



A Compensatory Fuzzy Approach to Multi-Objective Linear Transportation Problem with Fuzzy Parameters

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Abstract. In this paper, we focus on the solution procedure of the Multi-Objective Linear Transportation Problem (MOLTP) with fuzzy parameters, i.e. fuzzy cost coefficients, fuzzy supply quantities and fuzzy demand quantities. Because of several linear objectives and its fuzzy parameters, this transportation problem is very complicated and also due to the fuzziness in the costs this problem has non-linear structure. To overcome these difficulties, we gave an approach with three stages. By using linear solution techniques, our approach generates compromise solutions which are both compensatory and Pareto-optimal. In the first stage, the fuzziness in supply and demand quantities is eliminated by using Zimmermann's "min" operator to satisfy the balance condition. In the second stage, breaking points (i.e. the values of cost-satisfaction parameters that changed the optimal solution) and cost-satisfaction interval sets are obtained for each objective. In the third stage, considering cost-satisfaction interval sets of all objectives, an overall cost-satisfaction interval set is determined. And then for each member of this set, our approach generates compensatory compromise Pareto-optimal solutions using Werner's μ_{and} operator. To our knowledge, combining compensatory (μ_{and}) operator with MOLTP has not been published up to now. An illustrative numerical example is given to explain our approach.

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1. Introduction

The classical transportation problem is a special type of linear programming problem and it has wide practical applications in manpower planning, personnel allocation, inventory control, production planning, etc. The parameters of the transportation problem are unit costs (or

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profits), supply and demand quantities. The unit cost c_{ij} is the coefficient of the objective function and it could represent transportation cost, delivery time, number of goods transported unfulfilled demand, and many others. Thus multiple objectives may exist concurrently which lead to the research work on multi-objective transportation problems (MOTP). Also in practice, the parameters of MOTP are not always exactly known and stable. This imprecision may follow from the lack of exact information, changeable economic conditions, etc. A frequently used way of expressing the imprecision is to use the fuzzy numbers. It enables us to consider tolerances for the model parameters in a more natural and direct way. Therefore, MOTP with fuzzy parameters seems to be more realistic and reliable.

A lot of researches have been conducted on MOTP with fuzzy parameters. Hussein [8] dealt with the complete solutions of MOTP with possibilistic coefficients. Das *et al.* [6] focused on the solution procedure of the MOTP where all the parameters have been expressed as interval values by the decision maker. Ahlatcioglu *et al.* [1] proposed a model for solving the transportation problem that supply and demand quantities are given as triangular fuzzy numbers bounded from below and above, respectively. Basing on extension principle, Liu and Kao [12] developed a procedure to derive the fuzzy objective value of the fuzzy transportation problem where the cost coefficients, supply and demand quantities are fuzzy numbers. Using signed distance ranking, defuzzification by signed distance, interval-valued fuzzy sets and statistical data, Chiang [5] get the transportation problem in the fuzzy sense. Ammar and Youness [3] examined the solution of MOTP which has fuzzy cost, source and destination parameters. They introduced the concepts of fuzzy efficient and parametric efficient solutions. Islam and Roy [9] dealt with a multi-objective entropy transportation problem with an additional delivery time constraint, and its transportation costs are generalized trapezoidal fuzzy numbers. Chanas and Kuchta [4] proposed a concept of the optimal solution of the transportation problem with fuzzy cost coefficients and an algorithm determining this solution. Pramanik and Roy [14] showed how the concept of Euclidean distance can be used for modeling MOTP with fuzzy parameters and solving them efficiently using priority based fuzzy goal programming under a priority structure to arrive at the most satisfactory decision in the decision making environment, on the basis of the needs and desires of the decision making unit.

In this paper, we present a compensatory fuzzy approach to the Multi-objective Linear Transportation Problem (MOLTP). All the parameters of the problem are taken as triangular fuzzy numbers. Our approach has three stages. In the first stage, using Zimmermann's "min" operator, the fuzziness in supply and demand quantities is eliminated, that is, the crisp supply and demand quantities are obtained from fuzzy quantities to satisfy the balance condition. In the second stage, for each objective, breaking points and cost-satisfaction interval sets are determined. In the third stage, considering cost-satisfaction interval sets of all objectives, an overall cost-satisfaction interval set is found. And then for each member of this set, our approach generates compensatory compromise Pareto-optimal solutions using Werner's μ_{and} operator.

So, this paper is organized as follows. Next section presents the MOLTP formulation with fuzzy parameters. Section 3 introduces the compensatory fuzzy aggregation operators briefly. Section 4 explains our methodology using Werners' compensatory "fuzzy and" operator. Sec-

tion 5 gives an illustrative numerical example. Finally, Section 6 includes some results.

