2x^4+2z^2y^4+x^5z$$
$$+xz^5+\mathbf{2y^2x^4}+2x^2y^4+2x^2z^4+x^5y+xy^5+x^6+y^6+z^6+p^6+2x^3y^3+2x^3z^3$$
$$+2y^3z^3+2x^2p^4+x^5p+xp^5+2y^2p^4+y^5p+yp^5+2z^2p^4+z^5p+zp^5+2p^2x^4$$
$$+2p^2y^4+2p^2z^4+2x^3p^3+2y^3p^3+2z^3p^3$$

Each term of this expression on the right hand side corresponds to a fixed number of red, green, blue and yellow colors.

(a) The term $2x^4y^2$ says that there are 2 possible ways to color the cube with 4 red and 2 green faces.

(b) The term $3x^3y^2z$ says that there are 3 possible ways to color the cube with 3 red, 2 green and 1 blue faces.

(c) The term $8x^2yzp^2$ says that there are 8 possible ways to color the cube with 2 red, 1 green, 1 blue and 2 yellow faces.

Problem 3 (Vertex Coloring of a Cube). A cube has 8 vertices and the vertices are to be colored. How many inequivalent ways are there to color it with

(a) 6 black, 1 yellow and 1 green vertices?

(b) 1 black, 1 red, 4 blue and 2 yellow vertices?

(c) 3 black, 1 red, 1 green 1 blue and 2 yellow vertices?

Solution: We have to color the vertices of the cube. So recall Example 12. Then the cycle index is

$$Z(G,t_1,t_2,\ldots,t_8)=\frac{1}{24}(6t_4^2+9t_2^4+8t_1^2t_3^2+6t_2^3+t_1^8).$$

Now we will apply Pólya's Substitution method. Let the colors black, red, green, blue and yellow be denoted by x,y,z,p and v respectively.Then substituting $t_i \to (x^i+y^i+z^i+p^i+v^i)$ for each $i=\{1,2,3,4,5,6,7,8\}$ in the previous equation, we get

$$\begin{aligned}Z_G=&\frac{1}{24}(6(x^4+y^4+z^4+p^4+v^4)^2+9(x^2+y^2+z^2+p^2+v^2)^4\\&+8(x+y+z+p+v)^2(x^3+y^3+z^3+p^3+v^3)^2\\&+6(x^2+y^2+z^2+p^2+v^2)^3+(x+y+z+p+v)^8).\end{aligned}$$

We use Maple to expand and find that the expression on the right hand side is very long. Each term of this expression corresponds to a fixed number of black, red, green, blue and yellow colors.

(a) the term $3x^6zv$ contains cube with 6 black, 1 green and 1 yellow vertices. The coefficient of this term is 3. Hence there are 3 possible ways to color the cube.

(b) In this case the term is $35p^4v^2xy$ which contains cube 1 black, 1 red, 4 blue and 2 yellow vertices. So there are 35 ways to color the cube.

(c) For this we need to find out the coefficient of the term $140pv^2x^3yz$ which contains cube with 3 black, 1 red, 1 green 1 blue and 2 yellow vertices. There are 140 ways to color the cube.

Problem 4 (16 Bead Necklace)**.** Count the number of ways to arrange beads on a necklace, where 16 total beads arranged on the necklace with

(a) 2 different colors;

(b) 14 red and 2 black colors;

(c) 4 different colors;

Solution: With a necklace, we will obviously rotate it around and flip over it. We will also use the Definition 2 with the formula for the cyclic and dihedral groups to solve this problem. The cycle index of this group C_{16} from (1) is

$$\begin{aligned}
Z(C_{16}, t_1, \ldots, t_{16}) =& \frac{1}{16}\sum_{l/16}\phi(l)t_l^{16/l} \\
=& \frac{1}{16}(\phi(1)t_1^{16/1} + \phi(2)t_2^{16/2} + \phi(4)t_4^{16/4} + \phi(8)t_8^{16/8} + \phi(16)t_{16}^{16/16}) \\
=& \frac{1}{16}(t_1^{16} + t_2^8 + 2t_4^4 + 4t_8^2 + 8t_{16}^1)
\end{aligned}$$

Now we will illustrate the formula by computing the cycle index of D_{16}

$$\begin{aligned}
Z(D_{16}, t_1, \ldots, t_{16}) =& \frac{1}{2}Z(C_{16}, t_1, \ldots, t_{16}) + \frac{1}{4}(t_2^8 + t_1^2t_2^7) \\
=& \frac{1}{2}(\frac{1}{16}(t_1^{16} + t_2^8 + 2t_4^4 + 4t_8^2 + 8t_{16}^1)) + \frac{1}{4}(t_2^8 + t_1^2t_2^7) \qquad (4) \\
=& \frac{1}{32}(t_1^{16} + 9t_2^8 + 8t_1^2t_2^7 + 2t_4^4 + 4t_8^2 + 8t_{16}^1)
\end{aligned}$$

Let us consider 2 colors red and black denoted by x and y respectively. Now we will apply Pólya's Substitution Method. Substituting $t_i \to (x^i + y^i)$ for each $i = \{1, 2, 3, \ldots, 16\}$ in equation (4), we get

$$\begin{aligned}
Z(D_{16}, t_1, \ldots, t_{16}) =& \frac{1}{32}((x + y+)^{16} + 9(x^2 + y^2)^8 \\
& + 8(x + y)^2(x^2 + y^2)^7 + 2(x^4 + y^4)^4 \\
& + 4(x^8 + y^8)^2 + 8(x^{16} + y^{16})^1)
\end{aligned}$$

(i) Let $x = y = 1$. Then using Maple by the above equation we get the total count 2250.