2. The Formulation of MOLTP with Fuzzy Parameters

The MOLTP is formulated as follows:

$$\begin{aligned}
 \min \quad & z^k(x) = \sum_{i=1}^n \sum_{j=1}^m c_{ij}^k \cdot x_{ij}, \quad k = 1, 2, \dots, K \\
 \text{subject to} \quad & \sum_{j=1}^m x_{ij} = a_i, \quad i = 1, 2, \dots, n \\
 & \sum_{i=1}^n x_{ij} = b_j, \quad j = 1, 2, \dots, m \\
 & \forall x_{ij} \geq 0, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m
 \end{aligned} \tag{1}$$

Here, x_{ij} is decision variable which refers to product quantity that transported from supply point i to demand point j . a_i and b_j are the capacities of the supply and demand points, respectively. And c_{ij}^k is the unit cost for transporting the goods from supply point i to demand point j for the objective k , ($k = 1, 2, \dots, K$) where K is the number of the objective functions. When at least one of these parameters is assumed as fuzzy, then a MOLTP with Fuzzy Parameters arises. In that case $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$, and $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m$ are called as n fuzzy supply and m fuzzy demand quantities, respectively. Similarly, \tilde{c}_{ij}^k is called as fuzzy unit transportation cost from supply point i to demand point j for the objective k , ($k = 1, 2, \dots, K$). For our fuzzy transportation problem, the fuzzy numbers \tilde{a}_i , \tilde{b}_j , and \tilde{c}_{ij}^k are considered in the following triangular forms

$$\tilde{a}_i = (-\infty, a_i^2, a_i^3), \quad \tilde{b}_j = (b_j^1, b_j^2, \infty), \quad \tilde{c}_{ij}^k = (-\infty, c_{ij}^2, c_{ij}^3)$$

The membership function of the fuzzy number \tilde{a}_i and its figure are given in (2) and Figure 1, respectively. Similarly, the membership functions and figures of the fuzzy numbers and can be constructed according to their triangular definitions.

$$\mu_{a_i}(a_i) = \begin{cases} 1, & a_i < a_i^2 \\ \frac{a_i^3 - a_i}{a_i^3 - a_i^2}, & a_i^2 \leq a_i \leq a_i^3 \\ 0, & a_i > a_i^3 \end{cases} \tag{2}$$

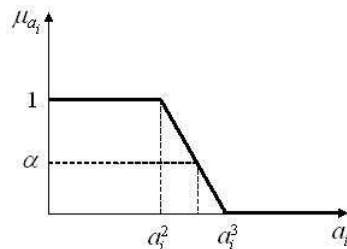


Figure 1: The membership function of the fuzzy supply quantity \tilde{a}_i .

Definition 1. [Pareto-optimal solution for MOLTP] Let S be the feasible region of (1). $x^* \in S$ is said to be a Pareto-optimal solution (strongly efficient or non-dominated) if and only if there does not exist another $x \in S$ such that $z^k(x) \leq z^k(x^*)$ for all $k = 1, 2, \dots, K$ and $z^k(x) \neq z^k(x^*)$ for at least one $k = 1, 2, \dots, K$.

Definition 2. [Compromise solution for MOLTP] A feasible solution $x^* \in S$ is called a compromise solution of (1) if and only if $x^* \in E$ and $z^k(x^*) \leq \bigwedge_{x \in S} z^k(x)$ where $z(x) = (z^1(x), z^2(x), \dots, z^K(x))$, \bigwedge stands for “min” operator and E is the set of Pareto-optimal solutions.

This definition imposes two conditions on the solution for it to be a compromise solution. First, the solution should be Pareto-optimal. Second, the feasible solution vector x^* should have the minimum deviation from the ideal point than any other point in S . That is, the compromise solution is the closest solution to the ideal one that maximizes the underlying utility function of the decision maker.

3. Compensatory Fuzzy Aggregation Operators

There are several fuzzy aggregation operators. The detailed information about them exists in [20] and [16]. The most important aspect in the fuzzy approach is the compensatory or non-compensatory nature of the aggregation operator. Several investigators [10, 13, 15, 20] have discussed this aspect.

Using the linear membership function, Zimmermann [19] proposed the “min” operator model to the Multi-objective linear problem (MOLP). It is usually used due to its easy computation. Although the “min” operator method has been proven to have several nice properties [13], the solution generated by min operator does not guarantee compensatory and Pareto-optimality [7, 11, 18]. The biggest disadvantage of the aggregation operator “min” is that it is non-compensatory. In other words, the results obtained by the “min” operator represent the worst situation and cannot be compensated by other members which may be very good. On the other hand, the decision modeled with average operator is called fully compensatory in the sense that it maximizes the arithmetic mean value of all membership functions.

Zimmermann and Zysno [21] show that most of the decisions taken in the real world are neither non-compensatory (min operator) nor fully compensatory and suggested a class of hybrid compensatory operators with γ compensation parameter.

Basing on the γ -operator, Werners [17] introduced the compensatory “fuzzy and” operator which is the convex combinations of min and arithmetical mean:

$$\mu_{and} = \gamma \min_i \mu_i + \frac{(1-\gamma)}{m} \left(\sum_i \mu_i \right),$$

where $0 \leq \mu_i \leq 1, i = 1, 2, \dots, m$ and the magnitude of $\gamma \in [0, 1]$ represent the grade of compensation.

Although this operator is not inductive and associative, this is commutative, idempotent, strictly monotonic increasing in each component, continuous and compensatory. Obviously, when $\gamma = 1$, this equation reduces to $\mu_{and} = \min$ (non-compensatory) operator. In literature,

it is showed that the solution generated by Werners' compensatory "fuzzy and" operator does guarantee compensatory and Pareto-optimality for MOLP [13, 15–18, 21]. Thus this operator is also suitable for our MOLTP. Therefore, due to its advantages, in this paper, we used Werners' compensatory "fuzzy and" operator.

4. A Compensatory Fuzzy Approach to MOLTP with Fuzzy Parameters

Our compensatory fuzzy approach has three stages. Stage 1 aims to convert the fuzzy supply and demand quantities to crisp ones. In Stage 2, fuzzy costs are written as depending on the cost-satisfaction parameter and then breaking points are obtained. After that cost-satisfaction interval sets are determined. In Stage 3, (1) is reduced to the MOLTP and it is solved by using Werners' compensatory "fuzzy and" operator. In the following subsections, our approach is explained in detail.

4.1. The First Stage

First of all, the fuzzy supply and demand quantities are converted to crisp forms to satisfy the balance condition. By using of the "min" fuzzy operator model proposed by Zimmermann [19], the problem

$$\begin{aligned}
 & \max \min \quad \mu_{a_i}, \mu_{b_j}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \\
 & \text{subject to} \quad \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = 0 \\
 & \quad \quad \quad \sum_{i=1}^n x_{ij} = b_j, \quad j = 1, 2, \dots, m \\
 & \quad \quad \quad \forall a_i, b_j \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n
 \end{aligned}$$

is solved for obtaining a solution which maximizes the least degree of satisfaction among all supply and demand quantities. By introducing the auxiliary variable β ,

$$\min \mu_{a_i}, \mu_{b_j} = \beta \quad \Rightarrow \quad \mu_{a_i} \geq \beta, \quad \mu_{b_j} \geq \beta,$$

this problem can be converted into the following equivalent maximization problem:

$$\begin{aligned}
 & \max \quad \beta \\
 & \text{subject to} \quad \mu_{a_i}(a_i) \geq \beta, \quad i = 1, 2, \dots, m \\
 & \quad \quad \quad \mu_{b_j}(b_j) \geq \beta, \quad j = 1, 2, \dots, n \\
 & \quad \quad \quad \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = 0 \\
 & \quad \quad \quad \forall a_i, b_j \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \\
 & \quad \quad \quad \beta \in [0, 1]
 \end{aligned} \tag{3}$$

By solving (3), crisp supply-demand quantities are determined at the maximum satisfactory degree $\bar{\beta}$ in order to get the following balance condition:

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j.$$

Since the membership function of each supply quantity is the strictly monotone decreasing for a_i in the closed interval $[a_i^2, a_i^3]$, and similarly the membership function of each demand quantity is the strictly monotone increasing for b_j in the closed interval $[b_j^1, b_j^2]$, from (3) we have

$$\begin{aligned} & \max \quad \beta \\ & \text{subject to} \quad \mu_{a_i}^{-1}(\beta) \geq a_i, \quad i = 1, 2, \dots, m \\ & \quad \quad \quad \mu_{b_j}^{-1}(\beta) \geq b_j, \quad j = 1, 2, \dots, n \\ & \quad \quad \quad \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = 0 \\ & \quad \quad \quad \forall a_i, b_j \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \\ & \quad \quad \quad \beta \in [0, 1] \end{aligned} \tag{4}$$

where $\mu_{a_i}^{-1}(\beta) = \inf\{a_i | \mu_{a_i}(a_i \geq \beta)\}$ and $\mu_{b_j}^{-1}(\beta) = \inf\{b_j | \mu_{b_j}(b_j \geq \beta)\}$. By introducing the membership functions of \tilde{a}_i and \tilde{b}_j to (4), we obtain:

$$\begin{aligned} & \max \quad \beta \\ & \text{subject to} \quad a_i + (a_i^3 - a_i^2)\beta \leq a_i^3, \quad i = 1, 2, \dots, m \\ & \quad \quad \quad b_j - (b_j^2 - b_j^1)\beta \leq b_j^1, \quad j = 1, 2, \dots, n \\ & \quad \quad \quad \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = 0 \\ & \quad \quad \quad \forall a_i, b_j \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \\ & \quad \quad \quad \beta \in [0, 1] \end{aligned} \tag{5}$$

By solving (5), the crisp values \bar{a}_i and \bar{b}_j are determined in order to satisfy the balance condition. In this way, the MOLTP with crisp supply - demand parameters but still fuzzy costs

is as follows:

$$\begin{aligned}
 \min \quad & z^k(x) = \sum_{i=1}^n \sum_{j=1}^m \tilde{c}_{ij}^k \cdot x_{ij}, \quad k = 1, 2, \dots, K \\
 \text{subject to} \quad & \sum_{j=1}^m x_{ij} = \bar{a}_i, \quad i = 1, 2, \dots, m \\
 & \sum_{i=1}^n x_{ij} = \bar{b}_j, \quad j = 1, 2, \dots, n \\
 & \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = 0 \\
 & \forall x_{ij} \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n
 \end{aligned} \tag{6}$$

We note that the transportation costs are still fuzzy numbers in (6). The fuzziness of the costs leads our problem to non-linear structure. We will overcome this difficulty in the next Stage.