(ii) Expanding the same equation we have by Maple

$$\begin{aligned}Z_{D_{16}} =& x^{15} + \mathbf{8x^{14}y^2} + 21x^{13}y^3 + 72x^{12}y^4 + 147x^{11}y^5 \\ &+ 280x^{10}y^6 + 375x^9y^7 + 440x^8y^8 + 375x^7y^9 \\ &+ 280x^6y^{10} + 147x^5y^{11} + 72x^4y^{12} + 21x^3y^{13} \\ &+ 8x^2y^{14} + xy^{15} + x^{16} + y^{16}\end{aligned}$$

The term $8x^{14}y^2$ contains 14 red and 2 black colors. The coefficient of this term is 8. So there are 8 ways to decorate a 16 beaded Necklace with 14 red and 2 black colors.

(iii) For this case consider 4 colors x, y, z, and p. We will again apply Pólya's Substitution Method. Substituting $t_i \to (x^i + y^i + z^i + p^i)$ for each $i = \{1, 2, 3, \ldots, 16\}$ in equation (4), we get

$$\begin{aligned}Z(D_{16}, t_1, \ldots, t_{16}) =& \frac{1}{32}((x + y + z + p)^{16} + 9(x^2 + y^2 + z^2 + p^2)^8 \\ &+ 8(x + y + z + p)^2(x^2 + y^2 + z^2 + p^2)^7 + 2(x^4 + y^4 + z^4 + p^4)^4 \\ &+ 4(x^8 + y^8 + z^8 + p^4)^2 + 8(x^{16} + y^{16} + z^{16} + p^{16})^1)\end{aligned}$$

Let $x = y = z = p = 1$. Then using Maple by the above equation we get the total count 134301715.

3. Generalization of Pólya's Theorem

In the preceding sections we considered mappings of X into Y, introduced by a permutation group G of X. We are now going to move on more general situation. Let us consider two mappings $f_1 \in Y^X$ and $f_2 \in Y^X$. They are equivalent if there exist elements $g \in G$ and $h \in H$ such that $f_1 g = h f_2$, that is $f_1(gx) = h f_2(x)$ for all $x \in X$.

Next we assume that each $f \in Y^X$ has a certain weight $W(f)$. Then we can also assume that equivalence functions have the same weight:

$$f_1 \sim f_2 \text{ implies } W(f_1) = W(f_2).$$

If F denotes a pattern, we define its weight $W(F)$ as the common value of all $W(f)$ with $f \in F$ [2].

Lemma 2. *The Pattern inventory is*

$$\sum W(F) = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{h\in H}\sum_{f}^{(g,h)} W(f)$$

where $\sum_f^{(g,h)} W(f)$ *means the sum of* $W(f)$ *extended over all* f *that satisfy* $fg = hf$ *[2].*

3.1. Patterns of One-to-one Mappings

We already have the finite sets X and Y, subject to the permutation groups G and H. Now we define the weight $W(f)$ of any $f \in Y^X$ by

$$W(f) = \begin{cases} 1 & \text{if } f \text{ is a one-to-one mapping} \\ 0 & \text{otherwise} \end{cases}$$

If $g \in G$ and $h \in H$, then the mapping hfg^{-1} is one-to-one if and only if f is one-to-one and thus $W(f_1) = W(f_2)$. The inventory $\sum W(F)$ is just the number of patterns of one-to-one function.

Now let g be of type $\{g_1, g_2, \ldots\}$ where g_i shows the number of cycles of length i. Similarly let h be of type $\{h_1, h_2, \ldots\}$. We will find the number of one-to-one mappings f of Y into X that satisfies $fg = hf$. Let x be any element in X, belonging to a cycle of length j. This cycle consists of elements $x, gx, g^2x, \ldots, g^{j-1}x$ and we have $g^jx = x$. Now $fg = hf$ implies

$$fg^2 = fgg = hfg = hhf = h^2 f, fg^3 = h^3 f, \ldots$$

Hence we have $h^j f x = f g^j x = f x$. It follows that the length of cycle of Y to which $f x$ belongs is a divisor of j. The number of elements of one-to-one mappings of a set g_j elements into a set of h_j elements equals $h_j(h_j - 1)(h_j - 2)\ldots(h_j - g_j + 1)$ which is zero if $h_j < g_j$. Therefore we have

$$\sum_f^{(g,h)} W(f) = \prod_j j^{g_j} h_j(h_j - 1)(h_j - 2)\ldots(h_j - g_j + 1).$$

This product runs over all j for $g_j > 0$; but if $g_j = 0$ we can take the over all values $j = 1, 2, 3, \ldots$ as well.

Next we can write $j^g h(h-1)(h-2)\ldots(h-g+1)$ as gth derivative of $(1 + jz)^h$ at the point $z = 0$. As the result of a number of partial differentiations with respect to variables $z_1, z_2, \ldots$ at the points $z_1 = z_2 = \ldots = 0$, we have

$$\left(\frac{\partial}{\partial z_1}\right)^{g_1} \left(\frac{\partial}{\partial z_2}\right)^{g_2} \left(\frac{\partial}{\partial z_3}\right)^{g_3} \ldots (1 + z_1)^{h_1}(1 + 2z_2)^{h_2}(1 + 3z_3)^{h_3} \ldots$$

The differential operator is obtained from a term of cycle index $Z_G(t_1, t_2, \ldots)$ of G upon substitution of $t_1 = \frac{\partial}{\partial z_1}$, $t_2 = \frac{\partial}{\partial z_2}, \ldots$ and the operand is obtained upon substitution of $t_1 = 1 + z_1$, $t_2 = 1 + 2z_2, \ldots$ into a term of the index $Z_H(t_1, t_2, ..)$ of H [2].

Theorem 3. *The number of patterns of one-to-one mappings of X into Y equals*

$$Z_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots\right) Z_H(1 + z_1, 1 + 2z_2, 1 + 3z_3, \ldots)$$

evaluated at $z_1 = z_2 = z_3 = \ldots = 0$.

If $|X| = |Y|$, then we always have $\sum_j g_j = \sum_j h_j$. So that we have either $g_1 = h_1$, $g_2 = h_2, \dots$ or at least once $g_j > h_j$. Thus we have [2]

$$\left(\frac{\partial}{\partial z_1}\right)^{g_1}\left(\frac{\partial}{\partial z_2}\right)^{g_2}\left(\frac{\partial}{\partial z_3}\right)^{g_3}\dots(z_1)^{h_1}(2z_2)^{h_2}(3z_3)^{h_3}\dots\dots$$

Theorem 4. *If $|X| = |Y|$, then the number of patterns is equal to*

$$Z_G\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\frac{\partial}{\partial z_3},\dots\right)Z_H(z_1,2z_2,3z_3,\dots)$$

evaluated at $z_1 = z_2 = z_3 = \dots = 0$.

The inverse mappings are one-to-one mappings of Y onto X. In this case we can easily find that the number of patterns is equal to

$$Z_H\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\frac{\partial}{\partial z_3},\dots\right)Z_G(z_1,2z_2,3z_3,\dots)$$

evaluated at $z_1 = z_2 = z_3 = \dots = 0$ [2].

Problem 5. How many geometrically different ways can the faces of a cube be arranged in cyclic order [2]?

Solution: We need to find out the number of patterns of one-to-one mappings. Let X be the set of faces of a cube and G be the cube induced by rotations(Example 11); Y be the 6th roots of unity and again H is the group of permutation induced by the rotations in the complex plane (Definition 2 with cycle index formula $Z(C_n)$). Then $|X| = |Y|$.

The cycle indexes are

$$Z_G = \frac{1}{24}(\mathbf{t_1^6} + \mathbf{6t_2^3} + \mathbf{8t_3^2} + 3t_1^2t_2^2 + 6t_1^2t_4)$$

$$Z_H = \frac{1}{6}(\mathbf{t_1^6} + \mathbf{t_2^3} + \mathbf{2t_3^2} + 2t_6)$$

The number of patterns is equal to

$$\begin{aligned}Z_G\left(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\frac{\partial}{\partial z_3}\right)Z_H(z_1,2z_2,3z_3) =& \frac{1}{24}\cdot\frac{1}{6}\left(\left(\frac{\partial}{\partial z_1}\right)^6(z_1)^6 + \left(\frac{\partial}{\partial z_2}\right)^3(2z_2)^3 + \left(\frac{\partial}{\partial z_3}\right)^2(3z_3)^2\right)\\ =& \frac{1}{24}\cdot\frac{1}{6}(6! + 6\cdot 2^3\cdot 3! + 16\cdot 3^2\cdot 2!)\\ =& 9\end{aligned}$$

Problem 6. How many geometrically different ways can the edges of a tetrahedron be arranged in cyclic order?

Solution: Here we also need to find out the number of patterns of one-to-one mappings. Again let X be the set of edges of a tetrahedron and G be the tetrahedron induced by rotations;Y be the 6th roots of unity and again H is the group of permutation induced by the rotations in the complex plane (Definition 2 with cycle index formula $Z(C_n)$). Then $|X| = |Y|$.