4.2. The Second Stage

To overcome the non-linearity, fuzzy costs are written as depending on a cost-satisfaction parameter α . Indeed, the membership function of \tilde{c}_{ij}^k is monotone decreasing in the interval $[0, 1]$. So, there exists at least one parameter $\alpha \in [0, 1]$ satisfying

$$\mu_{c_{ij}^k}(c_{ij}^k) = \alpha \quad \Rightarrow \quad c_{ij}^k = \mu_{c_{ij}^k}^{-1}(\alpha).$$

By introducing the membership function of \tilde{c}_{ij}^k , we get the following parametric expression of $c_{ij}^k (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$:

$$\alpha = \frac{c_{ij}^{3k} - c_{ij}^k}{c_{ij}^{3k} - c_{ij}^{2k}} \quad \Rightarrow \quad c_{ij}^k = (c_{ij}^{2k} - c_{ij}^{3k})\alpha + c_{ij}^{3k}. \tag{7}$$

Since c_{ij}^k depends on α , the optimal solution of (6) also depends on α . Here, for the cost c_{ij}^k corresponding to the value of $\alpha \in [0, 1]$, all possible optimal solutions of (6) are found. Let us define the breaking points as the value of α that changed the optimal solution set and give an algorithm for finding breaking points of each objective $z^k (k = 1, 2, \dots, K)$. Let T^k be the breaking points set of objective z^k .

The Breaking Points Algorithm (BPA)

Step 0: (Initialization) Set $T^k = \{0\}$, $I_p = \emptyset$, $p = 0$, ($t_0 = 0$).

Step 1: (Obtaining the first optimal solution) Solve the crisp transportation problem corresponding to $\alpha = 0$ and obtain its optimal solution set.

Step 2: (Finding the Breaking Point) Using Stepping Stone method, determine the breaking point as

$$t_{p+1} = \{\alpha \in [0, 1] \mid d_{ij}(\alpha) \geq 0 \text{ for all } d_{ij}(\alpha) \text{ which is empty cell, } \alpha > t_p\}$$

where $d_{ij}(\alpha)$ is the contribution of empty cell (i, j) to the objective (We note that the contribution $d_{ij}(\alpha)$ must be positive since the objective function is minimization).

Step 3: (Updating) $p = p + 1$, $T^k = T^k \cup \{t_p\}$. If $t_p = 1$, then STOP.

Step 4: (Obtaining the next optimal solution) Enter the empty cell (or non-basic variable) corresponding to t_p into the basis as the basic variable. Go to Step 2.

This algorithm is executed for each objective $z^k (k = 1, 2, \dots, K)$ and then the overall breaking points set

$$T_{overall} = \bigcup_{k \in K} T^k$$

is obtained. The consecutive elements of $T_{overall}$ constitute intervals those belong to the overall cost-satisfaction intervals set $I_{overall}$. For all possible values of α in each member of $I_{overall}$, the optimal solution set remains the same due to the definition of the breaking point. Thus, the optimal solution set of problem (6) is analyzed for every possible value of the costs for each objective.

4.3. The Third Stage

Since the optimal objective value doesn't change between consecutive breaking points, a representative point α can be chosen for each member of $I_{overall}$. Substituting this α value in each objective's costs, the fuzzy costs of MOLTP are converted to crisp form for the relevant interval.

After this, the membership functions of the objectives will be defined to apply our compensatory approach. Let L_k and U_k be the lower and upper bounds of the objective function z^k , respectively. L_k and U_k can be determined as follows: Solve the MOLTP as a single objective TP using each time only one objective and ignoring all others. Determine the corresponding values for every objective at each solution derived. And find the best (L_k) and the worst (U_k) values corresponding to the set of solutions.

Alternatively, by solving $2K$ single-objective TP, the lower and upper bounds L_k and U_k can also be determined for each objective $z^k(x) (k = 1, 2, \dots, K)$ as follows:

$$L_k = \min_{x \in S} z^k(x), \quad U_k = \max_{x \in S} z^k(x) \tag{8}$$

where S is the feasible solution space that is satisfied supply-demand and non-negativity constraints. For the sake of simplicity, we used the linear membership function:

$$\mu_k(z_k) = \begin{cases} 1, & z_k < L_k \\ \frac{U_k - z_k}{U_k - L_k}, & L_k \leq z_k \leq U_k \\ 0, & z_k > U_k \end{cases} \quad (9)$$

Here, $L_k \neq U_k, k = 1, 2, \dots, K$ and in the case of $L_k = U_k, \mu_k(z_k(x)) = 1$. The membership function $\mu_k(z_k)$ is linear and strictly monotone decreasing for z^k in the interval $[L_k, U_k]$. Using Zimmermann's minimum operator [19], MOLTP can be written as:

$$\begin{aligned} & \max_x \min_k \mu_k(z_k(x)) \\ & \text{subject to } x \in S \end{aligned} \quad (10)$$

By introducing an auxiliary variable λ , (10) can be transformed into the following equivalent conventional LP problem:

$$\begin{aligned} & \max \quad \lambda \\ & \text{subject to } \mu_k(z_k(x)) \geq \lambda, \quad k = 1, 2, \dots, K \\ & \quad x \in S \\ & \quad \lambda \in [0, 1] \end{aligned} \quad (11)$$

It is pointed out that Zimmermann's min operator model doesn't always yield a strongly-efficient solution [7, 11, 18]. By using Werners' operator, (11) is converted to as follows:

$$\begin{aligned} & \max \quad \mu_{and} = \lambda + \frac{(1-\gamma)}{K}(\lambda_1 + \lambda_2 + \dots + \lambda_K) \\ & \text{subject to } x \in S \\ & \quad \mu_k(z_k(x)) \geq \lambda + \lambda_k, \quad k = 1, 2, \dots, K \\ & \quad \lambda + \lambda_k \leq 1, \quad k = 1, 2, \dots, K \\ & \quad \gamma \in [0, 1] \end{aligned} \quad (12)$$

Now, corresponding to each member of $I_{overall}$, (12) will be constructed and relevant compromise pareto-optimal solutions will be obtained for different 11 values of the compensation parameter γ .

So, our compensatory model generates compensatory compromise Pareto-optimal solutions for MOLTP

We shall prove this assertion in the following theorem.

Theorem 1. *If (x, λ^x) is an optimal solution of problem (12), then x is a Pareto-optimal solution for MOLTP where $\lambda^x = (\lambda^x, \lambda_1^x, \lambda_2^x, \dots, \lambda_K^x)$*

Proof. Suppose, to the contrary, there exists a feasible solution (y, λ^y) such that $y \succ x$. This means

$$z_k(y) \leq z_k(x), \quad k = 1, 2, \dots, K, \text{ and } z_k(y) < z_k(x)$$

for some k . Thus, for the membership functions of objectives, it can be written as

$$\mu_k(z_k(y)) \geq \mu_k(z_k(x)), \quad \forall k = 1, 2, \dots, K$$

and

$$\mu_k(z_k(y)) > \mu_k(z_k(x)),$$

for some k . And this implies that there exist λ_k^x and λ_k^y satisfying

$$\mu_k(z_k(y)) = \lambda^* + \lambda_k^y \geq \mu_k(z_k(x)) = \lambda^* + \lambda_k^x, \quad \forall k = 1, 2, \dots, K$$

and

$$\mu_k(z_k(y)) = \lambda^* + \lambda_k^y > \mu_k(z_k(x)) = \lambda^* + \lambda_k^x,$$

for some k . Therefore, it holds that

$$\lambda_k^y \geq \lambda_k^x, \quad \forall k = 1, 2, \dots, K \text{ and } \lambda_k^y > \lambda_k^x, \text{ for some } k.$$

This means that $\sum_{k=1}^K \lambda_k^y > \sum_{k=1}^K \lambda_k^x$ and so

$$\mu_{and}(y) = \lambda^* + \frac{(1-\gamma)}{K} \left(\sum_{k=1}^K \lambda_k^y \right) > \lambda^* + \frac{(1-\gamma)}{K} \left(\sum_{k=1}^K \lambda_k^x \right) = \mu_{and}(x),$$

that is, $\mu_{and}(y, \lambda^y) > \mu_{and}(x, \lambda^x)$, and this is contradictory to the fact that (x, λ^x) is an optimal solution to problem (12). If required, Pareto-optimality test [2] can be applied to the solutions of (12) and it could be seen that these solutions are Pareto-optimal for MOLTP. \square

5. An Illustrative Example

Let us consider a MOLTP with the following characteristics:

Supplies: $\tilde{a}_1 = (-\infty, 40, 120), \tilde{a}_2 = (-\infty, 150, 220), \tilde{a}_3 = (-\infty, 100, 260),$

Demands: $\tilde{b}_1 = (60, 200, \infty), \tilde{b}_2 = (20, 80, \infty), \tilde{b}_3 = (100, 220, \infty), \tilde{b}_4 = (120, 240, \infty)$

Costs:

$$\tilde{C}_1 = \begin{bmatrix} (-\infty, 1, 2) & (-\infty, 1, 3) & (-\infty, 5, 9) & (-\infty, 4, 8) \\ (-\infty, 1, 2) & (-\infty, 7, 10) & (-\infty, 2, 6) & (-\infty, 3, 5) \\ (-\infty, 7, 9) & (-\infty, 7, 11) & (-\infty, 3, 5) & (-\infty, 3, 7) \end{bmatrix}$$

$$\tilde{C}_2 = \begin{bmatrix} (-\infty, 3, 5) & (-\infty, 2, 6) & (-\infty, 2, 4) & (-\infty, 1, 5) \\ (-\infty, 4, 6) & (-\infty, 7, 9) & (-\infty, 7, 10) & (-\infty, 9, 11) \\ (-\infty, 4, 8) & (-\infty, 1, 3) & (-\infty, 3, 6) & (-\infty, 1, 2) \end{bmatrix}$$

Stage 1: After constructing the membership functions of fuzzy supply-demand quantities, the crisp values of \tilde{a}_i and \tilde{b}_j are determined as follows by solving (5):

$$\overline{a}_1 = 88, \overline{a}_2 = 192, \overline{a}_3 = 196, \overline{b}_1 = 116, \overline{b}_2 = 44, \overline{b}_3 = 148, \overline{b}_4 = 168.$$