The cycle indexes are

$$\begin{aligned} Z_G =& \frac{1}{12}(\mathbf{t_1^6} + \mathbf{8t_3^2} + 3t_1^2 t_2^2) \\ Z_H =& \frac{1}{6}(\mathbf{t_1^6} + t_2^3 + \mathbf{2t_3^2} + 2t_6) \end{aligned}$$

The number of patterns is equal to

$$\begin{aligned} Z_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_3}\right) Z_H(z_1, 2z_2) =& \frac{1}{12} \cdot \frac{1}{6}\left(\left(\frac{\partial}{\partial z_1}\right)^6 (z_1)^6 + \left(\frac{\partial}{\partial z_3}\right)^2 (3z_3)^2\right) \\ =& \frac{1}{12} \cdot \frac{1}{6}(6! + 16 \cdot 3^2 \cdot 2!) \\ =& 14 \end{aligned}$$

Example 14. *We have 6 colored labels $l_1, l_2, l_3, l_4, l_5, l_6$ to be pasted on the faces of a cube, one on each side. Label l_1 and l_2 are yellow and black respectively. The labels l_3 and l_4 are both violet and can not be distinguished from each other. The same condition holds for l_5 and l_6 which are both purple. We are interested to find out the number of patterns.*

Now X be set of 6 labels and Y is the set of 6 faces. The group G consists of 8 permutations, characterized by the conditions that l_1 and l_2 are fixed and the subset $\{l_3, l_4\}$ mapped onto either itself or the set $\{l_5, l_6\}$ [2]. The cycle index of this group is

$$Z_G = \frac{1}{8}(\mathbf{t_1^6} + 2t_1^4 t_2 + \mathbf{3t_1^2 t_2^2} + \mathbf{2t_1^2 t_4})$$

and also given by

$$Z_H = \frac{1}{24}(\mathbf{t_1^6} + 6t_2^3 + 8t_3^2 + \mathbf{3t_1^2 t_2^2} + \mathbf{6t_1^2 t_4})$$

For the number of patterns we obtain

$$\begin{aligned} & Z_H\left(\frac{\partial}{\partial z_1}, \frac{\partial^2}{\partial z_1 \partial z_2}, \frac{\partial^2}{\partial z_1 \partial z_4}\right) Z_G(z_1, 2z_2, 4z_4) \\ =& \frac{1}{24} \cdot \frac{1}{8}\left(\left(\frac{\partial}{\partial z_1}\right)^6 (z_1)^6 + \left(\frac{\partial^2}{\partial z_1 \partial z_2}\right)^2 (z_1 2z_2)^2 + \left(\left(\frac{\partial}{\partial z_1}\right)^2 \left(\frac{\partial}{\partial z_1}\right)\right)(z_1^2 4z_4)\right) \\ =& \frac{1}{24} \cdot \frac{1}{8}(6! + 3 \cdot 3 \cdot 2!2^2 \cdot 2! + 6 \cdot 2 \cdot 2!4 \cdot 1!) \\ =& 5 \end{aligned}$$

Example 15. *Consider 6 colors $l_1, l_2, l_3, l_4, l_5, l_6$ to be colored on the edges of a tetrahedron, one on each edge by following the same conditions with Example 14. Then the cycle index of this group is*

$$Z_G = \frac{1}{8}(\mathbf{t_1^6} + 2t_1^4 t_2 + \mathbf{3t_1^2 t_2^2} + 2t_1^2 t_4)$$

and also given by

$$Z_H = \frac{1}{12}(\mathbf{t_1^6} + 8t_3^2 + \mathbf{3t_1^2 t_2^2})$$

For the number of patterns we obtain

$$\begin{aligned}
&Z_H\left(\frac{\partial}{\partial z_1}, \frac{\partial^2}{\partial z_1 \partial z_2}\right) Z_G(z_1, 2z_2) \\
=&\frac{1}{12}\cdot\frac{1}{8}\left(\left(\frac{\partial}{\partial z_1}\right)^6 (z_1)^6 + \left(\frac{\partial^2}{\partial z_1 \partial z_2}\right)^2 (z_1 2z_2)^2\right) \\
=&\frac{1}{12}\cdot\frac{1}{8}(6! + 3\cdot 3\cdot 2!2^2\cdot 2!) \\
=&9
\end{aligned}$$

3.2. The Total Number of Patterns

Again consider finite sets X and Y with permutation groups G and H of X and Y respectively. We will find the number of patterns. Again we will apply Lemma 2 to evaluate $\sum_f^{(g,h)} W(f)$. We will use the subsection 3.1. So we can easily compute the number of possibilities for f. For each cycle of X we will select an element, called the selected element. The number of possibilities for the element of Y onto which is selected element can be mapped by an f is $\sum_{j/i} jc_j$, where i is the length of the cycle of X to which the selected item belongs, c's are of type $\{c_1, c_2, \ldots\}$ of h and the sum is over all divisors j of i. Because the choices of function values for the various selected elements are independent and determine f completely. The number of f equals the product of $\sum_{j/i} jc_j$ taken over all selected elements. Since there are g_i cycles of length i, then we get

$$\begin{aligned}
\sum_f^{(g,h)} W(f) =& \prod(\sum_{j/i} jc_j)^{g_i} \\
=&(c_1)^{g_1}(c_1 + 2c_2)^{g_2}(c_1 + 3c_3)^{g_3}(c_1 + 2c_2 + 4c_4)^{g_4}\ldots
\end{aligned} \tag{5}$$

Note that a power with exponent 0 has to be interpreted as 1 in this context even if the base is zero. Therefore is no difficulty about the interpretation of the infinite product.