So, we obtain the problem corresponding to (6):

$$\begin{aligned} \min \quad & z^k(x) = \sum_{i=1}^3 \sum_{j=1}^4 \tilde{c}_{ij}^k \cdot x_{ij}, \quad k = 1, 2. \\ \text{subject to} \quad & \sum_{j=1}^4 x_{1j} = 88, \quad \sum_{j=1}^4 x_{2j} = 192, \quad \sum_{j=1}^4 x_{3j} = 196, \\ & \sum_{i=1}^3 x_{i1} = 116, \quad \sum_{i=1}^3 x_{i2} = 44, \quad \sum_{i=1}^3 x_{i3} = 148, \quad \sum_{i=1}^3 x_{i4} = 168, \\ & x_{ij} \geq 0, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4. \end{aligned}$$

Stage 2: Using (7), the parametric expression of $\tilde{c}_{ij}^k (i = 1, 2, 3; j = 1, 2, 3, 4; k = 1, 2)$ is constructed. For example, $\tilde{c}_{22}^1 = (-\infty, 7, 10) \Rightarrow c_{22}^1 = 10 - 3\alpha, \alpha \in [0, 1]$.

After this, the BPA is executed for two objectives, separately and the sets of breaking points

$$T^1 = \{0, 1\}, \quad T^2 = \{0, \frac{3}{4}, 1\}$$

are obtained. T^1 indicates that for all possible values of $\tilde{c}_{ij}^1 (i = 1, 2, 3; j = 1, 2, 3, 4)$ (i.e. $\forall \alpha \in [0, 1]$), objective function z^1 has the same optimal solution set X^1 ,

$$X^1 = \left\{ \begin{matrix} x_{11} = 44, & x_{12} = 44, & x_{21} = 72 \\ x_{24} = 120, & x_{33} = 148, & x_{34} = 48 \end{matrix} \right\}, \quad z^1 \in [1204, 2040]$$

We also note that although the optimal solution set remains the same, the objective function value is changing in the closed interval $[1204, 2040]$ due to the changing of α . Here, it can be said that, under the assumption of having only the objective z^1 , X^1 will be the optimal solution set for any value of $\tilde{c}_{ij}^1 (i = 1, 2, 3; j = 1, 2, 3, 4)$. Thus, the fuzziness of the cost parameters had been eliminated.

Similarly, T^2 indicates that for the possible values of $\tilde{c}_{ij}^2 (i = 1, 2, 3; j = 1, 2, 3, 4)$ corresponding to $\forall \alpha \in [0, \frac{3}{4}]$ and $\forall \alpha \in [\frac{3}{4}, 1]$, objective function z^2 has the optimal solution set

$$X^2 = \left\{ \begin{matrix} x_{13} = 88, & x_{21} = 116, & x_{22} = 16 \\ x_{23} = 60, & x_{32} = 28, & x_{34} = 168 \end{matrix} \right\}, \quad z^2 \in [1579, 2212],$$

and

$$X^3 = \left\{ \begin{matrix} x_{13} = 72, & x_{14} = 16, & x_{21} = 116 \\ x_{23} = 76, & x_{32} = 44, & x_{34} = 152 \end{matrix} \right\}, \quad z^2 \in [1352, 1579],$$

respectively. Under the assumption of having only the objective z^2 , these results pointed out that the amount of transported from supply point 2 to demand point 1 is 116 for any value of $\tilde{c}_{ij}^2 (i = 1, 2, 3; j = 1, 2, 3, 4)$. And also the amount of transported from supply point 1 to demand point 3 is at least 72. Similar explanations can be made for other variables.

After obtaining the breaking points set T^1 and T^2 , the overall cost-satisfaction intervals set, $I_{overall}$, of our example is obtained as

$$I_{overall} = \left\{ \left[0, \frac{3}{4} \right], \left[\frac{3}{4}, 1 \right] \right\}.$$

Stage 3: A representative point must be chosen from each member of $I_{overall}$. Let us determine the representative point of intervals by arithmetic mean, that is, $\alpha_1 = 0.375, \alpha_2 = 0.875$. Thus, all parameters of the MOLTP has been converted into their crisp form. For $\alpha_1 = 0.375$, using (8), the lower and upper bounds of the objectives are determined to construct the membership functions as follows:

$$L_1 = 1726.5, \quad U_1 = 3151.5, \quad L_2 = 1895.5, \quad U_2 = 3420.$$

Using (12), the compensatory model for $[0, \frac{3}{4}]$ is constructed:

$$\begin{aligned} \max \quad & \mu_{and} = \lambda + \frac{(1-\gamma)}{2}(\lambda_1 + \lambda_2) \\ \text{subject to} \quad & \sum_{j=1}^4 x_{1j} = 88, \quad \sum_{j=1}^4 x_{2j} = 192, \quad \sum_{j=1}^4 x_{3j} = 196, \\ & \sum_{i=1}^3 x_{i1} = 116, \quad \sum_{i=1}^3 x_{i2} = 44, \quad \sum_{i=1}^3 x_{i3} = 148, \quad \sum_{i=1}^3 x_{i4} = 168, \\ & z^1(x) + 1425\lambda + 1425\lambda_1 \leq 3151.5 \\ & z^2(x) + 1524.5\lambda + 1524.5\lambda_2 \leq 3420 \\ & \lambda + \lambda_1 \leq 1, \quad \lambda + \lambda_2 \leq 1, \\ & \lambda, \lambda_1, \lambda_2, \gamma \in [0, 1], \\ & x_{ij} \geq 0, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4. \end{aligned} \tag{13}$$

By solving (13), the results for different 11 values of the compensation parameter γ with 0.1 increment are obtained and given in Table 1. The results are: the compensatory compromise Pareto-optimal solution x ; the values of objective functions $z^k (k = 1, 2)$; the satisfactory levels of the objectives corresponding to solution x , (i.e. the values of membership functions $\mu_k (k = 1, 2)$); the most basic satisfactory level λ ; the compensation satisfactory level μ_{and} , respectively. As it can be seen from Table 1, our compensatory model produced the same results for $\gamma \in [0.1, 1]$.

In a similar way, for $\alpha_2 = 0.875$, the lower and upper bounds of the objectives are determined as follows:

$$L_1 = 1308.5, \quad U_1 = 2610.5, \quad L_2 = 1465.5, \quad U_2 = 2732.$$

Thus, the compensatory model for $[\frac{3}{4}, 1]$ is

$$\begin{aligned}
 \max \quad & \mu_{and} = \lambda + \frac{(1-\gamma)}{2}(\lambda_1 + \lambda_2) \\
 \text{subject to} \quad & \sum_{j=1}^4 x_{1j} = 88, \quad \sum_{j=1}^4 x_{2j} = 192, \quad \sum_{j=1}^4 x_{3j} = 196, \\
 & \sum_{i=1}^3 x_{i1} = 116, \quad \sum_{i=1}^3 x_{i2} = 44, \quad \sum_{i=1}^3 x_{i3} = 148, \quad \sum_{i=1}^3 x_{i4} = 168, \\
 & z_1(x) + 1302\lambda + 1302\lambda_1 \leq 2610.5 \\
 & z_2(x) + 1266.5\lambda + 1266.5\lambda_2 \leq 2732 \\
 & \lambda + \lambda_1 \leq 1, \quad \lambda + \lambda_2 \leq 1, \\
 & \lambda, \lambda_1, \lambda_2, \gamma \in [0, 1], \\
 & x_{ij} \geq 0, \quad i = 1, 2, 3; \quad j = 1, 2, 3, 4.
 \end{aligned} \tag{14}$$

Table 2 shows results obtained by solving (14) for different 11 values of the compensation parameter γ . In this case, for $\gamma \in [0.2, 1]$, the same results are obtained. These solutions and the values of all membership functions are offered to Decision Maker (DM). If DM is not satisfied with the proposed solution then he/she could assign the weights $w_i, (w_i > 0, \sum_i w_i = 1)$, on his/her objectives $z^k (k = 1, 2)$. In this case, the weights w_i are inserted to the compensatory model as the following manner:

$$\begin{aligned}
 \frac{\mu_k(z^k)}{w_k} &\geq \lambda + \lambda_k, (k = 1, 2) \\
 w_k(\lambda + \lambda_k) &\leq 1, (k = 1, 2)
 \end{aligned}$$

instead of the constraints

$$\begin{aligned}
 \mu_k(z^k) &\geq \lambda + \lambda_k, (k = 1, 2) \\
 \lambda + \lambda_k &\leq 1, (k = 1, 2).
 \end{aligned}$$

So, our compensatory model generates the following compensatory compromise Pareto-optimal solutions X^{1*} and X^{2*} for $\alpha \in [0, \frac{3}{4}]$ and also X^{3*} and X^{4*} for $\alpha \in [\frac{3}{4}, 1]$ for our MOLTP

$$\begin{aligned}
 X^{1*} &= \left\{ \begin{array}{cccc} x_{11} = 0, & x_{12} = 44, & x_{13} = 44, & x_{14} = 0 \\ x_{21} = 116, & x_{22} = 0, & x_{23} = 76, & x_{24} = 0 \\ x_{31} = 0, & x_{32} = 0, & x_{33} = 28, & x_{34} = 168 \end{array} \right\}, \quad \begin{array}{l} z^1(X^{1*}) = 2128.5 \\ z^2(X^{1*}) = 2034 \end{array} \\
 X^{2*} &= \left\{ \begin{array}{cccc} x_{11} = 0, & x_{12} = 44, & x_{13} = 0.5436, & x_{14} = 43.4564 \\ x_{21} = 116, & x_{22} = 0, & x_{23} = 76, & x_{24} = 0 \\ x_{31} = 0, & x_{32} = 0, & x_{33} = 71.4564, & x_{34} = 124.5436 \end{array} \right\}, \quad \begin{array}{l} z^1(X^{2*}) = 1998.1308 \\ z^2(X^{2*}) = 2186.0974 \end{array}
 \end{aligned}$$