As in subsection 3.1, we can interpret the previous equation as a derivative. A power a^b can be written as the bth derivative of e^{az} at $z = 0$; if $a = b = 0$, this still gives the desired value $0^\circ = 1$. So the previous expression can be written as

$$\left(\frac{\partial}{\partial z_1}\right)^{g_1}\left(\frac{\partial}{\partial z_2}\right)^{g_2}\left(\frac{\partial}{\partial z_3}\right)^{g_3}\ldots exp(\sum_i z_i \sum_{j/i} jc_j),$$

evaluated at $z_1 = z_2 = \ldots = 0$. The exponent can be expressed as

$$(\sum_i z_i \sum_{j/i} jc_j) = \sum_j jc_j(z_j + z_{2j} + z_{3j} + \ldots).$$

Now the differential operator is obtained from $t_1^{g_1} t_2^{g_2} t_3^{g_3} \ldots$ on substitution of $t_1 = \frac{\partial}{\partial z_1}$, $t_2 = \frac{\partial}{\partial z_2}$, $t_3 = \frac{\partial}{\partial z_3}4, \ldots$ and the operand is obtained from $t_1^{c_1} t_2^{c_2} t_3^{c_3} \ldots$ on substitution of [2]

$$t_1 = e^{z_1+z_2+z_3+\cdots}, t_2 = e^{2(z_2+z_4+z_6+\ldots)}, t_3 = e^{3(z_3+z_6+z_9+\ldots)}, \ldots.$$

Theorem 5. *The total number of patterns of mapping of X into Y equals*

$$Z_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots\right) Z_H[e^{z_1+z_2+z_3+\cdots}, e^{2(z_2+z_4+z_6+\ldots)}, e^{3(z_3+z_6+z_9+\ldots)}, \ldots]$$

evaluated at $z_1 = z_2 = z_3 = \ldots = 0$.

There is a second expression for the number of patterns, which sometimes may turn out to be simpler to handle. Note that (5) *is obtained by substituting $t_1 = c_1$, $t_2 = c_1 + 2c_2$, $\ldots$ into $t_1^{g_1} t_2^{g_2} \ldots$. Summing for g and dividing by $|G|$, we obtain*

$$Z_G(c_1, c_1 + 2c_2, c_1 + 3c_3, c_1 + 2c_2 + 4c_4, \ldots),$$

in which the ith argument is $\sum_{j/i} jc_j$. Thus again using Lemma 2 for the number of patterns we have

$$|H|^{-1} \sum_{h\in H} Z_G(c_1, c_1 + 2c_2, c_1 + 3c_3, c_1 + 2c_2 + 4c_4, \ldots)$$

Here it has to be remembered that $\{c_1, c_2, c_3, \ldots\}$ is of type h [2].

Now we are going to discuss some of the examples. In all cases, m and n stand for the number of elements of X and Y respectively.

Example 16. *Let G be a group consisting of the identity only, where $Z_G = t_1^m$. Then the patterns are called patterns of variation with repetition. Now let $\{1, 2, \ldots, m\}$ be a sequence of X then for each $f \in X^Y$, the sequence $\{f(1), f(2), \ldots, f(m)\}$ shows repetitions. For the number of patterns, theorem 4 gives the expression*

$$\left[\left(\frac{d}{dz}\right)^m Z_H(e^z, 1, 1, \ldots)\right]_{z=0} \tag{6}$$

A second specialization of (6) *can be obtained by keeping Y and H general but X has only one element d_1. Now two functions f_1 and f_2 are equivalent if and only if $f_1(d_1)$ are mapped to $f_2(d_1)$. So the number of patterns is the number of transitive sets. If we take $m = 1$ in* (6) *then we get $|H|^{-1}c_1$ and c_1 is the number of elements of Y that are invariant under the permutation h [2].*

Example 17. *Again let G be the symmetric group of X with H not specified. We are only interested in the number of elements of X that a function f maps onto a given element of Y. Thus our patterns of mappings ψ of Y into the set $P = \{0, 1, 2, \ldots\}$, with restriction $\sum_{r\in R} \psi(r) = m$,* *whereas patterns are formed with respect to H acting on Y. Let the weights of P be $1, w^1, w^2, \ldots$. If we apply theorem 4, we have as differential operator the coefficient of w^m in*

$$exp(w\frac{\partial}{\partial z_1} + \frac{1}{2}w^2\frac{\partial}{\partial z_2} + \ldots)$$

by Example 5. The effect of this operator on a function $\psi(z_1, z_2, \ldots)$ at $z_1 = z_2 = \ldots = 0$ is $\psi(w, \frac{1}{2}w^2, \frac{1}{3}w^3, \ldots)$. Now by substituting $z_1 = w, z_2 = \frac{1}{2}w^2, z_3 = \frac{1}{3}w^3, \ldots$ in theorem 4 we get

$$\begin{aligned} exp(z_1 + z_2 + \ldots) =& exp(w + \frac{1}{2}w^2 + \frac{1}{3}w^3 + \ldots) = (1-w)^{-1} \\ exp(2(z_2 + z_4 + \ldots)) =& exp(w^2 + \frac{1}{2}w^4 + \frac{1}{3}w^6 + \ldots) = (1-w^2)^{-1} \end{aligned}$$

etc. Thus we obtain $Z_H[(1-w)^{-1}, (1-w^2)^{-1}, \ldots]$ [2].

Example 18. *Next let us take for H the symmetric group of all permutations of Y and for G take the identity only. Now the patterns turn out to be class partition of X into at most m parts, where a class partition is defined as a set of disjoint subsets(classes) of X whose union is X. The previous result follows from the fact that if $f \in Y^X$, then f defines a class partition by putting into one class all d that are mapped by f onto one and the same element of Y. According to (a) and Example 5 we find that the number of patterns is the coefficient of w^m in*

$$\left[\left(\frac{d}{dz}\right)^m exp(we^z + \frac{1}{2}w^2 + \frac{1}{3}w^3 + \ldots)\right]_{z=0}$$

and we can easily find to be $m!$ times the coefficient of $z^m w^n$ in the expression of $(1-w)^{-1}e^{w(e^z-1)}$. Here the number of partitions can be at most n parts. So the total number of patterns equals $m!$ times the coefficients of z^m in e^{e^z-1} [2].

Example 19. *Now we are interested for H the symmetric group of X without any specialization of G. We get partitions of class partitions of X. It seems to us impossible to simplify the result of theorem 4. If we take G to the symmetric group of X, the partitions become class partitions of a set of unidentifiable objects. So we are only concern about the size of the classes. Our patterns can be brought into one-to-one correspondence with the partitions of m into at most n parts. A partition of m is a solution $\{g_1, g_2, \ldots\}$ of the equation*

$$g_1 + 2g_2 + 3g_3 + \ldots = m$$

in nonnegative integers $g_1, g_2, \ldots$ So m has been partitioned into g_1 1's, g_2 2's, ... and accordingly $g_1 + g_2 + g_3 + \ldots$ is called the number of parts of the partition.

The number of partitions can be obtained from Example 17 by specializing H to be the symmetric group. We obtain the coefficient of z^m in

$$Z_H[(1-z)^{-1}, (1-z^2)^{-1}, \ldots]$$

which becomes in this case, the coefficient of $z^m w^n$ *in*

$$exp[w(1-z)^{-1} + \frac{1}{2}w^2(1-z^2)^{-1} + \frac{1}{3}w^3(1-z^3)^{-1} + \ldots],$$

and this expression can be reduced to

$$exp[log(1-w)^{-1} + log(1-wz)^{-1} + log(1-wz^2)^{-1} + \ldots] = \prod_{k=1}^{\infty} \frac{1}{1-wz^k}.$$

This is a well-know result for the generating function of the number of partitions of a given number into a given number of components [2].

Example 20. *Let* H *be a symmetric group and let* G *be specialized by taking* $n = 2$*. So the patterns of class partitions into at most two parts. By* (5) *the number of these is*

$$\frac{1}{2}Z_G(2,2,2,\ldots) + \frac{1}{2}Z_G(0,2,0,2,\ldots),$$

since $Z_H = \frac{1}{2}t_1^2 + \frac{1}{2}t_2$*.*

If we compare this to the number of patterns of partitions into two labeled classes, $Z_G(2,2,2,\ldots)$*, we observe that the term* $Z_G(0,2,0,2,\ldots)$ *represents the number of symmetric patterns of* X *into two classes* X_1 *and* X_2*. That is,* $Z_G(0,2,0,2,\ldots)$ *represents the number of symmetric patterns of subsets* X_1 *that are equivalent to there complement, where equivalence is defined by the permutations of* G *and patterns are defined by this equivalence.*

Let X *be the set of 10 roots of unity and* G *be the group of 10 rotations. The cycle index (see Definition 2 with cyclic groups formula* (1)*) is*

$$Z_G = \frac{1}{10}(t_1^{10} + t_2^5 + 4t_5^2 + 4t_{10})$$

and therefore [2]

$$Z_G(0,2,0,2,\ldots) = \frac{1}{10}((2)^5 + 4.2) = 4.$$

Theorem 6 (Pólya)**.** *We have*

$$Z_{G[H]}(t_1,t_2,\ldots) = Z_G[Z_H(t_1,t_2,t_3,\ldots), Z_H(t_2,t_4,t_6,\ldots),\ldots]$$

where $Z_{G[H]}$ *is in Kranz group and the the right hand side is obtained on substitution of* $x_j = Z_H(t_j, t_{2j}, t_{3j}, \ldots)$ *into* $Z_G(x_1,x_2,x_3,...)$ *[2].*