$$X^{3*} = \left\{ \begin{array}{cccc} x_{11} = 0, & x_{12} = 44, & x_{13} = 0, & x_{14} = 44 \\ x_{21} = 116, & x_{22} = 0, & x_{23} = 76, & x_{24} = 0 \\ x_{31} = 0, & x_{32} = 0, & x_{33} = 72, & x_{34} = 124 \end{array} \right\}, \quad \begin{array}{l} z^1(X^{3*}) = 1591.5 \\ z^2(X^{3*}) = 1612 \end{array}$$

$$X^{4*} = \left\{ \begin{array}{cccc} x_{11} = 0, & x_{12} = 44, & x_{13} = 0, & x_{14} = 44 \\ x_{21} = 116, & x_{22} = 0, & x_{23} = 57.7152, & x_{24} = 18.2848 \\ x_{31} = 0, & x_{32} = 0, & x_{33} = 90.2848, & x_{34} = 105.7152 \end{array} \right\}, \quad \begin{array}{l} z^1(X^{4*}) = 1536.6456 \\ z^2(X^{4*}) = 1687.4248 \end{array}$$

All of these solutions pointed out that for all possible values of \tilde{c}_{ij}^k ($i = 1, 2, 3; j = 1, 2, 3, 4; k = 1, 2$), the certainly transported amounts are:

$$\left\{ \begin{array}{cc} x_{11} = 0, & x_{12} = 44, \\ x_{21} = 116, & x_{22} = 0, \\ x_{31} = 0, & x_{32} = 0, \end{array} \right\}.$$

For $\gamma = 1$, μ_{and} equals to min(non-compensatory) operator that is $\mu_{and} = 0.8094$ and gives the solution X^{2*} for $\alpha \in [0, \frac{3}{4}]$. This solution remains the same from $\gamma = 1$ to $\gamma = 0.1$. As seen, the minimal satisfactory level of all objectives is equal to 0.8094.

For $\gamma = 0$, μ_{and} equals to average operator (full-compensatory) operator that is $\mu_{and} = 0.8135$ and gives the solution X^{1*} for $\alpha \in [0, \frac{3}{4}]$. As seen, the satisfactory levels of the objectives are $\mu_1 = 0.7179$ and $\mu_2 = 0.9092$, respectively. The efficiency of the transportation process for our MOLTP is averagely 0.8135.

6. Conclusion

In this paper, we deal with MOLTP whose costs and supply-demand quantities are given as fuzzy numbers. MOLTP is a well-known problem in the literature. We proposed a solution procedure with three stages that aims to defuzzificate the parameters of MOLTP. Firstly, fuzzy supply and demand quantities are converted to their crisp forms to satisfy the balance condition. And then the optimal solution sets are analyzed for every possible value of the fuzzy costs for each objective to obtain the overall breaking points set. So, through the breaking points, an analysis could be presented to DM. Finally, using Werner's μ_{and} operator, our approach generates a solution for each member of this set. This compromise solution of MOLTP with fuzzy parameters is both compensatory and Pareto-optimal. It is known that Zimmerman's "min" operator is not compensatory and also does not guarantee to generate the Pareto-optimal solutions. Werner's μ_{and} operator is useful about computational efficiency and always generates Pareto-optimal solutions. And to our knowledge, combining compensatory (μ_{and}) operator with MOLTP has not been published up to now. The proposed approach also makes it possible to overcome the non-linear nature owing to the fuzziness in the costs. This paper discussed MOLTP with fuzzy parameters. Further work will involve the multi-index form of this problem. The necessity of considering multi-index form arises when there exist different type of product and also when heterogeneous transportation modes called conveyances (i.e. trucks, air freights, freight trains, ships etc.) are available for the shipments of goods. Thus, in real world applications, the multi-index form of transportation problem becomes more important.

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Table 1: The results of our compensatory model for $[0, \frac{3}{4}]$, i.e. $\alpha_1 = 0.375$

Value of γ	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	z^1	z^2	μ^1	μ^2	λ	μ_{and}
$\gamma = 0^*$	0	44	44	0	116	0	0	0	0	0	28	168	2128.5	2034	0.7179	0.9092	0.7179	0.8135
$\gamma = 0.1 - \gamma = 1$	0	44	0.5436	43.4564	116	0	76	0	0	0	71.4564	124.5436	1998.1308	2186.0974	0.8094	0.8094	0.8094	0.8094

*Alternate optimal solutions exist.

Table 2: The results of our compensatory model for $[\frac{3}{4}, 1]$, i.e. $\alpha_1 = 0.875$

Value of γ	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	z^1	z^2	μ^1	μ^2	λ	μ_{and}
$\gamma = 0^*$	0	44	0	44	116	0	76	0	0	0	72	124	1591.5	1612	0.7826	0.8843	0.7826	0.8335
$\gamma = 0.1$	0	44	0	44	116	0	76	0	0	0	72	124	1591.5	1612	0.7826	0.8843	0.7826	0.8284
$\gamma = 0.2 - \gamma = 1$	0	44	0	44	116	0	57.7152	18.2848	0	0	90.2848	105.7152	1536.6456	1687.4248	0.8248	0.8248	0.8248	0.8248

*Alternate optimal solutions exist.