Example 21. *Consider* n *cubes and we want to color the faces with Red and Yellow. We are interested for the number of ways to do this when equivalences are defined by permutations of the sets of cubes and rotations of the separate cubes.*

The group is considered as $S_n[G]$ *where* S_n *is the symmetric group of degree* n *and* G *is the cube-face group of Example 11. Then we have to substitute* $t_1 = t_2 = \ldots = 2$ *in to its cycle index to find the required number. If we make this substitution in any of the polynomials*

$$Z_G(t_1,t_2,t_3,\ldots), Z_G(t_2,t_4,t_6,\ldots), Z_G(t_3,t_6,t_9,\ldots),\ldots, \tag{7}$$

we always get $Z_G(2,2,2,\ldots)$ *and hence*

$$Z_G(2,2,2,\ldots) = \frac{1}{24}(6(2)^2(2) + 3(2)^2(2)^2 + 8(2)^2 + 6(2)^3 + 2^6) = 10$$

Thus the answer of the question is $S_n(10,10,10,\ldots)$*. This number equals the coefficient of* w^n *in the development of*

$$exp(10w + \frac{1}{2}10w^2 + \frac{1}{3}10w^3 + \ldots) = (1-w)^{-10}$$

therefore

$$S_n(10,10,10,\ldots) = \frac{(n+9)!}{n!9!}$$

How many of the preceding patterns have the property that they do not change if we interchange the color?

According to Example 20 this number is found by substituting

$$\begin{aligned} t_1 =& t_3 = t_5 = ... = 0, \\ t_2 =& t_4 = t_6 = ... = 2 \end{aligned}$$

into the cycle index. Under the substitution the polynomials (7) *become* $Z_G(0,2,0,2,\ldots)$ *and* $Z_G(2,2,2,\ldots)$*, alternately. As* $Z_G(0,2,0,2,\ldots) = 2$*. We obtain*

$$Z_{S_n[G]}(0,2,0,2,\ldots) = Z_{S_n}(2,10,2,10,\ldots)$$

This is the coefficient of w^n *in*

$$\begin{aligned} & exp(2w + \frac{1}{2}10w^2 + \frac{1}{3}2w^3 + \frac{1}{4}10w^4 + \ldots.) \\ =& exp[2log(1-w)^{-1} + 8log(1-w^2)^{-1/2}] \\ =& (1-w)^{-2}(1-w^2)^{-4} \\ =& (1+2w+w^2)(1-w^2)^{-6} \\ =& 1 + 2w + 7w^2 + 12w^3 + 27w^4 + 42w^5 \\ & + 77w^6 + 112w^7 + 182w^8 + 252w^9 + \ldots \end{aligned}$$

For example if $n = 9$ *the required number of patterns is 252 [2].*

Example 22. *Consider n tetrahedrons and we want to color the edges with Red and Yellow. By following problem 1 we get from edge coloring of a Tetrahedron*

$$Z_G(2,2,2,\ldots) = 12$$

Thus the answer of this question is $S_n(12,12,12,\ldots)$*. This number equals the coefficient of* w^n *in the development of*

$$exp(12w + \frac{1}{2}12w^2 + \frac{1}{3}12w^3 + \ldots) = (1-w)^{-12}$$

therefore

$$S_n(12,12,12,\ldots) = \frac{(n+11)!}{n!11!}$$

Now we want to find the number of the preceding patterns having the property that they do not change if we interchange the color. Again following problem 1 we get

$$Z_{S_n[G]}(0,2,0,2,\ldots) = Z_{S_n}(2,12,2,12,\ldots)$$

This is the coefficient of w^n in

$$\begin{aligned}
&exp(2w + \frac{1}{2}12w^2 + \frac{1}{3}2w^3 + \frac{1}{4}12w^4 + \ldots.)\\
&=exp[2log(1-w)^{-1} + 10log(1-w^2)^{-1/2}]\\
&=(1-w)^{-2}(1-w^2)^{-5}\\
&=(1+2w+w^2)(1-w^2)^{-7}\\
&=(1+2w+w^2)(1+7w^2+28w^4+\ldots)\\
&=1+2w+8w^2+14w^3+35w^4+\ldots
\end{aligned}$$

For example if $n=4$ the required number of patterns is 35.

Problem 7. Let us take n cubes and we want to color the faces with Red, Yellow and green. We are interested for the number of ways to do this when equivalences are defined by permutations of the sets of cubes and rotations of the separate cubes.
Solution: The group is considered as $S_n[G]$ where S_n is the symmetric group of degree n and G is the cube-face group of Example 11. Then we have to substitute $t_1 = t_2 = \ldots = 3$ in to its cycle index to find the required number. If we make this substitution in any of the polynomials

$$Z_G(t_1,t_2,t_3,]\ldots), Z_G(t_2,t_4,t_6,\ldots), Z_G(t_3,t_6,t_9,\ldots),\ldots,$$

we always get $Z_G(3,3,3,\ldots)$ and hence

$$Z_G(3,3,3,\ldots) = \frac{1}{24}(6(3)^2(2) + 3(3)^2(3)^2 + 8(3)^2 + 6(3)^3 + 3^6) = 57$$

Thus the answer of the question is $S_n(57,57,57,\ldots)$. This number equals the coefficient of w^n in the development of

$$exp(57w + \frac{1}{2}57w^2 + \frac{1}{3}57w^3 + \ldots) = (1-w)^{-57}$$

therefore

$$S_n(57,57,57,\ldots) = \frac{(n+56)!}{n!(56)!}$$

How many of the preceding patterns have the property that they do not change if we interchange the color?

By (5), this number is found by substituting

$$\begin{aligned} t_1 =& t_2 = t_4 = t_5 \ldots = 0, \\ t_3 =& t_6 = t_9 = \ldots = 3 \end{aligned}$$

into the cycle index. Under the substitution the polynomials become $Z_G(0,0,3,0,0,3,\ldots)$ and $Z_G(3,3,3,\ldots)$, alternately. As $Z_G(0,0,3,0,0,3,\ldots) = 3$. We obtain

$$Z_{S_n[G]}(0,0,3,0,0,3,\ldots) = Z_{S_n}(3,3,57,3,3,57,\ldots)$$

This is the coefficient of w^n in

$$\begin{aligned} & exp(3w + \frac{1}{2}3w^2 + \frac{1}{3}57w^3 + \frac{1}{4}3w^4 + ...) \\ =& exp[3log(1-w)^{-1} + 54log(1-w^3)^{-1/3}] \\ =& (1-w)^{-3}(1-w^3)^{-18} \\ =& (1+3w+6w^2+10w^3\ldots)(1+18w^3+171w^6+\ldots) \\ =& 1+3w+6w^2+28w^3+\ldots \end{aligned}$$

For example if $n = 3$ the required number of patterns is 28.

Problem 8. Let us take n cubes and we want to color the faces with Red, black, Yellow and green. How many of the patterns have the property that they do not change if we interchange the color?

Solution: With the help of (5), this number is found by substituting

$$\begin{aligned} t_1 =& t_3 = t_5 = t_7 \ldots = 0, \\ t_2 =& t_6 = t_{10} = \ldots = 4, \\ t_4 =& t_8 = \ldots = 4 \end{aligned}$$

into the cycle index. Under the substitution the polynomials become $Z_G(0,0,0,4,0,0,0,4,\ldots)$, $Z_G(0,4,0,4,0,4,0,4,\ldots)$ and $Z_G(4,4,4,\ldots)$, alternately. Now

$$Z_G(4,4,4,\ldots) = \frac{1}{24}(6(4)^2(4) + 3(4)^2(4)^2 + 8(4)^2 + 6(4)^3 + 4^6) = 240$$

and

$$Z_G(0,4,0,4,\ldots) = \frac{1}{24}(6(0)^2(4) + 3(0)^2(4)^2 + 8(0)^2 + 6(4)^3 + 0^6) = 16$$

As $Z_G(0,0,0,4,0,0,0,4,\ldots) = 0$. We obtain

$$Z_{S_n[G]}(0,0,0,4,0,0,\ldots) = Z_{S_n}(0,16,0,240,0,16,0,240,\ldots)$$

This is the coefficient of w^n in

$$exp(0w + \frac{1}{2}16w^2 + \frac{1}{3}0w^3 + \frac{1}{4}240w^4 + \ldots)$$

$$
\begin{aligned}
&= exp(\frac{1}{2}16w^2+\frac{1}{4}240w^4+\frac{1}{6}16w^6+\frac{1}{8}240w^8\ldots)\\
&= exp[16log(1-w^2)^{-1/2}+224log(1-w^4)^{-1/4}]\\
&=(1-w^2)^{-8}(1-w^4)^{-56}\\
&=(1+8w^2+36w^4+\ldots)(1+56w^4+1596w^8+\ldots)\\
&=1+8w^2+92w^4+\ldots
\end{aligned}
$$

For example if $n=4$ the required number of patterns is 92.

4. Conclusion

Counting symmetries is a vast and challenging topic. Many Mathematicians have already researched on it. In this study it has already observed that Pólya's Theory has remarkable and numerous applications in counting symmetries. This theory has been used on Colorings on Cube (Faces and Vertices) and Tetrahedron. Furthermore we have also discussed on Generalization of Pólya's Theorem with some problems and examples; but there are penalty of other uses as well. Pólya and Reade[9] mentioned the other following applications without exploring the details

(i) Counting Latin squares, which are $n \times n$ arrays filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

(ii) Counting the number of essentially different propositions of n statements, and showing that the problem is equivalent to coloring the vertices of a hypercube.

(iii) Counting finite automata and certain binary matrices.

(iv) Counting graphs in statistical mechanics.

There are certainly more which have not been mentioned. Hence we can conclude that Pólya's Theory is one of the most powerful and useful tools in counting symmetries.

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